CAPITAL- AND LABOR-SAVING TECHNICAL CHANGE IN AN AGING ECONOMY

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Abstract: Does population aging affect economic growth ? The answer is given in a novel analytical framework that allows for population aging to affect endogenous capital- and labor-saving technical change. Population aging is equivalent to an increase in the old-age dependency ratio of an OLG-economy with two-period lived individuals facing a survival probability. The short-run analysis reveals that population aging induces more labor- and less capital-saving technical change as it increases the relative scarcity of labor with respect to capital. Due to external contemporaneous knowledge spill-overs across innovating firms induced technical change has a first-order effect on current aggregate income. Capital-saving technical change implies that the economy's steady-state growth rate is independent of its age structure: neither a higher life-expectancy nor a decline in fertility affects economic growth in the long run.

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1 Introduction

Does population aging affect economic growth ? Currently, this question is at the heart of many policy debates both in the developing and the developed world. The reason is at least twofold. First, population aging, defined as the process by which older individuals become a proportionally larger fraction of the total population, has been the predominant demographic phenomenon in many countries over the last decades and is predicted to reach unprecedented heights. Table 1 shows actual data and forecasts of the United Nations concerning the old-age dependency ratio (OADR) for several countries and regions.[1] Roughly speaking, between 2005 and 2050 the OADR is projected to double in Europe and the United States of America. For China, India, Japan and the entire planet its predicted increase is even more pronounced. Second, these developments pose serious challenges for many important fields of economic policy including health care systems, pension schemes, or public debt (see, e. g., Bloom, Canning, and Fink (2008)).

In the political arena economic growth is often seen as a means to solve, or at least to alleviate, these problems.[2] To gauge this prospect it is fundamental to understand the causal link between population aging and economic growth. This is the topic of the present paper. More precisely, I argue that population aging has an effect on the investment behavior of firms. These investments are a crucial determinant of the speed of technical progress and, eventually, of the growth performance of an economy.

I address this issue in a novel endogenous growth model that allows for technical change to be capital- and labor-saving. Arguably, allowing for capital-saving technical change is the major contribution to the existing literature. This feature turns out to substantially modify the predicted effect of population aging on economic growth. The production side of the economy builds on and extends ideas of the so-called 'induced innovations' literature (see, e. g., Hicks (1932) and Drandakis and Phelps (1966)). Population aging affects the relative scarcity of labor with respect to capital, relative factor prices, and induces technical change.[3]

[1]The predictions appear in United Nations (2011) as the 'medium variant'. The old-age dependency ratio is the ratio of the population aged 65 or over to the population aged 15-64. This ratio is stated as the number of dependants per 100 persons of working age (15-64). I focus on the OADR as an indicator of population aging since this measure has a natural counterpart in the theoretical analysis that follows.

[2]See, e. g., Chancellor Merkel's government declaration of November 10, 2009 (Merkel (2009)).

[3]This chain of reasoning connects the phenomenon of population aging to the famous con-

Table 1: Old-Age Dependency Ratios in Selected Countries and Regions (United Nations (2011)).

Year	World	Europe	USA	China	India	Japan
2005	11	23	18	11	7	30
2050	26	47	35	38	20	74

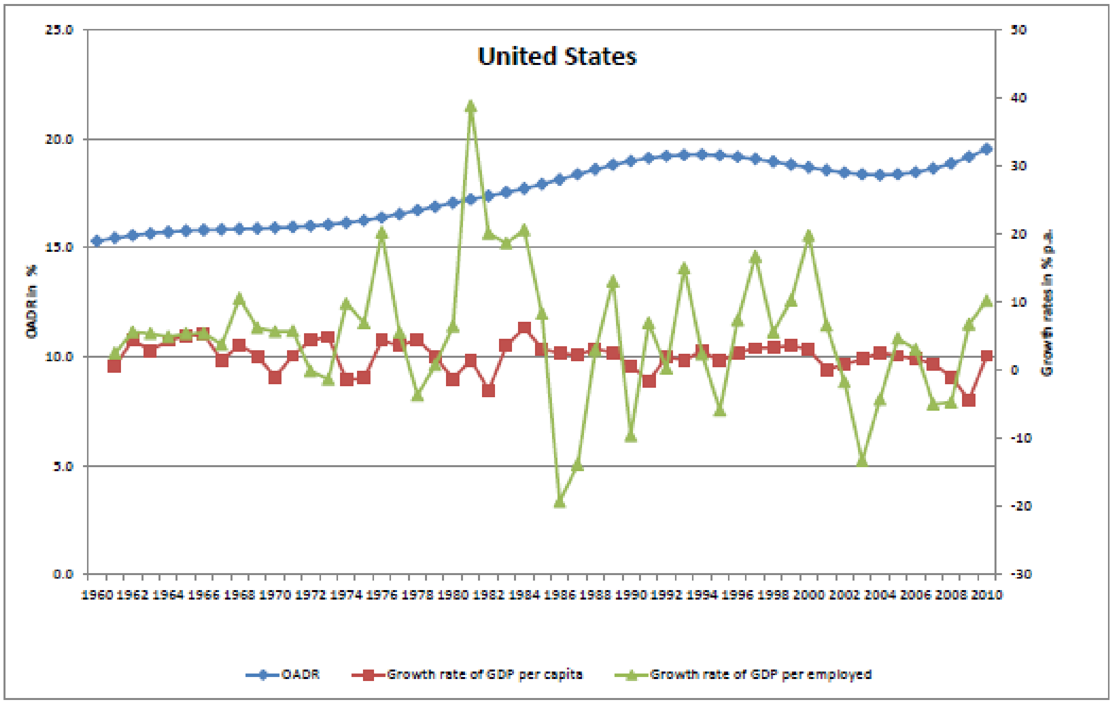
However, unlike this literature, the investment behavior of competitive firms is fully micro-founded in the present paper. The household side has two-period lived overlapping generations as in Allais (1947), Samuelson (1958), or Diamond (1965). At the onset of old age individuals face a survival probability. This framework allows for a straightforward representation of population aging as an increase in the OADR. Both, a decline in the growth rate of the young and an increase in the survival probability augment this ratio. These demographic changes capture the tendency shown in Table 1.

My analysis derives two main sets of results concerning the relationship between population aging and economic growth. The first set concerns the *short-run* effects of population aging that, by definition, occur between two adjacent periods. These results are based on the comparison of two initially identical economies with a differing demographic evolution between the two periods in question.

My findings do not support the view that population aging necessarily slows down growth in the short run. Moreover, they highlight that the source of aging matters for the growth effects. For instance, aggregate income is predicted to grow faster in the older economy if population aging is due to a higher life-expectancy. Anticipating this, households increase their savings. This leads to a

tention of John Hicks (Hicks (1932), p. 124-125) on induced inventions according to which "A change in the relative prices of the factors of production is itself a spur to invention, and to invention of a particular kind - directed to economising the use of a factor which has become relatively expensive. The general tendency to a more rapid increase of capital than labour which has marked European history during the last few centuries has naturally provided a stimulus to labour-saving inventions."

Figure 1.1: Old-Age Dependency Ratio (OADR) and Annual Growth Rates of Per-capita GDP and GDP per Person Employed: US, 1960-2010.



larger capital stock and induces faster labor productivity growth. Both channels increase aggregate income. These forces also drive per-capita income growth. However, their impact is mitigated in the older economy since a higher life-expectancy increases total population.

The second set of results is related to the *long-run* effects of population aging. As a main finding, I establish that the steady-state growth rate of the economy is independent of population aging. This property is due to the presence of endogenous capital-saving technical change. It implies that a variant of Uzawa's Steady-State Growth Theorem (Uzawa (1961)) applies to the economy. Therefore, in the steady state, capital-saving technical progress vanishes and the growth rate of per-capita variables is equal to the growth rate of labor-saving technical change. To support these growth rates, the state variables of the dynamical system adjust such that the investment behavior of profit-maximizing firms is consistent with it. These adjustments are shown to be independent of the demographic environment.

There are at least two ways to link these findings to the recent experience of the US economy. First, as shown in Figure 1.1 for the time span between 1960 and 2010, the evolution of the OADR, of per-capita GDP growth, and of labor productivity growth appear uncorrelated.[4] This is the predicted effect of population aging on steady-state growth. Second, as shown by Klump, McAdam, and Willman (2007) for the period 1953 to 1998, the US economy exhibited exponential labor-saving technical change whereas capital-saving technical change faded away. This is the predicted behavior of technical change in the neighborhood of the steady state.[5]

This paper is organized as follows. Section 2 discusses the link to the existing literature. Section 3 presents the details of the model. Section 4 studies the intertemporal general equilibrium and establishes the dynamical system. Section 5 contains the main results of this paper on the short-run and the long-run implications of population aging for economic growth. Additional results are derived in Section 6. In Section 6.1, I revisit the role of capital-saving technical change and identify this form of technical change as the main reason for why steady-state growth is independent of population aging. The following sections allow for various new features. They highlight the robustness of my results for the long-run and discuss necessary modifications for the short-run analysis. Section 6.2 shows that all qualitative results remain valid if capital-saving investments generate external contemporaneous knowledge spill-overs in the spirit of Frankel (1962) and Romer (1986). Section 6.3 allows for three generations to be alive in each period and for expectations of survival rates to be myopic. Section 6.4 endogenizes the supply of labor, Section 6.5 sketches the role of endogenous fertility. Section 7 concludes. Proofs are relegated to Appendix A.

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[4]In Figure 1.1 the old-age dependency ratio (OADR) is the ratio of the population aged 65 and older per 100 persons of the working-age population aged 15-64. GDP per capita is gross domestic product divided by midyear population. Data on these variables are from The World Bank (2012). Data on GDP per person employed is taken from European Commission - Economic and Financial Affairs - AMECO (2012). The impression of zero correlation is confirmed when per-capita GDP or GDP per person employed is regressed on the OADR and a time trend. Then, the impact of the OADR on the outcome is statistically not different from zero. The p-value in both regressions is larger than 0.6. The detailed regression results are available from the author upon request.

[5]To the extent that population aging is due to a decline in population growth my results for the long run support what Bloom, Canning, and Sevilla (2003) call the "neutralist view": population aging has no effect on economic growth. These authors support the neutralist view with the assessment that cross-country evidence provides little evidence for population growth to either foster or hamper economic growth (ibidem, p. 17). In their empirical studyAcemoglu and Johnson (2007) find a negative but small causal effect an increased life-expectancy on economic growth. Ashraf, Lester, and Weil (2009) confirm these findings in a simulation exercise. Overall, the empirical literature provides mixed results on the effect of life expectancy on economic growth (see, e. g., Bloom, Canning, and Sevilla (2004), Lorentzen, McMillan, and Wacziarg (2008), Cervellati and Sunde (2011)).

2 Related Literature

This paper builds on and contributes to several strands of the literature. First, it relates to a large and growing literature on the economic consequences of population aging in endogenous growth economies. Here, it is closest to the strand that emphasizes the link between population aging and technical change (see, e. g., Futagami and Nakajima (2001), Futagami, Iwaisako, and Nakajima (2002), Heer and Irmen (2009), Prettner (2011)).[6] However, to the best of my knowledge, the present paper is the first that studies population aging in the context of endogenous capital- and labor-saving technical change. The discussion of Section 6.1 shows why the inclusion of capital-saving technical change substantially modifies the predicted effects of population aging in endogenous growth models.

Second, this paper makes a contribution to the theory of induced capital- and labor-saving technical change that has its roots in the so-called 'induced innovations' literature of the 1960s (see, e. g., Drandakis and Phelps (1966)). More recent contributions to this literature focus on the question whether and why purely labor-saving technical change is required for the existence of a balanced growth path (see, e. g., Acemoglu (2003) or Jones (2005)). The answer suggested by the present paper is that the Steady-State Growth Theorem of Uzawa (1961) also has implications for endogenous growth models. As shown in Section 5.2, the main implication of Uzawa's Theorem applies in the present context since equilibrium net output has constant returns to scale in capital and labor.[7] As a consequence, a steady-state path that starts in finite time must have zero growth of capital-saving technical change.

