Educ Stud Math
DOI 10.1007/s10649-011-9373-7

Commentary 2: knowledge integration in mathematics learning: the case of inversion

Michael Schneider

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1 Why is inversion important?

In their most common form, inversion problems have the form of $a+b-b=?$. The correct answer in this case is a. It can be derived without computation because addition and subtraction of the same number cancel each other out. This solution method, that is, finding the answer without computation, is called the *inversion-shortcut strategy*. Many researchers use inversion problems, related principles, and related problem types to investigate understanding by the learners of the inverse relation between mathematical operations. In the following, I use the terms "inversion" and "area of inversion" to refer to the principles, strategies, problem types etc. that are related to the concept of inversion. Researchers frequently emphasize the importance of learning about inversion. However, they rarely explain why inversion is important, even though this is hardly self-evident.

There are at least three reasons to doubt the importance of learning about inversion. First, empirical studies show that it is possible to be good at arithmetic problem solving without having a good understanding of inversion and vice versa (Bryant, Christie, & Rendu, 1999; Gilmore & Papadatou-Pastou, 2009). To my knowledge, there is not a single experiment that shows a direct positive causal effect of an understanding of inversion on arithmetic or other mathematical skills (cf. Schneider & Stern, 2009). Second, inversion problems rarely occur in everyday life. The reader of this article might try to remember when he or she last encountered a problem of the form $a+b-b$ (with arbitrary numbers for a and b) in their life. Most people have troubles coming up with even just one or two examples. Third, learning about the inversion shortcut might be an inefficient use of learning time. Based on the existing studies, it can be estimated that using the inversion shortcut instead of left-to-right computation saves roughly one second, at least if relatively small and easy numbers are involved. Consequently, if a teacher used only 90 min on inversion instruction, a learner would have to solve $90\times60=5,400$ inversion problems by using the shortcut strategy before he gained more time by using the shortcut than he invested by learning about inversion.

M. Schneider (✉)
Department of Psychology, University of Trier and ETH Zurich, 54286 Trier, Germany
e-mail: m.schneider@uni-trier.de

Published online: 29 December 2011

Of course, this argument is provocative. It can and should be questioned. But it demonstrates that the question “What is learning about inversion good for?” has to be taken seriously. The answer is not obvious, at least not from the perspective of everyday life, which is the perspective taken by many students.

The articles in this special issue give some clues as to why an understanding of inversion might be an important learning goal. The authors argue that the field of mathematical inversion is ideal for teaching students about how to competently choose and execute an arithmetical strategy (Robinson & LeFevre, 2011, this issue; Nunes, Bryant, Bell, Evans, & Barros, 2011 this issue), about how to know and use multiple strategies in a domain (Peltenburg, van den Heuvel-Panhuizen, & Robitzsch, 2011, this issue), about deciding adaptively between alternative strategies (Peters, De Smedt, Torbeyns, Ghesquière, & Verschaffel, 2011, this issue), about acquiring well-organized mathematical knowledge, about mathematical modeling, and about solving mathematical problems with understanding (Greer, 2011, this issue; Selter, Prediger, Nührenbörger, & Hußmann, 2011, this issue).

All of the enumerated learning goals have one thing in common: They can be formulated without reference to inversion. They are all based on the idea that inversion is a relationally rich area of mathematics, which bodes well for exemplifying general competences and insights that are important in *other* mathematical and non-mathematical areas as well. For example, “deciding adaptively between alternative strategies” is a learning goal that is important not only for solving inversion problems but also for solving mathematical problems not related to inversion. This explains—and justifies—why educational researchers and some practitioners pay so much attention to mathematical inversion.

However, this argument raises two questions: First, is inversion really a highly relational content area that provides a multitude of learning opportunities? Second, do students use these learning opportunities to build up correct and highly relational knowledge about inversion? We can expect a positive effect of learning about inversion only if both questions have to be answered with “yes”. We will discuss these questions in the following two sections.