Third, there is a contribution to the theory of competitive endogenous growth. The production sector of the economy under scrutiny here builds on and substantially extends the one of Hellwig and Irmen (2001) and Irmen (2005). Moreover, my paper is related to the class of competitive endogenous growth models including Champernowne (1961) and Zeira (1998) studied and classified in Acemoglu (2007) and Acemoglu (2010). However, the presence of capital-saving technical change implies that Acemoglu's results do not carry over to the present setting. The reason is that the technology here is two-dimensional and net output of the final good exhibits neither increasing nor decreasing differences in labor and the

[6] A second strand focusses on the aging-education nexus (see, e. g., de la Croix and Licandro (1999), Zhang, Zhang, and Lee (2001), or Boucekkine, de la Croix, and Licandro (2002)). Prettner and Prskawetz (2010) provide a recent survey of both strands.

[7] A general proof of Uzawa's Theorem applied to an economy where technical change requires resources can be found in Irmen (2012b).

vector of technologies. Therefore, in the taxonomy of Acemoglu, the technology is neither *strongly labor saving* nor *strongly capital-saving*.

Finally, this paper also broadens and complements the theory of OLG–models (de la Croix and Michel (2002)). Indeed, I show in Section 4.2 that the dynamical system with endogenous capital- and labor-saving technical change nests several other specifications. They include the economy where endogenous technical change must be labor-saving and the model of Diamond (1965), either with or without exogenous technical change.

3 The Basic Model

The economy has a household sector and a final-good sector in an infinite sequence of periods $t = 1, 2, ..., \infty$. The household sector comprises two-period lived individuals facing a survival probability. There are three objects of exchange. The *manufactured final good* can be consumed or invested. If invested it may either become future capital or serve as an input in current capital- or labor-saving investments. Households supply *labor* and *capital*. Labor is 'owned' by the young, the old own the capital stock. Capital is the only asset in the economy and, without loss of generality, fully depreciates after one period.[8] Each period has markets for all three objects of exchange. The final good serves as numéraire.

3.1 Households

Individuals live for possibly two periods, young and old age. When young, they work, consume, and save. At the onset of old age, they face a survival probability $\nu \in (0, 1)$. Survivors retire and consume their wealth. The population at t consists of L_t young and νL_{t-1} old individuals. Due to births and other demographic factors, the amount of young individuals between two adjacent periods grows at rate $\lambda > (-1)$. For short, I shall refer to λ as the fertility rate.

[8] My setup is mute on the question as to who owns the infinitely-lived firms in the economy. As individual preferences are defined over the single consumption good of each period, property rights of firms do not matter for the equilibrium allocation. Moreover, I consider competitive equilibria where maximized per-period profits are zero such that the expected present discounted value of dividends associated with any ownership share is zero, too. It is well known that these considerations are not sufficient to exclude equilibria with bubbles since the number of potential traders is infinite in the OLG-framework (Tirole (1985)). In what follows, I disregard this possibility and focus on equilibria without bubbles.

Young individuals are endowed with a labor endowment equal to unity. It is inelastically supplied to the labor market. Hence, L_t is the labor supply at t and λ its growth rate. The OADR at t is then

$$OADR_t \equiv \frac{\nu L_{t-1}}{L_t} = \frac{\nu}{1+\lambda}. \tag{3.1}$$

According to this measure, there is population aging between period $t-1$ and t if $OADR_t > OADR_{t-1}$. A decline in the fertility rate of generation $t-1$ and/or an increase in the survival probability of this generation leads to population aging.

Preferences of a member of cohort t are homothetic and defined over the level of consumption when young and old, c_t^y and c_{t+1}^o, respectively. Normalizing $u(0) = 0$ to be the utility after death, expected lifetime utility is

$$U_t = u\left(c_t^y\right) + \nu u\left(c_{t+1}^o\right), \tag{3.2}$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is a per-period utility function. It is $\mathcal{C}^2$ on $\mathbb{R}_{++}$ and satisfies $u'(c) > 0 > u''(c)$ as well as $\lim_{c\to 0} u'(c) = \infty$.

I follow, e. g., Yaari (1965) or Blanchard (1985), and assume a perfect annuity market for insurance against survival risk. At the end of their young age, individuals of cohort t deposit their entire savings with mutual funds. These funds rent savings out as capital to the firms producing in $t+1$. The latter pay a real rental rate R_{t+1} per unit of capital. Perfect competition among mutual funds assures a gross return to a surviving old at $t+1$ of R_{t+1}/ν.

Hence, the maximization of (3.2) is subject to the per-period budget constraints $c_t^y + s_t \leq w_t$ and $c_{t+1}^o \leq s_t\, R_{t+1}/\nu$, where s_t denotes savings and w_t the real wage at t. Given the vector of prices, $(w_t, R_{t+1}) \in \mathbb{R}_{++}^2$, standard arguments reveal that the optimal plan of a member of cohort t, $\left(c_t^y, s_t, c_{t+1}^o\right)$, includes a continuous and partially differentiable function[9]

$$s_t = s\left(R_{t+1}, \nu\right) w_t, \quad \text{with} \quad s_R\left(R_{t+1}, \nu\right) \gtreqless 0 \quad \text{and} \quad s_\nu\left(R_{t+1}, \nu\right) > 0. \tag{3.3}$$

3.2 Firms

At all t, the production sector has a continuum $[0, 1]$ of competitive firms. Without loss of generality, their behavior may be analyzed through the lens of a competitive representative firm. To save on heavy notation, I shall use this perspective throughout the analysis if not indicated otherwise.

[9] See, e. g., Bloom, Canning, and Graham (2003) for empirical support for the positive effect of longevity on the savings rate. These authors also find that the effect of the interest rate on the savings rate is small.

3.2.1 Technology

Two types of tasks have to be performed to produce output. The first type needs capital, the second labor as the only input. Let $m \in \mathbb{R}_+$ index a task performed by capital and $n \in \mathbb{R}_+$ a task performed by labor. With m_t and n_t denoting the total 'number' of tasks of each type performed at t, I have $m \in [0, m_t]$ and $n \in [0, n_t]$. The production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ assigns the maximum output, Y_t, to each pair $(m_t, n_t) \in \mathbb{R}^2_+$, i. e.,

$$Y_t = F(m_t, n_t). \tag{3.4}$$

The function F is $\mathcal{C}^2$ on $\mathbb{R}^2_{++}$ with $F_1 > 0 > F_{11}$ and $F_2 > 0 > F_{22}$. Moreover, it exhibits constant-returns-to-scale with respect to both task types.[10] For further reference, let κ_t denote the period-t task intensity of the firm, i. e.,

$$\kappa_t = \frac{m_t}{n_t}. \tag{3.5}$$

The production function in intensive form is then $F(\kappa_t, 1) \equiv f(\kappa_t)$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$, with $f'(\kappa_t) > 0 > f''(\kappa_t)$ for all $\kappa_t > 0$.

At t, a task m requires $k_t(m) = 1/b_t(m)$ units of capital, a task n needs $l_t(n) = 1/a_t(n)$ units of labor. Hence, $b_t(m)$ and $a_t(n)$ denote the productivity of capital and labor, respectively. They are equal to

$$\begin{aligned} b_t(m) &= B_{t-1}(1-\delta)(1+q_t^B(m)), \\ a_t(n) &= A_{t-1}(1-\delta)\left(1+q_t^A(n)+\eta^A e_t^A\right); \end{aligned} \tag{3.6}$$

here B_{t-1} and A_{t-1} denote aggregate indicators of the level of technological knowledge at $t-1$, and $\delta \in (0,1)$ is the rate of depreciation of technological knowledge between any pair of periods $t-1$ and t. Accordingly, the terms $B_{t-1}(1-\delta)$ and $A_{t-1}(1-\delta)$ represent the level of technological knowledge to which the firm at t has access for free. Then, $q_t^B(m) \in \mathbb{R}_+$ and $q_t^A(n) \in \mathbb{R}_+$ are indicators of productivity growth associated with task m and task n, respectively. Finally, the productivity of labor in task n hinges on an external effect, $e_t^A \geq 0$. It captures external contemporaneous knowledge spill-overs associated with the creation of labor-saving technical change. This externality is equal to the average productivity growth rates achieved in all tasks using labor, i. e.,

$$e_t^A \equiv \frac{1}{n_t}\int_0^{n_t} q_t^A(n)dn.$$

[10] To include, e. g., the CES production function, I make no assumptions on the limits of the function F and its derivatives for $m_t \to 0$, $n_t \to 0$, $m_t \to \infty$, and $n_t \to \infty$.

The parameter $\eta^A \in \mathbb{R}_+$ measures its strength.

To achieve productivity growth rates $q_t^B(m) > 0$ and $q_t^A(n) > 0$, the firm must invest $i(q_t^B(m)) > 0$ and $i(q_t^A(n)) > 0$ units of final output in period t. To fix ideas, suppose that a task has to be performed by a mainframe computer that is part of a firm's capital stock. Then, any equipment investment that reduces the time this computer needs to accomplish the task in question generates capital-saving technical change. Similarly, for a task performed by labor, one may think of labor-saving technical change as the result of an equipment investment that reduces the amount of time a worker needs to accomplish the considered task.[11] In addition, labor-saving technical change may also reflect investments of the firm in the human capital of its workforce. As, e. g., in Lucas (1988), the creation of human capital motivates the presence of the positive external effect whenever $\eta^A > 0$.

The function $i : \mathbb{R}_+ \to \mathbb{R}_+$ is the same for all tasks, time invariant, $\mathcal{C}^2$ on $\mathbb{R}_{++}$, increasing and strictly convex. Hence, higher rates of productivity growth require ever larger investments. Moreover, with the notation $i'\left(q^j\right) \equiv di\left(q^j\right)/dq^j$ for $j = A, B$, it satisfies

$$i(0) = 0, \quad \lim_{q^j \to 0} i'(q^j) = 0, \quad \text{and} \quad \lim_{q^j \to \infty} i\left(q^j\right) = \lim_{q^j \to \infty} i'\left(q^j\right) = \infty. \tag{3.7}$$

At the level of the individual firm, I assume that any new piece of technological knowledge is proprietary knowledge of an investing firm only in t, i. e., in the period when it occurs. Subsequently, the advancement of technological knowledge becomes embodied in aggregate task specific productivity indicators (A_t, B_t), (A_{t+1}, B_{t+1}), ..., with no further scope for proprietary exploitation. The evolution of these indicators will be specified below. If the firm decides not to make an investment for a task m or n then it has access to the production technique represented by $A_{t-1}(1-\delta)$ and $B_{t-1}(1-\delta)$ such that $a_t(n) = A_{t-1}(1-\delta)$ and $b_t(m) = B_{t-1}(1-\delta)$.

[11] At the semantic level, the effect of technical change associated with equipment investments motivates the terminology 'capital-saving' and 'labor-saving'. Hicks (1932), p. 121-122, classified technical change according to its effect on the ratio of the marginal product of capital to that of labor and called technical change *labor-saving* (*capital-saving*) if it increases (decreases) this ratio. The results established Irmen (2012a) show that the Hicksian definition applies to the present model, too.

3.2.2 Profit-Maximization

The firm takes the sequence $\{w_t, R_t, A_{t-1}, B_{t-1}, e_t^A\}_{t=1}^{\infty}$ of real wages, real rental rates of capital, of aggregate productivity indicators, and of the knowledge externality as given and chooses a production plan $\left(m_t, n_t, k_t(m), l_t(n), q_t^B(m), q_t^A(n)\right)$ for $m \in [0, m_t]$, $n \in [0, n_t]$ and all t. This plan maximizes the sum of the present discounted values of profits in all periods. Since an investment generates proprietary knowledge only in the period when it is made, the inter-temporal maximization boils down to the maximization of per-period profits. Hence, for each period t, the firm needs to find the plan that maximizes turnover minus total costs

$$\begin{aligned} \Pi_t &= F(m_t, n_t) - C_t, \\ C_t &= \int_0^{m_t} \left[R_t k_t(m) + i(q_t^B(m))\right] dm + \int_0^{n_t} \left[w_t l_t(n) + i(q_t^A(n))\right] dn; \end{aligned} \tag{3.8}$$

here, C_t sums up the costs per task of both task types. With (3.6) I have

$$k_t(m) = \frac{1}{B_{t-1}(1-\delta)(1+q_t^B(m))} \quad \text{and} \quad l_t(n) = \frac{1}{A_{t-1}(1-\delta)\left(1+q_t^A(n)+\eta^A e_t^A\right)}.$$

Therefore, at all $t = 1, 2, ..., \infty$, the firm's problem may be split up in two parts. First, it chooses for each $n \in [0, n_t]$ and $m \in [0, m_t]$ the values $\left(q_t^A(n), q_t^B(m)\right) \in \mathbb{R}_+^2$ that minimize C_t. Second, it determines the number of tasks $(n_t, m_t) \in \mathbb{R}_+^2$ that maximize Π_t. The respective first-order (sufficient) conditions for an interior solution are[12]

[12] Sufficiency follows since Π_t is strictly concave in $\left(q_t^A(n), q_t^B(m)\right)$ and exhibits constant returns to scale in (n_t, m_t).

$$q_t^A(n) \quad : \quad \frac{-w_t}{A_{t-1}(1-\delta)(1+q_t^A(n)+\eta^A e_t^A)^2} + i'\left(q_t^A(n)\right) = 0, \forall n \in [0, n_t], \tag{3.9}$$

$$q_t^B(m) \quad : \quad \frac{-R_t}{B_{t-1}(1-\delta)(1+q_t^B(m))^2} + i'\left(q_t^B(m)\right) = 0, \forall m \in [0, m_t], \tag{3.10}$$

$$n_t \quad : \quad f(\kappa_t) - \kappa_t f'(\kappa_t) - i\left(q_t^A(n_t)\right) - \frac{w_t}{a_t(n_t)} = 0, \tag{3.11}$$

$$m_t \quad : \quad f'(\kappa_t) - i\left(q_t^B(m_t)\right) - \frac{R_t}{b_t(m_t)} = 0. \tag{3.12}$$

For each task of the respective type, conditions (3.9) and (3.10) equate the marginal reduction of the firm's wage bill/capital cost to the marginal increase in its investment costs. Hence, these conditions assure that each task is performed at minimum cost. Assuming $w_t > 0$ and $R_t > 0$, the convexity of the innovation cost function and the fact that $\lim_{q^j \to 0} i'(q^j) = 0$, $j = A, B$, imply that these conditions determine a unique $q_t^A(n) = q_t^A > 0$ and $q_t^B(m) = q_t^B > 0$ for either task type. Accordingly, $a_t(n) = a_t$, $b_t(m) = b_t$, and $e_t^A = q_t^A$.[13]

Conditions (3.11) and (3.12) assure that the number of tasks of each type is chosen optimally. For the marginal task it must hold that the difference between its value product and its cost vanishes. The former is expressed in terms of the task intensity, the latter is the sum of the investment outlays and the wage, respectively capital cost of the marginal task.

Observe that Π_t has constant returns to scale in (m_t, n_t) at $q_t^A(n) = q_t^A$ and $q_t^B(m) = q_t^B$. Therefore, conditions (3.11) and (3.12) leave the number of tasks undetermined. They will be pinned down by market clearing conditions. For the same reason, profit-maximization implies zero profits. Combining the first-order conditions at the optimum delivers the following important result.

Lemma 1 *If (3.9) - (3.12) hold at t, then there are maps, $g^A : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ and $g^B : \mathbb{R}_{++} \to \mathbb{R}_{++}$, such that $q_t^A = g^A(\kappa_t, \eta^A)$ and $q_t^B = g^B(\kappa_t)$ satisfy*

$$g_\kappa^A(\kappa_t, \eta^A) > 0 > g_\kappa^B(\kappa_t) \quad \textit{and} \quad g_{\eta^A}^A(\kappa_t, \eta^A) < 0 \quad \textit{for all } \kappa_t > 0. \tag{3.13}$$

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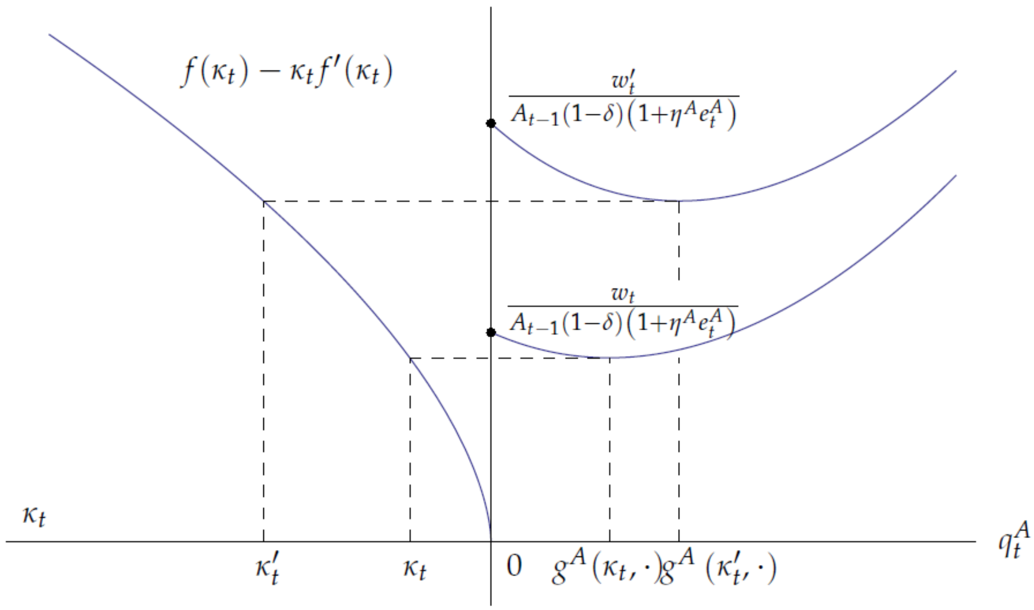
[13]Upon dividing (3.9) by (3.10) and rearranging reveals that the incentives to minimize (total) costs constitute an essential part of Hicks' conjecture quoted in Footnote 3. Indeed, one obtains $w_t/R_t = A_{t-1}\left(1 + \left(1+\eta^A\right) q_t^A\right)^2 i'\left(q_t^A\right) / \left[B_{t-1}(1+q_t^B)^2 i'\left(q_t^B\right)\right]$. Since the numerator of the right-hand side increases in q_t^A and the denominator increases in q_t^B, an increase in the relative price of labor induces, ceteris paribus, relatively more labor-saving technical change, i. e., a hike in w_t/R_t means a greater ratio q_t^A/q_t^B.

Figure 3.1: The Link between κ_t, $q_t^A = g^A\left(\kappa_t, \eta^A\right)$ and w_t.



Moreover, there are maps $w : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ *and* $R : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$, *such that the real wage and the rental rate of capital satisfy*

$$\begin{aligned} w_t &= w(\kappa_t, A_{t-1}) > 0, \quad \textit{with} \quad w_\kappa(\kappa_t, A_{t-1}) > 0, \quad w_A(\kappa_t, A_{t-1}) > 0, \\ R_t &= R(\kappa_t, B_{t-1}) > 0, \quad \textit{with} \quad R_\kappa(\kappa_t, B_{t-1}) < 0, \quad R_B(\kappa_t, B_{t-1}) > 0. \end{aligned} \tag{3.14}$$

Lemma 1 states two key properties of the production sector. First, the equilibrium incentives to engage in labor- and capital-saving technical change depend on the task intensity. The incentive to engage in labor-saving technical change increases with this intensity, the incentive to engage in capital-saving technical change decreases with it. Moreover, the positive externality reduces the incentives to invest in labor-saving technical change. Second, factor prices may be expressed as a function of the task intensity and the respective technology indicators, A_{t-1} and B_{t-1}.[14] While the real wage increases with the task intensity, the real rental rate of capital declines with it. Both factor prices increase in the technology indicators.

Intuitively, the effect of changing κ_t on investment incentives and factor prices reflects two sides of the same coin. To see this, consider a triple $\left(\kappa_t, g^A\left(\kappa_t, \eta^A\right), w_t\right)$

[14]One readily verifies that the function w also depends on η^A with w_{η^A} being indeterminate in general. Since this argument is of little interest for what follows I shall suppress it.

that satisfies (3.9) and (3.11). As shown in Figure 3.1, for these values the cost per task is minimized (right panel) and this minimum is equal to the marginal value product of task n_t (left panel). The latter increases as κ_t increases to κ_t' so that the minimum cost of task n_t must increase. This requires the higher wage w_t', hence $w_\kappa(\kappa_t, A_{t-1}) > 0$. At w_t', the marginal benefit of faster productivity growth is higher. Therefore, the minimum cost is attained at a higher q_t^A, hence $g_\kappa^A(\kappa_t, \eta^A) > 0$. The effect of κ_t on R_t and q_t^B may be derived in an analogous manner. However, in this case an increase in κ_t reduces the marginal value product of task m_t in (3.12). Therefore R_t must decline, and so does q_t^B in accordance with (3.10). Hence, $R_\kappa(\kappa_t, B_{t-1}) < 0$ and $g_\kappa^B(\kappa_t) < 0$.

Finally, note that the external effect increases in η^A. Therefore, a larger η^A weakens the marginal cost advantage of labor-saving innovation investments. Accordingly, the cost per task reaches its minimum at a lower level of q_t^A. This explains why $g_{\eta^A}^A(\kappa_t, \eta^A) < 0$.

3.3 The Evolution of Technological Knowledge

The evolution of the economy's level of technological knowledge is given by the evolution of the aggregate task specific productivity indicators A_t and B_t. An important question is then how these indicators are linked to the productivity enhancing investments associated with all performed tasks. In what follows I associate the highest level of labor and capital productivity attained across all tasks of a respective type with A_t and B_t, i. e.,

$$\begin{aligned} A_t &= \max\{a_t(n) = A_{t-1}(1-\delta)\left(1 + q_t^A(n) + \eta^A e_t^A\right) \mid n \in [0, n_t]\} \\ B_t &= \max\{b_t(m) = B_{t-1}(1-\delta)\left(1 + q_t^B(m)\right) \mid m \in [0, m_t]\}. \end{aligned} \tag{3.15}$$

Since profit-maximization implies $q_t^A(n) = q_t^A = e_t^A$, $q_t^B(m) = q_t^B$, $a_t(n) = a_t$, and $b_t(m) = b_t$, I have

$$\begin{aligned} A_t &= a_t = A_{t-1}(1-\delta)\left(1 + \left(1 + \eta^A\right) q_t^A\right) \\ B_t &= b_t = B_{t-1}(1-\delta)\left(1 + q_t^B\right) \end{aligned} \tag{3.16}$$

for all $t = 1, 2, ..., \infty$, with $A_0 > 0$ and $B_0 > 0$ as initial conditions.

4 Inter-temporal General Equilibrium

4.1 Definition

A *price system* corresponds to a sequence $\{w_t, R_t\}_{t=1}^{\infty}$. An *allocation* is a sequence

$$\{c_t^y, s_t, c_t^o, Y_t, n_t, m_t, q_t^A(n), q_t^B(m), a_t(n), b_t(m), l_t(n), k_t(m), e_t^A, L_t, K_t\}_{t=1}^{\infty}$$

for all tasks $n \in [0, n_t]$ and $m \in [0, m_t]$. It comprises a strategy $\{c_t^y, s_t, c_{t+1}^o\}_{t=1}^{\infty}$ for all cohorts, consumption of the old at $t = 1$, c_1^o, and a strategy for the production sector $\{Y_t, n_t, m_t, q_t^A(n), q_t^B(m), a_t(n), b_t(m), l_t(n), k_t(m)\}_{t=1}^{\infty}$.

For an exogenous evolution of the labor force, $L_t = L_1 (1+\lambda)^{t-1}$ with $L_1 > 0$ and $\lambda > (-1)$, a given survival probability ν for all cohorts $t = 2, 3, ..., \infty$, a given initial level of capital, $K_1 > 0$, and initial values of technological knowledge, $A_0 > 0$ and $B_0 > 0$, an *inter-temporal general equilibrium with perfect foresight* corresponds to a price system, an allocation, and a sequence $\{A_t, B_t\}_{t=1}^{\infty}$ of aggregate productivity indicators that satisfy the following conditions for all $t = 1, 2, ..., \infty$:

(E1) The young of each period save according to (3.3) and supply L_t units of labor.

(E2) The production sector satisfies Lemma 1.

(E3) The market for the final good clears, i. e.,

$$L_{t-1}c_t^o + L_t c_t^y + I_t^K + I_t^A + I_t^B = Y_t, \tag{4.1}$$

where I_t^K is aggregate capital investment, I_t^A and I_t^B denote aggregate innovation investments in labor- and capital-saving technical change.

(E4) There is full employment of labor and capital, i. e.,

$$\int_0^{n_t} l_t(n)dn = L_t \quad \text{and} \quad \int_0^{m_t} k_t(m)dm = K_t. \tag{4.2}$$

(E5) The productivity indicators A_t and B_t evolve according to (3.16).