2 Is inversion a highly relational content area?

Many authors give examples of how multiple concepts, solution strategies, and experiences can be intertwined and complement each other in the area of inversion. In line with the cognitive literature, we are using the words concept and principle synonymously with each other throughout this article. The same applies to the words strategy and procedure. *Concepts* related to inversion are the inversion principle ($a+b-b=a$), the subtractive negation principle ($a-a=0$), the subtractive identity principle ($a-0=a$), and the complement principle ($a+b=c$ is equivalent to $a=c-b$) (Baroody, Torbeyns, & Verschaffel, 2008), which directly relate to each other. Inversion is also closely linked to the concept of mathematical groups (Greer, 2011, this issue). Important computational *strategies* include left-to-right arithmetic computations as well as the various shortcut strategies that apply to problem types such as $5+2-2=?$ or $8-8=?$. Partly different strategies might be needed for addition and subtraction (e.g., $5+2-2=?$) and multiplication and division (e.g., $5\times3/3=?$) (Robinson & LeFevre, 2011, this issue), for whole-number arithmetic and for fraction arithmetic, as well as for arithmetic problems (e.g., $7+3-3=?$) and algebraic problems (e.g., $12+b-b=?$) (cf. Selter et al., 2011, this issue). Related *everyday-life experiences* are, for example, that making a shirt dirty and cleaning it puts it back in its original state, that after having taken three cookies out of a jar putting three cookies in later restores the original number of cookies in the jar, and that splitting a collection of objects into two subsets and rejoining the sets restores the original number of objects (cf. Bryant et al., 1999; Sherman & Bisanz, 2007).

Given this high number of pieces of knowledge which all relate to the idea of inversion, it seems plausible that inversion is a content area in which learners can acquire many insights about how mathematics works in general. For example, Greer (2011, this issue) writes "Inversion is a fundamental relational building-block both within mathematics as the study of structures and within people's physical and social experience, linked to many other key elements such as equilibrium, invariance, reversal, compensation, symmetry, and balance." However, learners first need to understand mathematical inversion before they can generalize and transfer this knowledge and use it as a "building-block" in other mathematical, physical, and social content areas. Ill-understood and ill-structured knowledge can hardly be transferred (Wagner, 2006). Do learners really understand how the diverse principles, strategies, and experiences in this area are connected on a mathematical level? Unfortunately, many empirical findings suggest otherwise.

3 Do students build up highly relational knowledge about inversion?

A number of studies demonstrate that learners frequently fail to see how different pieces of knowledge relate to each other in the area of inversion. They do not transfer insights between situations or problem types and, thus, do not generalize their knowledge. For example, learners frequently demonstrate selective knowledge of only some pieces of knowledge, while they lack knowledge about closely related principles or strategies. Other findings are low inter-correlations of measures assessing related pieces of knowledge and unexpected developmental orderings.

Schneider and Stern (2009) provide an extensive overview of empirical studies indicating the fragmentation of children's knowledge about inversion. For example, in a cross-sectional study, children demonstrated an understanding of the subtractive negation principle and the subtractive identity principle already at the age 4 years but did not show an understanding of the inversion principle before the age of 6 years (Baroody, Lai, Li, & Baroody, 2008), although all three principles are closely related (e.g., $a+b-b=a$ implies that $b-b=0$). Contrary to theoretical expectations, preschool children's counting competence does not seem to contribute to their performance on inversion problems (Sherman & Bisanz, 2007). Several studies used a wide range of methods including a meta-analysis of 14 studies (Gilmore & Papadatou-Pastou, 2009) show that arithmetic skills and an understanding of inversion were largely independent of each other.

Consequently, learners' knowledge about inversion consists of many pieces or facets which are acquired and stored in memory partly independent of each other. Even though inversion is a highly relational content area from a theoretical point of view, learners' actual knowledge of inversion is fragmented, heterogeneous, and multifaceted.