(E1) guarantees optimal behavior of the household sector under perfect foresight. Since the surviving old own the capital stock, their consumption at $t = 1$ is $\nu_0 L_0 c_1^o = R_1 K_1$. (E2) assures optimal behavior of the production sector and zero profits. (E3) states the resource constraint in (4.1). It reflects the fact that capital fully depreciates after one period.

Full employment of labor and capital, i. e., (E4), and Lemma 1 imply that in equilibrium

$$n_t = a_t L_t \qquad \text{and} \qquad m_t = b_t K_t, \tag{4.3}$$

$$I_t^A = a_t L_t i\left(q_t^A\right) \qquad \text{and} \qquad I_t^B = b_t K_t i\left(q_t^B\right), \tag{4.4}$$

i. e., for each task type the number of performed tasks is equal to the respective input in efficiency units. In other words, technical change is factor augmenting and $Y_t = F(b_t K_t, a_t L_t)$. Moreover, aggregate investment in labor- and capital-saving technical change is proportionate to the respective input in efficiency units.

Observe that the task intensity of (3.5) and the full employment condition (4.3) imply that $\kappa_t = m_t/n_t = b_t K_t / a_t L_t$. Hence, in equilibrium the task intensity is equal to the 'efficient capital intensity' defined as the amount of efficient capital per unit of efficient labor. However, from Lemma 1 the efficient capital intensity itself depends on the task intensity, i. e.,

$$\kappa_t = \frac{B_{t-1}\left(1 + g^B(\kappa_t)\right) K_t}{A_{t-1}\left(1 + \left(1 + \eta^A\right) g^A\left(\kappa_t, \eta^A\right)\right) L_t}. \tag{4.5}$$

Therefore, I have to make sure that a value $\kappa_t > 0$ exists that satisfies (4.5). To address this issue, denote by

$$\varepsilon_\kappa^A\left(\kappa_t, \eta^A\right) \equiv \frac{d \ln\left(1 + \left(1 + \eta^A\right) g^A\left(\kappa_t, \eta^A\right)\right)}{d \ln \kappa_t} > 0, \; \varepsilon_\kappa^B(\kappa_t) \equiv \frac{-d \ln\left(1 + g^B(\kappa_t)\right)}{d \ln \kappa_t} > 0$$

the elasticities of the respective productivity growth factors with respect to the efficient capital intensity.

Lemma 2 *There is a map $\kappa : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that*

$$\kappa_t = \kappa\left(\frac{B_{t-1} K_t}{A_{t-1} L_t}\right) > 0 \tag{4.6}$$

satisfies (4.5). Moreover,

$$\varepsilon^\kappa(\kappa_t) \equiv \frac{d \ln \kappa_t}{d \ln(K_t/L_t)} = \frac{1}{1 + \varepsilon_\kappa^A\left(\kappa_t, \eta^A\right) + \varepsilon_\kappa^B(\kappa_t)} \in (0,1). \tag{4.7}$$

Hence, there is a unique solution $\kappa_t > 0$ to (4.5).[15] Moreover, an increase in the capital-labor ratio, K_t/L_t, implies a higher efficient capital intensity. Due to induced innovation investments, this increase is less than proportionate.

[15]For ease of notation, the dependency of the functions $\kappa(\cdot)$ and $\varepsilon^\kappa(\cdot)$ from η^A is suppressed.

4.2 The Dynamical System

The equilibrium conditions (E1) - (E5) require savings to equal capital investment, i. e.,

$$I_t^K = s_t L_t = K_{t+1}, \quad \text{for } t = 1, 2, ..., \infty. \tag{4.8}$$

The evolution of the economy may then be characterized by means of two state variables, namely the efficient capital intensity, κ_t, and the level of the aggregate productivity indicator B_t. Let $\tilde{w}(\kappa_t) \equiv w(\kappa_t, A_{t-1})/a_t$ denote the real wage per efficiency unit. Moreover, use (3.3), (3.14), and (4.5) to define the elasticities

$$\varepsilon_R^s(\kappa_t) \equiv \frac{d\ln s\left(R\left(\kappa_t, B_{t-1}\right), \nu\right)}{d\ln R_t} \gtreqless 0, \quad \text{and} \quad \varepsilon_\kappa^R(\kappa_t) \equiv \frac{d\ln R(\kappa_t, B_{t-1})}{d\ln \kappa_t} < 0.$$

With $\varepsilon\left(\kappa_{t+1}\right) \equiv \left[\varepsilon_R^s\left(\kappa_{t+1}\right)\right]\left[\varepsilon_\kappa^R\left(\kappa_{t+1}\right)\right]\left[\varepsilon^\kappa\left(\kappa_{t+1}\right)\right]$, the dynamical system may be stated as follows.

Proposition 1 *Given* $(K_1, L_1, A_0, B_0) > 0$ *as initial conditions, there is a unique equilibrium sequence* $\{\kappa_t, B_t\}_{t=1}^{\infty}$ *determined by*

$$\frac{s\left(R\left(\kappa_{t+1}, B_t\right), \nu\right)}{1+\lambda}\tilde{w}\left(\kappa_t\right) = \frac{\kappa_{t+1}}{B_t}\frac{1+\left(1+\eta^A\right)g^A\left(\kappa_{t+1}, \eta^A\right)}{1+g^B\left(\kappa_{t+1}\right)}, \tag{4.9}$$

and

$$B_t = B_{t-1}\left(1-\delta\right)\left(1+g^B\left(\kappa_t\right)\right), \tag{4.10}$$

if

$$\varepsilon\left(\kappa_{t+1}\right) < 1 \quad \textit{for all } \kappa_{t+1} > 0. \tag{4.11}$$

For $t = 1$, κ_1 *is given by*

$$\kappa_1 = \frac{B_0\left(1+g^B\left(\kappa_1\right)\right)K_1}{A_0\left(1+\left(1+\eta^A\right)g^A\left(\kappa_1, \eta^A\right)\right)L_1} > 0. \tag{4.12}$$

According to Proposition 1, the dynamical system may be stated as a two-dimensional system of first-order, autonomous, non-linear difference equations. The equation of motion for the efficient capital intensity is (4.9). It restates the condition for savings to equal capital investment, i. e., (4.8), where K_{t+1} is replaced by an update of (4.5). For any given pair $(\kappa_t, B_t) \in \mathbb{R}_{++}^2$, (4.9) assigns a unique value $\kappa_{t+1} > 0$ if (4.11) holds. This value of κ_{t+1} is then used to derive B_{t+1} from (4.10). Since K_1, L_1, A_0, and B_0 are initial conditions, κ_1 is pinned down by (4.5) for $t = 1$.

Observe that the dynamical system of Proposition 1 nests several OLG–models with either endogenous or exogenous economic growth. For instance, the case of an economy with only endogenous labor-saving technical change obtains when $q_t^B(m) = 0$ and $B_t = 1$ are fixed for all t. I discuss this case in Section 6.1 to further elucidate the role of endogenous capital-augmenting technical change. Assuming in addition that labor productivity growth is equal to $q^A > (-1)$ for all t and costless, i. e., setting $i(q^A) = 0$, turns the production side into the one of the neoclassical growth model with exogenous labor-saving technical change. Then (4.9) reduces to $s\left(R\left(\kappa_{t+1}\right), \nu\right) \tilde{w}\left(\kappa_t\right) / \left(1+\lambda\right) = \kappa_{t+1}\left(1+\left(1+\eta^A\right) q^A\right)$, with $\kappa_1 = K_1 / \left[L_1 A_0 (1-\delta)(1+\left(1+\eta^A\right) q^A)\right]$. If moreover $q^A = \delta / \left[(1-\delta)\left(1+\eta^A\right)\right]$, then (4.9) collapses to Diamond's difference equation for the capital intensity (Diamond (1965)).

From this perspective, condition (4.11), which states the permissible percentage change of the savings rate induced by an increase in K_{t+1}/L_{t+1} at κ_{t+1}, may be seen as a generalization of a condition for the existence and the uniqueness of the inter-temporal equilibrium under perfect foresight in an OLG-economy without technical change (see, e. g., de la Croix and Michel (2002), p. 20 ff.). It allows for the savings rate to decline in response to an increase in the rental rate of capital. However, this decline should not be too pronounced.

5 Population Aging and Economic Growth

Now, I turn to the implications of the preceding results for the relationship between population aging and economic growth. I start with the short-run implications before I turn to the long run.

5.1 Population Aging and Economic Growth in the Short-Run

By definition, the short-run effects of population aging arise if the old-age dependency ratio increases between two adjacent periods. Taking the fertility and the mortality channel separately, this is either due to a decline in the fertility of the young of the first of these periods and/or to an (anticipated) increase in their survival probability. Both demographic changes increase the old-age dependency ratio in the second of the two periods. The following proposition compares the evolution of two initially identical economies that experience differing patterns of population aging between t and $t+1$.

Proposition 2 *Consider two economies with identical initial conditions* $(K_t, L_t, A_{t-1}, B_{t-1})$ *at some period* $t \geq 1$*. If the cohorts t of these economies have different fertility rates,* $\lambda > \lambda'$*, and/or different survival probabilities,* $\nu' > \nu$*, then* $(K_{t+1}/L_{t+1})' > K_{t+1}/L_{t+1}$*. Moreover,*

$$\left(q_{t+1}^{A}\right)' > q_{t+1}^{A}, \quad \left(q_{t+1}^{B}\right)' < q_{t+1}^{B}, \quad w_{t+1}' > w_{t+1}, \quad \textit{and} \quad R_{t+1}' < R_{t+1}. \tag{5.1}$$

According to Proposition 2 it is the 'older' economy that has a greater capital-labor ratio in $t+1$. Therefore, it experiences faster labor-saving technical change and slower capital-saving technical change. Moreover, its real wage is higher and its real rental rate of capital is lower.

The intuition for these results comes in two steps. The first step concerns the inter-temporal channel through which population aging affects the capital-labor ratio. It justifies why $(K_{t+1}/L_{t+1})' > K_{t+1}/L_{t+1}$. To see this, consider the capital accumulation equation (4.8). With savings of (3.3), Lemma 1, and Lemma 2 it can be expressed in terms of K_{t+1}/L_{t+1}, λ, and ν. For periods t and $t+1$ this gives

$$\frac{s\left(R\left(\kappa\left(\frac{B_t K_{t+1}}{A_t L_{t+1}}\right), B_t\right), \nu\right)}{1+\lambda} w_t = \frac{K_{t+1}}{L_{t+1}}. \tag{5.2}$$

Hence, for a given savings rate, a lower λ requires a larger capital-labor ratio in $t+1$. However, anticipating this and the ensuing implications for the efficient task intensity and the real rental rate of capital, individuals adjust there savings behavior.[16] Similarly, for a given real rental rate of capital, a higher survival probability implies a higher K_{t+1}/L_{t+1}. This induces an anticipated adjustment of the rental rate of capital and the savings rate. Total differentiation of (5.2) captures these repercussions and delivers

$$\frac{\partial\left(K_{t+1}/L_{t+1}\right)}{\partial\lambda} = -\frac{s\left(R_{t+1}, \nu\right) w_t / \left(1+\lambda\right)^2}{1-\varepsilon} < 0, \tag{5.3}$$

$$\frac{\partial\left(K_{t+1}/L_{t+1}\right)}{\partial\nu} = \frac{s_v\left(R_{t+1}, \nu\right) w_t / (1+\lambda)}{1-\varepsilon} > 0, \tag{5.4}$$

[16]There are at least two scenarios where this second channel is mute since savings do not respond to a changing rental rate of capital. First, this is the case if the inter-temporal elasticity of substitution is equal to one. Second, if expectations of generation t are not rational but 'myopic' (Michel and de la Croix (2000)), then the expected rental rate of capital is $R_{t+1} = R\left(\kappa_t, B_{t-1}\right)$, i. e., savings in t do not reflect changes of the economic environment that may happen between t and $t+1$.

where all elasticities are evaluated at κ_{t+1}. The signs follow from (4.11) and $s_v > 0$. Hence, an increase in the OADR induced either by a decline in fertility and/or by an increase in the survival probability leads to a higher capital-labor ratio.

The second step reflects the static adjustments in $t+1$ to an increase in the capital-labor ratio. According to Lemma 2, the (λ', ν') – economy exhibits a greater efficient capital intensity, i. e., $\kappa'_{t+1} > \kappa_{t+1}$. In accordance with Lemma 1, the latter induces the technology and price adjustments stated in (5.1).[17]

What are the implications of Proposition 2 for the growth rate of aggregate and per-capita income? To address this question, let V_t denote aggregate equilibrium income, or 'income' for short. Since $\Pi_t = 0$ in equilibrium, income is equal to the difference between the output of the final good and total investment outlays. More precisely, from (3.8), the market-clearing conditions (4.3) and (4.4), I obtain

$$V_t \equiv V(b_t K_t, a_t L_t) = F(b_t K_t, a_t L_t) - a_t L_t i\left(q_t^A\right) - b_t K_t i\left(q_t^B\right), \tag{5.5}$$

and per-capita income at t is $v_t \equiv V_t / (\nu L_{t-1} + L_t)$.

To prepare for the analysis of how population aging affects income growth, it proves useful to describe how changing factor supplies affect V_t through induced technical change. I denote this effect by E_t^L if it is due to a change in the labor force at t and by E_t^K if it is due to a change in the capital stock at t. Using (3.16) and Lemma 1 in (5.5) delivers

$$\begin{aligned} E_t^L &= \left[\frac{\partial V_t}{\partial q_t^A} g_\kappa^A\left(\kappa_t, \eta^A\right) + \frac{\partial V_t}{\partial q_t^B} g_\kappa^B(\kappa_t)\right] \frac{\partial \kappa_t}{\partial L_t}, \\ E_t^K &= \left[\frac{\partial V_t}{\partial q_t^A} g_\kappa^A\left(\kappa_t, \eta^A\right) + \frac{\partial V_t}{\partial q_t^B} g_\kappa^B(\kappa_t)\right] \frac{\partial \kappa_t}{\partial K_t}. \end{aligned} \tag{5.6}$$

Since $g_\kappa^A\left(\kappa_t, \eta^A\right) > 0 > g_\kappa^B(\kappa_t)$, changing factor endowments induce technical adjustments of opposite sign. In spite of this, the following lemma shows that these adjustments have unequivocal effects on income.

Lemma 3 *In equilibrium, it holds that*

$$\frac{\partial V_t}{\partial q_t^A} = \frac{\eta^A w_t L_t}{1 + (1+\eta^A) q_t^A} > 0, \quad \textit{and} \quad \frac{\partial V_t}{\partial q_t^B} = 0. \tag{5.7}$$

[17] Observe that both steps taken together provide a straightforward link between population aging and Hicks' contention mentioned in Footnote 3: population aging leads to a higher capital-labor ratio, i. e., the relative scarcity of labor increases. This induces a higher real wage, a lower real rental rate, more labor- and less capital-saving technical change.

Moreover,

$$E_t^L < 0 \quad and \quad E_t^K > 0. \tag{5.8}$$

According to the first result of Lemma 3, a small increase in q_t^A, evaluated at the equilibrium, increases V_t, whereas such an increase in q_t^B leaves V_t unchanged. The reason for this asymmetry lies in the presence of $\eta^A > 0$. Void of a contemporaneous knowledge externality associated with capital-saving investments, the competitive economy chooses the level q_t^B that maximizes income, i. e., $\partial V_t / \partial q_t^B = 0$.[18] The positive knowledge externality associated with labor-saving investments leads to under-investment in equilibrium. Therefore, having more of q_t^A increases V_t at the margin.[19]

The second result of Lemma 3 uses the reasoning above to conclude from (5.6) that $E_t^L < 0$ and $E_t^K > 0$. Hence, both a decline in the labor force and an increase in the capital stock induce more labor-saving technical change that increases income.

The following proposition compares the evolution of aggregate and per-capita income between period t and $t+1$ in the economies of Proposition 2. Denote the equilibrium income obtained under λ or ν by $V_{t+1} = V\left(b_{t+1}K_{t+1}, a_{t+1}L_{t+1}\right)$ and the one obtained under λ' or ν' by $V'_{t+1} = V\left(b'_{t+1}K'_{t+1}, a'_{t+1}L'_{t+1}\right)$.

Proposition 3 *Consider two economies with identical initial conditions* $(K_t, L_t, A_{t-1}, B_{t-1})$ *at some period* $t \geq 1$.

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[18]See Irmen (2012a) a for a detailed discussion. Related results appear in the competitive economies studied in Zeira (1998) and Acemoglu (2010).

[19]To see by how much, a *productivity effect* and an *investment effect* of opposite sign must be considered. There is a productivity effect since labor becomes more productive. Therefore, more tasks must be performed to satisfy the full employment condition (4.2), $n_t = A_{t-1}(1-\delta)(1+(1+\eta^A)q_t^A)L_t$. This gives $A_{t-1}(1-\delta)(1+\eta^A)L_t$ additional tasks. As each of these contributes $w_t l_t = w_t/a_t$ to income, the productivity effect accounts for $A_{t-1}(1-\delta)(1+\eta^A)L_t w_t/a_t$ additional income.

The investment effect reduces income since for each task already performed, the investment outlays increase by $i'(q_t^A)$. Therefore, total additional investment outlays increase by $A_{t-1}(1-\delta)(1+(1+\eta^A)q_t^A)L_t i'(q_t^A)$. Since firms minimize costs, I have from (3.9) that $(1+(1+\eta^A)q_t^A)i'(q_t^A) = w_t/a_t$. Hence, investment outlays increase by $A_{t-1}(1-\delta)L_t w_t/a_t$.

The difference between both effects is $\eta^A A_{t-1}(1-\delta)L_t w_t/a_t$. With (3.16) this is equal to η^A times economy's wage bill discounted by the growth factor of labor productivity, i. e., $\eta^A w_t L_t/(1+(1+\eta^A)q_t^A)$.

1. *If the cohorts t of these economies have different fertility rates, $\lambda > \lambda'$, that are not too far apart, then*

$$V'_{t+1} \lesseqgtr V_{t+1} \quad \Leftrightarrow \tag{5.9}$$

$$\left[\left(R_{t+1} + E^K_{t+1}\right)\left(\frac{K_{t+1}}{L_{t+1}}\right)\left(\frac{-\varepsilon}{1-\varepsilon}\right) + \left(w_{t+1} + E^L_{t+1}\right)\right](\lambda' - \lambda) \lesseqgtr 0,$$

and

$$v'_{t+1} \lesseqgtr v_{t+1} \quad \Leftrightarrow \tag{5.10}$$

$$\left[\left(R_{t+1} + E^K_{t+1}\right)\left(\frac{K_{t+1}}{L_{t+1}}\right)\left(\frac{-\varepsilon}{1-\varepsilon}\right) + \left(w_{t+1} + E^L_{t+1} - v_{t+1}\right)\right](\lambda' - \lambda) \lesseqgtr 0,$$

where ε is evaluated at κ_{t+1}.

2. *If the cohorts t of these economies have different survival probabilities between t and $t+1$, $\nu' > \nu$, that are not too far apart, then*

$$V'_{t+1} > V_{t+1} \quad \textit{since} \tag{5.11}$$

$$\left(R_{t+1} + E^K_{t+1}\right)\left(\frac{s_\nu\left(R_{t+1}, \nu\right) w_t L_t}{1-\varepsilon}\right)(\nu' - \nu) > 0,$$

and

$$v'_{t+1} \gtreqless v_{t+1} \quad \Leftrightarrow \tag{5.12}$$

$$\left[\left(R_{t+1} + E^K_{t+1}\right)\left(\frac{s_\nu\left(R_{t+1}, \nu\right) w_t}{1-\varepsilon}\right) - v_{t+1}\right](\nu' - \nu) \gtreqless 0,$$

where ε is evaluated at κ_{t+1}.

To understand the effects that drive the findings of Proposition 3, note that (5.9) - (5.12) are derived from first-order Taylor approximations with respect to either

λ or ν of V'_{t+1} at V_{t+1}. This procedure involves two steps. The first step approximates V'_{t+1} at (K_{t+1}, L_{t+1}). This gives

$$V'_{t+1} \approx V_{t+1} + \frac{dV_{t+1}}{dK_{t+1}}(K'_{t+1} - K_{t+1}) + \frac{dV_{t+1}}{dL_{t+1}}(L'_{t+1} - L_{t+1}). \tag{5.13}$$

Here, dV_{t+1}/dK_{t+1} and dV_{t+1}/dL_{t+1} denote the total effect of changing K_{t+1} and L_{t+1} on V_{t+1}, respectively. They comprise a direct effect equal to the respective factor price and the induced effect of (5.6), i. e.,[20]

$$\begin{aligned} \frac{dV_{t+1}}{dK_{t+1}} &= \frac{\partial V_{t+1}}{\partial K_{t+1}} + E^K_{t+1} = R_{t+1} + E^K_{t+1} > 0, \\ \frac{dV_{t+1}}{dL_{t+1}} &= \frac{\partial V_{t+1}}{\partial L_{t+1}} + E^L_{t+1} = w_{t+1} + E^L_{t+1} > 0. \end{aligned} \tag{5.14}$$

Hence, both effects are positive. From Lemma 3, the induced effect associated with K_{t+1} reinforces the partial effect whereas the induced effect of L_{t+1} weakens the partial effect without dominating it.[21]

The second step derives first-order Taylor approximations with respect to either λ or ν of K'_{t+1} at K_{t+1} to express $\left(K'_{t+1} - K_{t+1}\right)$ in (5.13). Moreover, it uses $L'_{t+1} - L_{t+1} = L_t(\lambda' - \lambda) < 0$.

Roughly speaking, the results of (5.9) - (5.12) reveal that the effect of population aging on the growth rate of aggregate and per-capita income hinges on the sign and the strength of the induced effects and on the response of savings. Only the effect of a higher survival probability on income growth has an unequivocal positive sign. This is so since a higher survival probability induces more savings. The resulting higher capital stock contributes positively to output (direct effect) and leads to more labor-saving technical change (induced effect). As shown in (5.12), the effect of life-expectancy on per-capita income is not unequivocal since more survivors increase the population in $t + 1$.

The effect of a decline in fertility is more involved since it triggers both a decline in the work force and an adjustment of the capital stock. The former channel reduces income in the λ'–economy since $w_{t+1} + E^L_{t+1} > 0$. Hence, for $V'_{t+1} < V_{t+1}$, it is

---

[20]From (5.5), the direct effect of K_{t+1} on V_{t+1} is $\partial V_{t+1}/\partial K_{t+1} = b_{t+1}\left[F_1(\cdot) - i\left(q^B_{t+1}\right)\right]$. It has an interpretation as the marginal contribution of capital to income given q^A_{t+1} and q^B_{t+1}. From (3.12), it is equal to R_{t+1}. Similarly, $\partial V_{t+1}/\partial L_{t+1} = a_{t+1}\left[F_2(\cdot) - i\left(q^A_{t+1}\right)\right]$ is the marginal contribution of labor to income. From (3.11), the latter is equal to w_{t+1}.

[21]Using Lemma 1 and Lemma 2, one readily verifies that the sign of $\partial V_{t+1}/\partial L_{t+1}$ follows since $w_{t+1}\left[1 - \left(\eta^A/\left(1+\eta^A\right)\right)\left(\varepsilon^A_\kappa(\kappa_{t+1}, \eta^A)/\left(1 + \varepsilon^A_\kappa(\kappa_{t+1}, \eta^A) + \varepsilon^B_\kappa(\kappa_{t+1})\right)\right)\right] > 0$.

sufficient to have $\varepsilon_R^s \geq 0$. Given that the latter is the most likely response of the savings rate to changes in the rental rate of capital, the prediction is that the λ'–economy has a lower growth rate of income. However, this does not imply that a lower fertility rate slows down growth of per-capita income. Condition (5.10) highlights that $v'_{t+1} > v_{t+1}$ is easier to satisfy since the population at $t+1$ is smaller under λ'. For instance, if the response of savings to the real rental rate is negligible a decline in fertility increases the growth rate of per-capita income in the short run if $w_{t+1} + E_{t+1}^L - v_{t+1} < 0$.

5.2 Population Aging and Economic Growth in the Long-Run

The focus here is on the effects of population aging on economic growth in the steady state of the dynamical system of Proposition 1. As will become clear below, to derive interpretable results at a high level of generality, I need to impose more structure. Therefore, I shall strengthen condition (4.11) and henceforth assume that individuals do not decrease their savings in response to an increase in the real rental rate of capital, i. e., $s_R\left(R_{t+1}, \nu\right) \geq 0$ or $\varepsilon_R^s(\kappa_t) \geq 0$.[22]

Moreover, to study the local stability properties of a steady state let me denote the two sets $\{(B_t, \kappa_t) \,|\kappa_{t+1} - \kappa_t = 0\}$ and $\{(B_t, \kappa_t) \,|B_{t+1} - B_t = 0\}$ by $\Delta\kappa_t = 0$ and $\Delta B_t = 0$, respectively. I assume that the evolution of κ_t is stable in the vicinity of $\Delta\kappa_t = 0$. This assumption allows for a meaningful comparison of steady states in a world with and without capital-saving technical change (see, e. g., Section 6.1).