4 The knowledge integration perspective

As described earlier, many authors imply that understanding inversion is a worthwhile learning goal because this content area is highly relational, and that understanding mathematical relations in this area might help to understand mathematical relations in general. This perspective has been termed *knowledge dissociation perspective* (Schneider & Stern, 2009), because from this perspective empirically found dissociations between pieces of knowledge are considered to be a central unexplained problem. An alternative, and more plausible, perspective is the knowledge integration perspective (Baroody, 2003; Linn, 2006; Schneider & Stern, 2009) which acknowledges that learners' knowledge is usually fragmented to some degree and which can explain why this is the case. From this latter perspective, knowledge integration cannot be taken as given but has to be fostered by instruction.

At least four lines of research support the knowledge integration perspective. First, studies on conceptual change and development traced learners' knowledge by means of fine-grained interview questions and category systems. They found that knowledge fragmentation occurs frequently and in various age groups (diSessa, 2008; diSessa, Gillespie, & Esterly, 2004). The degree of knowledge fragmentation decreased with learners' increasing experience with a domain (Clark, 2006; Straatemeier, van der Maas, & Jansen, 2008). Most of this research has been conducted in the domain of physics, but at least two empirical studies (Izsák, 2005; Wagner, 2006) demonstrated fragmented knowledge about mathematics as well.

Second, according to studies on the acquisition of expertise, the organization of knowledge in memory is an important characteristic that distinguishes novices from experts in a domain (Chase & Simon, 1973; Gobet, 1998). Learners pick up observations in superficially different situations. Experts have abstract background knowledge that helps them to understand how these pieces of knowledge relate to each other on a conceptual level. Novices lack this expert knowledge and, thus, are forced to focus on superficial differences between their observations (Chi, Feltovich, & Glaser, 1981). Thus, the same content domain can look highly relational and integrated for teachers, researchers, and other experts and simultaneously look highly heterogeneous and fragmented for students and other novices.

Third, the limited capacity of learners' working memory can explain why it takes much time and effort (cf. Ericsson, Krampe, & Tesch-Römer, 1993) to build up well-integrated knowledge structures in long-term memory required for expertise. Long-term memory has a virtually unlimited capacity and can store a vast amount of knowledge. However, new knowledge can only enter long-term memory through working memory, which can store only a few elements (Baddeley, 1994). Therefore, it is generally not possible to instantaneously acquire a large and well-integrated network of conceptual knowledge and to store it in long-term memory. Instead, learners need many cycles in which they load prior knowledge and new information into their working memory, compare and integrate them, and finally store the result in long-term memory. During this process the bottle-neck nature of working memory actually has a useful function: It forces learners to focus on the most important pieces of knowledge in a situation and prevents them from drowning in a flood of largely irrelevant information (Cowan, 2010).

Finally, as implied by proponents of the situated cognition view (Greeno, Moore, & Smith, 1993), low degrees of abstraction and transfer in learners have at least one advantage. They prevent learners from over-generalizing their knowledge. For example, blueberries are small, round, blue, tasty, and nutritious. Ivy berries, on the other hand, are small, round, blue, bitter, and poisonous. Thus, learners who generalize from their experiences with blueberries that all small, round, and blue berries are good for them will be surprised. The same applies to mathematics: $5+2$ equals $2+5$ and 5×2 equals 2×5; but this cannot be generalized to mathematics in general. For example, $5-2$ does not equal $2-5$. The term $a\times b$ equals the term $b\times a$ when a and b are rational numbers but not when a and b are matrices. Learners who are careful to avoid these and similar over-generalizations might pay the price of missing some important opportunities for actually adequate generalizations, for example, from inversion in arithmetic to inversion in algebra.

These examples show that knowledge fragmentation (a) occurs frequently, in particular, in novices in a domain, (b) results naturally from characteristics of the human information processing system, and (c) has the advantage of preventing learners from potentially harmful over-generalizations. This explains why many studies have demonstrated fragmented knowledge about inversion. It also raises the question why almost no studies investigate ways to instructionally support the integration of children's knowledge about inversion.

5 Implications for theory and practice

On the level of learning theories, there are at least two possible reasons why so far research on inversion, and research on mathematics learning in general, has largely ignored the problem of knowledge fragmentation and integration. First, mathematics is widely seen as a system of abstract inter-relations; and many mathematics educators have a solid background in mathematics. This might make it hard for them to imagine that the area of inversion can look completely different, that is, like a heterogeneous collection of partly unrelated pieces of knowledge, from the viewpoint of a novice. Second, inversion is closely related to Piaget's notion of conservation. Accordingly, research on inversion was strongly influenced by Piaget's structuralist view of cognitive development (Bryant et al., 1999), which is best known for its focus on domain-general stages of development and on logical structures. This might have diverted researchers from the ill-structured and seemingly illogical nature of many learners' knowledge in this area. The issue whether Piaget's theory can explain knowledge fragmentation at all, for example, by referring to the mechanism of horizontal decalage, is under dispute (Kreitler & Kreitler, 1989). In contrast, as explained above, information processing theories of development and some conceptual change theories can explain the finding of knowledge fragmentation in detail and very naturally (Schneider, 2012). Future studies will have to examine their power as theoretical frameworks for further research on inversion.

From a practical perspective, instructional support of knowledge integration is highly important (Linn, 2006). Students are not blank slates; they already have prior knowledge at the beginning of an instructional unit. Therefore, teachers have to assess students' prior knowledge and monitor its changes during instruction. This formative assessment (Sadler, 1989) can indicate when and where knowledge integration is needed. Linn (2006) conducted a number of case studies on how instruction can support knowledge integration. She suggests a four-step approach. First, instruction should stimulate students to elicit their ideas and make them explicit. Second, teachers should help students to acquire additional ideas which are more correct from a normative point of view. Third, the students need to develop criteria for the evaluation of their own ideas. Finally, knowledge integration requires the sorting out of irrelevant or contradictory ideas from the students' knowledge base.

Knowledge integration can also be supported by all the approaches that foster knowledge transfer and generalization, because they help learners to see abstract similarities between superficially different contexts. For example, diagrams can help to explicate the abstract structures underlying different contexts (Novick & Hmelo, 1994). Curricula can aid the acquisition of well-structured knowledge by providing meaningfully structured learning environments (Hardy, Jonen, Möller, & Stern, 2006).

6 The number line as tool for knowledge integration

Several articles in this special issue point to a representational tool for knowledge integration, which might be particularly useful in mathematics learning: this is the number line. Selter and colleagues (2011, this issue) use number lines repeatedly throughout their article to illustrate subtraction strategies for mental arithmetic, written arithmetic, arithmetic with negative numbers, and algebra. Likewise, Peltenburg and colleagues (2011, this issue), Peters and colleagues (2011, this issue), Nunes and colleagues (2011, this issue), and Greer (2011, this issue) mention the number line as a potentially effective external representation for instruction.

This interest in the number line is not specific to research on inversion. As revealed by cognitive studies, people use analog mental representations akin to mental number lines to store and process numerical magnitudes. The location in the brain and the neural mechanisms underlying the functioning of the mental number line are already rather well known (Ansari, 2008). Measures of the mental number line predict mathematical achievement in elementary school children (De Smedt, Verschaffel, & Ghesquière, 2009). A wide range of studies uses the number line estimation task to assess children's representations of numbers. Children are presented with a number and have to find its position on a number line which has labels and tick marks only at the beginning and at the end. Performance on this task correlates with arithmetic skill and mathematical achievement. This holds true for whole numbers as well as for fractions (Siegler, Thompson, & Schneider, 2011). Interventions designed to improve children's estimates on the number line have led to subsequent increases in arithmetic skill, which indicates a causal link between number line estimation and arithmetic (Booth & Siegler, 2008). Children's performance on number line estimation tasks can easily be improved by letting them play numerical board games, where they throw dice and have to count the number of fields they can move forward (Ramani & Siegler, 2008).

These many links of the number line with inversion but also with magnitude representation, arithmetic, and mathematical achievement both for whole numbers and for fractions suggest that the number line is one of the most useful tools for knowledge integration throughout mathematics. Further studies will have to test this potential.

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