Define a steady state as a trajectory along which all variables grow at a constant rate. I deduce from (4.10) that a trajectory with $B_{t+1}/B_t - 1 = const.$ requires $\kappa_t = \kappa_{t+1} = \kappa^*$. Moreover, according to (4.9), the latter needs $B_{t+1} = B_t = B^*$. Hence, a steady state is a solution to

$$\frac{s\left(R\left(\kappa^*, B^*\right), \nu\right)}{1+\lambda}\tilde{w}\left(\kappa^*\right) = \frac{\kappa^*}{B^*}\left(1-\delta\right)\left(1+\left(1+\eta^A\right)g^A\left(\kappa^*\right)\right). \tag{5.15}$$

$$g^B\left(\kappa^*\right) = \frac{\delta}{1-\delta} \tag{5.16}$$

Proposition 4 *(Steady State)*

[22] Without this assumption, neither the existence nor the uniqueness of a steady state may be established. Indeed, one readily verifies that, given κ^*, equation (5.15) below may determine none or several values $B^* > 0$ if $s_R\left(R_{t+1}, \nu\right)$ was allowed to be negative.

1. *There is a unique steady state involving $\kappa^* \in (0, \infty)$ and $B^* \in (0, \infty)$ if and only if*

$$\lim_{\kappa \to 0} f'(\kappa) > \frac{i'\left(\delta/(1-\delta)\right)}{1-\delta} + i\left(\delta/(1-\delta)\right) > \lim_{\kappa \to \infty} f'(\kappa). \tag{5.17}$$

2. *The steady-state growth rate of the economy is*

$$g^* \equiv \frac{A_{t+1}}{A_t} = (1-\delta)\left(1 + (1+\eta^A) g^A\left(\kappa^*, \eta^A\right)\right) - 1.$$

Moreover, along a steady-state path, I have

a) $$\frac{v_{t+1}}{v_t} = \frac{a_{t+1}}{a_t} = \frac{w_{t+1}}{w_t} = \frac{c^y_{t+1}}{c^y_t} = \frac{c^o_{t+1}}{c^o_t} = \frac{s_{t+1}}{s_t} = 1 + g^*,$$

b) $$\frac{V_{t+1}}{V_t} = \frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}}{K_t} = \frac{n_{t+1}}{n_t} = \frac{m_{t+1}}{m_t} = (1+g^*)(1+\lambda),$$

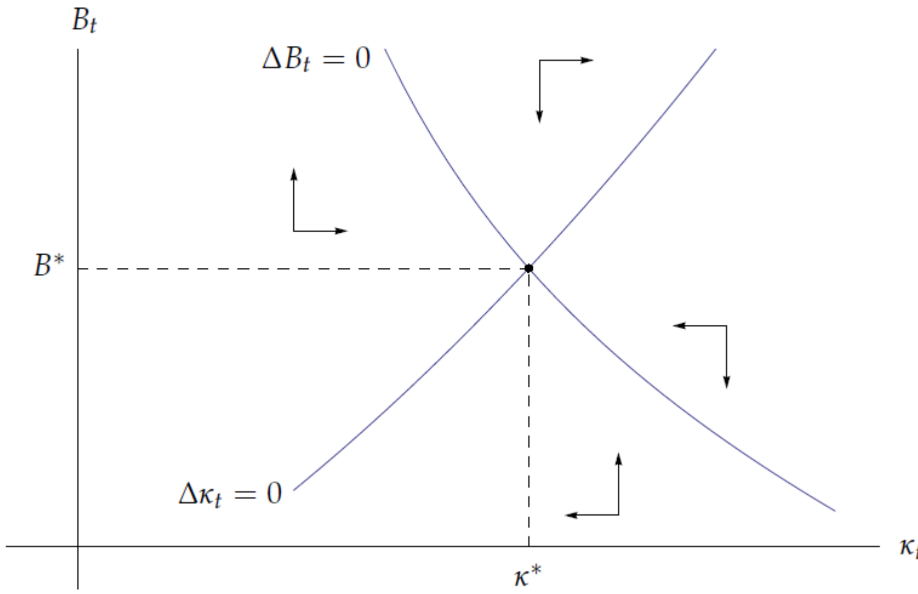
c) $$B_t = b_t = B^*, \quad R^* = B^* \frac{i'\left(\delta/(1-\delta)\right)}{1-\delta}, \quad k_t = k^* = \frac{1}{B^*}, \quad \frac{l_{t+1}}{l_t} = \frac{1}{1+g^*},$$

3. *If the set $\Delta\kappa_t = 0$ is stable in the vicinity of (κ^*, B^*), then the steady state is either a stable node, a focus, or a clockwise spiral sink. A typical phase diagram looks like the one of Figure 5.1.*

Proposition 4 states important properties of a steady state. According to Statement 1, a finite and unique steady state exists iff, at the equilibrium allocation, a small (large) stock of efficient capital has a sufficiently high (low) marginal value product. This mimics the role of diminishing returns for the existence of a steady state in the neoclassical growth model of Solow (1956) and Swan (1956). However, here the intuition is quite different. In fact, condition (5.17) assures that $q^B_t = g^B(\kappa_t)$ defined in Lemma 1 may take on the value $\delta/(1-\delta)$ for some $\kappa^* \in (0, \infty)$. For this to be possible the first-order conditions (3.10) and (3.12) has to hold for some $m_t \in (0, \infty)$ at $q^B_t = \delta/(1-\delta)$. Then, (3.10) implies that $i'(\delta/(1-\delta))/(1-\delta)$ is equal to the capital cost of task m_t so that $i'(\delta/(1-\delta))/(1-\delta) + i(\delta/(1-\delta))$ is its total cost. Using the latter in (3.12) reveals that (5.17) is the desired condition for $\kappa^* \in (0, \infty)$. Notice that such a value would always exist if I had imposed the usual Inada conditions on F.

Statement 2 of Proposition 4 gives the steady-state evolution of all admissible variables. The steady-state growth rate of the economy is equal to the growth

Figure 5.1: The Phase-Diagram of the Locally Stable Steady State (κ^*, B^*).



rate of the stock of labor-saving technological knowledge. Per-capita income, labor productivity, the real wage, and individual consumption and individual savings grow at this rate. Since κ^* is determined by the production side, an important conclusion is that the steady-state growth rate is independent of population aging.[23]

Aggregate variables such as Y_t or K_t grow at rate $g^* + \lambda$. Hence, an older economy experiences slower growth of economy-wide variables. There is no growth of capital-saving technological knowledge. Therefore, the rental rate of capital is constant.[24] With these results at hand, it is straightforward to see that the steady state is consistent with Kaldor's facts (Kaldor (1961)) as long as $g^* > 0$.

The conceptual underpinning of Statement 2 of Proposition 4 is the so-called Steady-Steady Growth Theorem of Uzawa (Uzawa (1961)). To see this, recall aggregate income of (5.5). A steady-state trajectory as defined above for all $t \geq$

[23]What then explains different steady-state growth rates in the present model? It is not difficult to show that a higher η^A unequivocally increases g^* since the (positive) direct effect dominates the (negative) indirect effect through $g^A(\kappa^*, \eta^A)$. Moreover, if I replace the production function (3.4) by $Y_t = \Gamma F(m_t, n_t)$, where $\Gamma > 0$ reflects cross-country differences in geography, technical or social infrastructure, then g^* is strictly increasing in $\Gamma > 0$.

[24]One readily verifies that the steady-state functional income distribution is constant and also independent of population aging.

$\tau \geq 1$ requires $q_t^A = q^A$ and $q_t^B = q^B$. Then, Uzawa's theorem applies since $V(b_t K_t, a_t L_t)$ has constant returns to scale in K_t and L_t as well as positive and diminishing marginal products of K_t and L_t, and $I_t^K > 0$. Hence, $b_\tau = b^*$, $q^B = \delta/(1-\delta)$, and technical progress is only labor-saving. Clearly, technical progress is exogenous in Uzawa's setting. With endogenous capital-saving technical change the efficient capital intensity has to adjust such that firms find it optimal to invest what is required by the steady-state growth theorem, i. e., the amount that guarantees zero net growth of capital-saving technological knowledge. This is guaranteed by (5.16).

According to Statement 3 of Proposition 4, the steady state is locally stable. Figure 5.1 shows the phase diagram with both loci, $\Delta\kappa_t = 0$ and $\Delta B_t = 0$, being stable. Hence, a one-time increase in the OADR due to a variation in λ or ν leaves the steady state of an economy unaffected. The convergence to the steady state may either be monotonic or oscillatory depending on the extent to which the OADR affects the eigenvalues of the dynamical system. The following proposition gives the steady-state effects of a permanent increase in the OADR.

Proposition 5 *Consider two economies with identical initial conditions* (K_1, L_1, A_0, B_0).

1. *If these economies differ only with respect to their fertility rates, such that* $L_t = L_1(1+\lambda)^{t-1} > L_t' = L_1(1+\lambda')^{t-1}$ *for* $t > 1$. *Then, their steady states satisfy* $\kappa^* = \kappa^{*\prime}$, $B^* > B^{*\prime}$, *and* $R^* > R^{*\prime}$.

2. *If these economies differ only with respect to their survival probability, such that* $\nu' > \nu$, *then their steady states satisfy* $\kappa^* = \kappa^{*\prime}$, $B^* > B^{*\prime}$, *and* $R^* > R^{*\prime}$.

While population aging does not affect the steady-state growth rate, it implies adjustment in steady-state levels. According to Proposition 5, the 'older' economy has a lower steady-state level of capital-saving technological knowledge and, therefore, a lower steady-state rental rate of capital. Intuitively, both a decline in the fertility rate and/or a higher survival probability increase, ceteris paribus, $s(R(\kappa^*, B^*), \nu)/(1+\lambda)$. Since κ^* is fixed, the necessary adjustment in (5.15) occurs through a decline in B^* and the concurrent fall of the steady-state real rental rate of capital.

6 Discussion and Extensions

6.1 The Role of Capital-Saving Technical Change

To further explore the role of capital-saving technical change, it proves useful to establish and compare the effect of population aging in an economy without it. To accomplish this, recall that the dynamical system of Proposition 1 nests the case of an economy with endogenous labor-saving technical change only. It obtains when, for all t, $b_t = B_t = 1$ is fixed and $q_t^B(m) = 0$. Then, given $(K_1, L_1, A_0, 1) > 0$ as initial conditions and (4.11), the dynamical system determines a unique equilibrium sequence $\{\kappa_t\}_{t=1}^{\infty}$ that satisfies

$$\frac{s\left(R\left(\kappa_{t+1}\right),\nu\right)}{1+\lambda}\,\tilde{w}\left(\kappa_t\right) = \kappa_{t+1}\left(1-\delta\right)\left(1+\left(1+\eta^A\right)g^A\left(\kappa_{t+1}\right)\right),$$

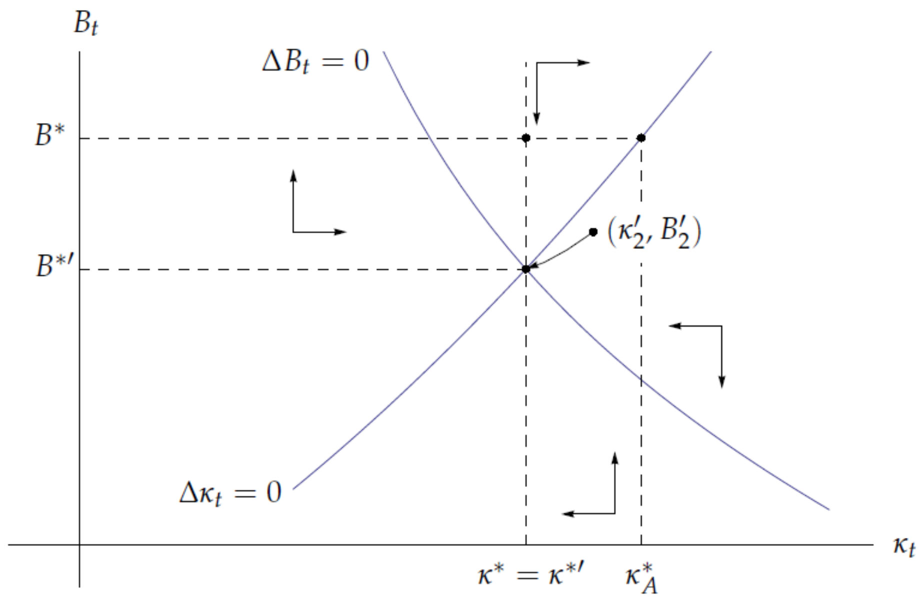
with κ_1 given by

$$\kappa_1 = \frac{K_1}{A_0\left(1-\delta\right)\left(1+\left(1+\eta^A\right)g^A\left(\kappa_1\right)\right)L_1} > 0.$$

Does this economy respond to population aging in the same way as the one with capital-saving technical change? To see that the answer is no, consider the phase diagram of Figure 6.1. Suppose the economy starts in the steady state (κ^*, B^*) before it experiences a one-time and permanent decline in λ or increase in ν. This shock shifts the $\Delta\kappa_t = 0$ and the $\Delta B_t = 0$ loci downwards to intersect at $(\kappa^{*\prime}, B^{*\prime})$. The economy without capital-saving technical change converges to the new steady state at κ_A^* since the $\Delta\kappa_t = 0$ locus is stable and $B_t = B^*$ remains constant. Intuitively, along the transition there is continuous capital deepening inducing more and more labor-saving technical change. As a consequence, population aging is associated with an increase in the steady-state growth rate.[25] However, if capital-saving technical change is possible, (κ_A^*, B^*) cannot be a steady state: to the right of κ^*, the growth rate of B_t is strictly negative. Following the initial shock the economy lands on a trajectory with $\kappa_t > \kappa^*$ and $B_t < B^*$ and, as shown in Figure 6.1, may converge to the new steady state $(\kappa^*, B^{*\prime})$.

[25]These two forces are behind the steady-state analysis that appears, e. g., in Heer and Irmen (2009). A similar mechanism drives the results of population aging on the steady-state growth rate in Futagami and Nakajima (2001). The AK-model in Li, Zhang, and Zhang (2007) exhibits a jump that corresponds to a shift from (κ^*, B^*) to (κ_A^*, B^*) without a transition period.

Figure 6.1: Comparative Statics and Dynamics of a One-Time and Permanent Decline (Increase) in the Growth Rate of the Labor Force (Survival Probability). The Case of a Stable Node.



6.2 Capital-Saving Investments with Contemporaneous Knowledge Spill-Overs

Thus far, the external contemporaneous knowledge spill-overs are confined to labor-saving investments. Authors like Frankel (1962) or Romer (1986) advocate the presence of such spill-overs in the context of capital investments. One way to capture this is to replace $b_t(m)$ of (3.6) by

$$
\begin{aligned}
b_t(m) &= B_{t-1}(1-\delta)\left(1+q_t^B(m)+\eta^B e_t^B\right), \\
e_t^B &\equiv \frac{1}{m_t}\int_0^{m_t} q_t^B(m)dm,
\end{aligned} \tag{6.1}
$$

where e_t^B is the external knowledge spill-over and $\eta^B \in \mathbb{R}_+$ measures its strength.

The incorporation of these knowledge spill-overs requires few modifications. For instance, the reasoning that led to Lemma 1 gives now rise to the map $g^B : \mathbb{R}_{++} \to \mathbb{R}_{++}$, where $q_t^B = g^B\left(\kappa_t, \eta^B\right)$ with $g^B_{\eta^B}(\kappa_t, \eta^B) < 0$. Proposition 2 for the short run as well as all qualitative findings for the long-run remain valid. The most striking new aspect concerns the role of population aging for the evolution of income. Due to the new external knowledge spill-over I have for the same reasons that explain Lemma 3

$$\frac{\partial V_t}{\partial q_t^B} = \frac{\eta^B R_t K_t}{1 + (1 + \eta^B)\, q_t^B} > 0. \tag{6.2}$$

The consequence is that the induced effects that appear in both lines of (5.6) are now of opposite sign. Hence, E_t^L and E_t^K can no longer be signed. However, the signs for dV_{t+1}/dK_{t+1} and dV_{t+1}/dL_{t+1} as established in (5.14) remain valid.[26] As a result, the qualitative results of Proposition 3 remain true.

6.3 Changing Demographic Features

Three Generations Following, e. g., Bommier and Lee (2003), one may argue that a representation of economic life by three periods, childhood, adulthood, and old age, is called for to study the implications of population aging. To incorporate this consider three generations alive at each period $t \geq 1$, i. e., νL_{t-1} retired old, L_t working adults, and L_{t+1} children. At the beginning of period t, each working adult gives birth to $(1 + \lambda_{t+1})$ offspring and maximizes his expected lifetime utility (3.2) having the offspring consumption included in c_t^y.[27]

However, for two reasons the relevance of this extension is rather limited in the present context. First, for the short-run effects the crucial link is between fertility of adults at t and the supply of labor at $t+1$. Adding a period of childhood does not affect this link. Therefore, the consequences of a decline in fertility stated in Proposition 2 and Proposition 3 remain valid. To the extent that fewer children reduce total population, an increase in per-capita income becomes more likely if fertility declines in two successive periods, i. e., condition (5.10) will be easier to satisfy for $v'_{t+1} > v_{t+1}$. Second, the steady state is independent of the economy's demographic structure. Hence, in the long run, the amount of periods an individual is supposed to live through does not affect the growth rate of the economy.

[26]To see this formally, denote E_t^L and E_t^K of (5.6) by $E_t^L(\eta^A, \eta^B)$ and $E_t^K(\eta^A, \eta^B)$, respectively. Then, (6.2) implies $E_t^L(\eta^A, \eta^B) > E_t^L(\eta^A, 0)$. Hence, $w_t + E_t^L(\eta^A, \eta^B) > 0$. Similarly, $E_t^K(\eta^A, \eta^B) > E_t^K(0, \eta^B)$. Therefore, applying the reasoning set out in Footnote 21 to $E_t^K(0, \eta^B)$ reveals that $R_t + E_t^K(\eta^A, \eta^B) > 0$.

[27]This setup has been studied by, e. g., Li, Zhang, and Zhang (2007). Here, the total dependency ratio may be a more appropriate indicator of economic dependency. It is equal to the sum of the old-age dependency ratio and the child dependency ratio, i. e., $(\nu L_{t-1} + L_{t+1})/L_t = \nu/(1 + \lambda_t) + (1 + \lambda_{t+1})$.

Underestimation of Survival Rates Perfect foresight includes the assumption that young individuals correctly foresee their survival rates. This seems unrealistic, especially in times where this rate considerably increases between generations.[28] One crude way to account for this is to allow for myopic foresight where the expected survival rate of cohort t is ν_{t-1}, i. e., the known survival rate of their parents. Then, an actual increase in the survival rate of cohort t has no effects on their savings and the important effect of changes in ν on K_{t+1}/L_{t+1} derived in (5.4) vanishes. As a consequence, the effects of an increase in ν stated in Proposition 2 are just postponed by one period. The same is true for Proposition 3: the savings channel is mute between t and $t+1$ so that $V'_{t+1} = V_{t+1}$ in (5.11) and $v'_{t+1} < v_{t+1}$ in (5.12). Underestimation of the survival rates leaves the steady-state growth rate unaffected.

6.4 Endogenous Labor Supply

Individuals may want to increase their labor supply in anticipation of a higher wage. However, at the level of economic aggregates a higher individual labor supply reduces, ceteris paribus, the capital-labor ratio. This weakens the incentive to increase labor productivity in accordance with Lemma 1 and Lemma 2 and reduces the wage. To address this tension assume that the individual labor supply is an increasing function of the current wage in efficiency units. To be precise, denote $\tau_t \in [0,1]$ the fraction of an individual's time endowment that she supplies to the labor market in t. Assume further that $\tau_t = \tau(\tilde{w}_t)$ where $\tau : \mathbb{R}_{++} \to [0,1]$ with $\tau'(\tilde{w}_t) > 0 > \tau''(\tilde{w}_t)$. Then, with Lemma 1, I have $\tau_t = \tau\left(\tilde{w}\left(\kappa_t\right)\right)$ and

$$\varepsilon_\kappa^\tau(\kappa_t) \equiv \frac{d\ln\tau_t}{d\ln\kappa_t} > 0.$$

Aggregate labor supply becomes $\tau\left(\tilde{w}\left(\kappa_t\right)\right)L_t$. This affects Lemma 2 to the extend that the response of κ_t to changes in K_t/L_t is weaker. In fact, adding the new term in the denominator of (4.5) reveals that the response of $\kappa_t = \kappa\left(B_{t-1}K_t/A_{t-1}L_t\right)$ to changes in the capital-labor ratio is now

$$\varepsilon^\kappa\left(\kappa_t\right) \equiv \frac{d\ln\kappa_t}{d\ln\left(K_t/L_t\right)} = \frac{1}{1+\varepsilon_\kappa^A\left(\kappa_t,\eta^A\right)+\varepsilon_\kappa^B\left(\kappa_t\right)+\varepsilon_\kappa^\tau\left(\kappa_t\right)} \in (0,1). \quad (6.3)$$

Intuitively, at the extensive margin, the scarcity of labor increases with K_t/L_t. However, at the intensive margin individuals supply more labor at the resulting

[28]See, e. g., Groneck, Ludwig, and Zimper (2011) for a discussion of behavioral biases in individual forecast errors of survival rates and their explanations.

higher wage. Hence, labor becomes more abundant. Nevertheless, the effect through the extensive margin remains dominant.

As a consequence, Proposition 1 remains valid. Moreover, the short-run effects of aging have to include the fact that now a change in factor endowments not only induces technical change as in (5.6) but also affects labor supply. Accordingly, (5.14) becomes

$$\begin{aligned}\frac{dV_{t+1}}{dK_{t+1}} &= R_2 + E^K_{t+1} + F_2\, a_{t+1}L_{t+1}\tau'\left(\tilde{w}_{t+1}\right)\tilde{w}_\kappa(\kappa_{t+1}))\frac{\partial \kappa_{t+1}}{\partial K_{t+1}},\\ \frac{dV_{t+1}}{dL_{t+1}} &= w_2 + E^L_{t+1} + F_2\, a_{t+1}L_{t+1}\tau'\left(\tilde{w}_{t+1}\right)\tilde{w}_\kappa(\kappa_{t+1}))\frac{\partial \kappa_{t+1}}{\partial L_{t+1}},\end{aligned} \tag{6.4}$$

where the argument of F_2 is $(b_{t+1}K_{t+1}, a_{t+1}\tau_{t+1}L_{t+1})$. The new effect on labor supply strengthens dV_{t+1}/dK_{t+1} and weakens dV_{t+1}/dL_{t+1}. Incorporating this in Proposition 2 reveals that the positive effect of a higher survival probability on aggregate income becomes more pronounced. Moreover, its effect on per-capita income is more likely to be positive. With an endogenous labor supply, the effect of a decline in fertility becomes more involved as long as $\varepsilon^s_R \neq 0$. In the special case where $\varepsilon^s_R = 0$ endogenous labor supply makes it more likely that income increases in absolute and per-capita terms with population aging as individuals will also work longer hours. Clearly, this extension does not affect the qualitative results concerning the long run.

Alternatively, labor supply may depend positively on the expected survival probability if people want to earn and save more expecting a longer period of retirement. To account for this let $\tau_t = \tau(\nu)$ where $\tau : \mathbb{R}_{++} \to [0,1]$ with $\tau'(\nu) > 0$. Accounting for this in (4.5) reveals that Lemma 2 must be extended since $\kappa_t = \kappa(\cdot, \nu)$ with $\kappa_\nu < 0$. Hence, a higher survival probability reduces the equilibrium task intensity since it increases the supply of labor. Then (5.2) becomes

$$\frac{s\left(R\left(\kappa\left(\frac{B_t K_{t+1}}{A_t L_{t+1}}, \nu\right), B_t\right), \nu\right)}{1+\lambda} w_t = \frac{K_{t+1}}{L_{t+1}\tau(\nu)}. \tag{6.5}$$

Moreover, as long as $s_R \geq 0$, it holds that $\partial\left(K_{t+1}/L_{t+1}\right)/\partial\nu > 0$ so that the qualitative results of Proposition 2 remain valid.

Proposition 3 must be modified since now ν induces technical change that affects aggregate income. To account for this use the same steps that lead to (5.6) and obtain

$$E^\nu_t = \left[\frac{\partial V_t}{\partial q^A_t} g^A_\kappa\left(\kappa_t, \eta^A\right) + \frac{\partial V_t}{\partial q^B_t} g^B_\kappa\left(\kappa_t\right)\right]\frac{\partial \kappa_t}{\partial \nu} < 0, \tag{6.6}$$

where the sign follows with Lemma 3. Intuitively, a higher life-expectancy means a larger labor supply and, therefore, less induced labor-saving technical change. Hence, output falls. This effect will pop up in the modified statements of (5.11) and (5.12) and weaken the prospect that population aging fosters aggregate and per-capita income growth. Again, this extension leaves the qualitative results on steady-state growth unaffected.

6.5 Endogenous Fertility

The question about the the repercussions between the evolution of the economy and its fertility rate is among the most challenging. A satisfactory discussion would certainly require the incorporation of a fully endogenous fertility choice. However, this is beyond the scope of this paper.[29] Nevertheless let me capture the idea and stipulate a functional relationship where the fertility rate of generation t depends in a negative way on its survival probability, i. e., $\lambda_t = \lambda(\nu)$, where $\lambda : [0, 1] \to (-1, \infty)$ with $\lambda'(\nu) < 0$. This specification is consistent with the recent experience of the industrialized world where fertility declines as life expectancy increases. This relationship may be driven, e. g., by advances in the life and the medical sciences.

With this specification the old-age dependency ratio is $\nu / \left(1 + \lambda(\nu)\right)$, and the effect of increasing ν becomes more pronounced as the current young reduce their fertility in anticipation of a higher survival probability. The most important modification of this feature concerns the clear-cut prediction that a rise in the survival probability increases aggregate income made in Proposition 3. Going through the steps that lead to (5.11), one now finds

$$V'_{t+1} \gtreqless V_{t+1} \quad \text{if}$$

$$\begin{aligned} & \left[\left(R_{t+1} + E^K_{t+1}\right)\left(\frac{K_{t+1}}{1+\lambda(\nu)}\frac{-\varepsilon}{1-\varepsilon}\right) + \left(w_{t+1} + E^L_{t+1}\right)\right]\lambda'(\nu)(\nu' - \nu) \\ + \quad & \left(R_{t+1} + E^K_{t+1}\right)\left(\frac{s_\nu\left(R_{t+1}, \nu\right) w_t}{1-\varepsilon}\right)(\nu' - \nu) \gtreqless 0, \end{aligned} \tag{6.7}$$

The first line of (6.7) states the new effect on income that a rise in the survival probability induces through a decline in fertility. From the discussion of (5.9)

[29] See, e. g., Becker, Murphy, and Tamura (1990) or Galor (2011) for a discussion of the fertility-growth nexus.

this effect is negative if $s_R \geq$. Accordingly, the positive effect of an increase in life-expectancy on economic growth through an increase in savings is weakened. Clearly, the steady-state growth rate will not depend on whether fertility is endogenous or not.

7 Concluding Remarks

This paper studies the role of population aging for economic growth. It claims that demographic change affects the investment behavior of firms with implications for technical progress and economic growth. A central result of the analysis is that the type of technical progress is crucial for the link between population aging and economic growth. To make this point, this paper extends the existing literature and allows for two types of endogenous technical progress, i. e., capital- and labor-saving technical change.

Capital-saving technical change is found to imply that the steady-state growth rate is independent of the economy's age structure. Hence, in the long run, economic growth is independent of demographic features. However, population aging affects economic growth along the transition since it accentuates the relative scarcity of labor with respect to capital. This leads to more labor- and less capital-saving technical change. Due to external contemporaneous knowledge spill-overs, this induced change in the direction of technical change is found to have first-order effects on the evolution income.

The present analysis suggests several routes for future research. They include the question about how population aging affects economic growth if the age structure of the labor force matters for the direction of technical change. Case studies like Nishimura, Minetaki, Shirai, and Kurokawa (2002) for Japan, Prskawetz, Kögel, Sanderson, and Scherbov (2007) for India, or the panel study of Feyrer (2007) suggest a role of population aging for economic growth beyond its effect on the scarcity of the entire labor force relative to capital.

Second, one may want to inquire into the scope for design of growth policies to meet the economic challenges of population aging. This will require an analysis of the welfare properties of the equilibrium. Due to the presence of contemporaneous and inter-temporal external effects the equilibrium will in general not be efficient. This leaves room for fiscal policy measures to improve the normative properties of the competitive equilibrium. However, the main insight of this paper suggests that the effects and the desirability of any growth policy will crucially depend on whether technical change is capital-saving in addition to being labor-saving. I leave these questions for future research.

A Proofs

A.1 Proof of Lemma 1

Consider (3.9) - (3.12) at $q_t^A(n) = q_t^A = e_t^A$ and $q_t^B(m) = q_t^B$. Upon substitution of (3.9) in (3.11) and (3.10) in (3.12) I obtain

$$\begin{aligned} f(\kappa_t) - \kappa_t f'(\kappa_t) &= \left(1 + \left(1 + \eta^A\right) q_t^A\right) i'\left(q_t^A\right) + i\left(q_t^A\right), \\ f'(\kappa_t) &= \left(1 + q_t^B\right) i'\left(q_t^B\right) + i\left(q_t^B\right). \end{aligned} \tag{A.1}$$

Denote $RHS(q, \eta)$ the right-hand side of both conditions with the understanding that $\eta = 0$ in the second one. In view of the properties of the function i given in (3.7), $RHS(q, \eta)$ is a mapping $RHS : \mathbb{R}_+^2 \to \mathbb{R}_+$ with $\lim_{q\to 0} RHS(q, \eta) = 0$, $RHS_q(q, \eta) > 0$ for $q > 0$, $\lim_{q\to 0} RHS_q(q, \eta) = i''(q) \geq 0$, and $\lim_{q\to\infty} RHS(q, \eta) = \infty$. Moreover, the properties of the function $f(\kappa_t)$ imply that the left-hand side of both conditions is strictly positive for $\kappa_t > 0$. Hence, for each $\kappa_t > 0$ there is a unique $q_t^j > 0$, $j = A, B$, that satisfies the respective condition stated in (A.1). I denote these maps by $q_t^A = g^A\left(\kappa_t, \eta^A\right)$ and $q_t^B = g^B(\kappa_t)$, respectively.

An application of the implicit function theorem to (A.1) gives $dq_t^A/d\kappa_t \equiv g_\kappa^A\left(\kappa_t, \eta^A\right) > 0$, $dq_t^A/d\eta^A \equiv g_{\eta^A}^A\left(\kappa_t, \eta^A\right) < 0$, and $dq_t^B/d\kappa_t \equiv g_\kappa^B(\kappa_t) < 0$. The respective signs follow from the properties of the functions f and i.

With $q_t^A = g^A\left(\kappa_t, \eta^A\right)$ and $q_t^B = g^B(\kappa_t)$, I may express w_t and R_t using the respective first-order condition of (3.9) and (3.10). Then,

$$\begin{aligned} w_t &= A_{t-1}(1-\delta)\left(1 + \left(1 + \eta^A\right) g^A\left(\kappa_t, \eta^A\right)\right)^2 i'\left(g^A\left(\kappa_t, \eta^A\right)\right) \equiv w(\kappa_t, A_{t-1}), \\ R_t &= B_{t-1}(1-\delta)\left(1 + g^B(\kappa_t)\right)^2 i'\left(g^B(\kappa_t)\right) \equiv R(\kappa_t, B_{t-1}). \end{aligned} \tag{A.2}$$

The properties stated in (3.14) are immediate. For further reference, notice that the sign of the partial derivatives $w_\kappa(\kappa_t, A_{t-1})$ and $R_\kappa(\kappa_t, B_{t-1})$ carry over to the respective factor prices in efficiency units, i. e., to $\tilde{w}(\kappa_t) \equiv w(\kappa_t, A_{t-1})/a_t = \left(1 + \left(1 + \eta^A\right) g^A(\kappa_t)\right) i'\left(g^A(\kappa_t)\right)$ and $\tilde{R}(\kappa_t) \equiv R(\kappa_t, B_{t-1})/b_t = \left(1 + g^B(\kappa_t)\right) i'\left(g^B(\kappa_t)\right)$. ■

A.2 Proof of Lemma 2

Equation (4.5) is a fixed-point problem with a unique solution $\kappa\left(B_{t-1}K_t/A_{t-1}L_t, \eta^A\right) > 0$. To see this, write the right-hand side of (4.5) as $RHS\left(\kappa_t, B_{t-1}K_t/A_{t-1}L_t, \eta^A\right)$. Given the properties of $g^A\left(\kappa_t, \eta^A\right)$ and $g^B(\kappa_t)$ as stated in (3.13) and $B_{t-1}K_t/A_{t-1}L_t > 0$, $RHS\left(\kappa_t, B_{t-1}K_t/A_{t-1}L_t, \eta^A\right)$ is a function which is continuous, strictly decreasing, and strictly positive for all $\kappa_t > 0$. Hence, $\lim_{\kappa_t\to 0} RHS(\kappa_t, \cdot) > 0$. Accordingly, there is a unique $\kappa > 0$ such that $\kappa = RHS(\kappa, \cdot)$. Implicit differentiation of (4.5) delivers (4.7). ■

A.3 Proof of Proposition 1

Given $(K_t, L_t, A_{t-1}, B_{t-1})$, it is straightforward to establish that the variables κ_t and B_t are indeed state variables of the economy at t. To prove the existence of a unique equilibrium sequence $\{\kappa_t, B_t\}_{t=1}^{\infty}$ let me first derive equation (4.9). To do so solve (4.5) for K_{t+1} and substitute the resulting expression into (4.8). Using $s_t = s\left(R\left(\kappa_{t+1}, B_t\right), \nu\right) w\left(\kappa_t\right)$ and $\tilde{w}\left(\kappa_t\right) \equiv w(\kappa_t)/a_t$ gives (4.9) after some straightforward manipulations. The difference equation (4.10) is from (3.16) where q_t^B is replaced by $g^B\left(\kappa_t\right)$ in accordance with Lemma 1. In the first period, κ_1 is pinned down by (4.5) for given initial values $(K_1, L_1, A_0, B_0,) > 0$. From Lemma 2, there is a unique solution $\kappa_1 > 0$.

To prove the uniqueness of the sequence $\{\kappa_t, B_t\}_{t=1}^{\infty}$ it is useful to introduce

$$\Psi\left(\kappa_t, B_{t-1}\right) \equiv \frac{1 + \left(1+\eta^A\right) g^A\left(\kappa_t, \eta^A\right)}{1 + g^B\left(\kappa_t\right)} \frac{\kappa_t(1+\lambda)}{s\left(R\left(\kappa_t, B_{t-1}\right), \nu\right)}. \tag{A.3}$$

Then, (4.9) may be written as

$$B_t\, \tilde{w}\left(\kappa_t\right) = \Psi\left(\kappa_{t+1}, B_t\right). \tag{A.4}$$

From (A.2) the left-hand side of (A.4) is strictly positive for any $(\kappa_t, B_t) \in \mathbb{R}_{++}^2$. Hence, there is a unique value $\kappa_{t+1} > 0$ that satisfies (A.4) if $\Psi\left(\kappa_{t+1}, B_t\right)$ is strictly positive, continuous, monotone in κ_{t+1}, and may take on all values in $\mathbb{R}_{++}^2$ as κ_{t+1} varies.

To see that the latter properties are indeed fulfilled, observe that $\Psi\left(\kappa, B_t\right) > 0$ for $\kappa > 0$. This follows from the definition of Ψ, Lemma 1, and $s\left(R_{t+1}, \nu\right) \in (0, 1)$. Let me simplify the notation for the remainder of this proof and write $s\left(R(\kappa)\right)$ for $s\left(R(\kappa, B_t), \nu\right)$. Then, one readily verifies that $\Psi_\kappa\left(\kappa, B_t\right) > 0$ for $\kappa > 0$ is assured if

$$\left(s_R\left(R(\kappa)\right) \frac{R(\kappa)}{s\left(R(\kappa)\right)}\right)\left(R_\kappa(\kappa) \frac{\kappa}{R(\kappa)}\right) < 1 + \varepsilon_\kappa^A\left(\kappa, \eta^A\right) + \varepsilon_\kappa^B(\kappa). \tag{A.5}$$

Observe with (3.3) and (3.14) that the left-hand side of (A.5) is $\varepsilon_R^s(\kappa) \cdot \varepsilon_\kappa^R(\kappa)$. From (4.5), the right-hand side is equal to $1/\varepsilon^\kappa$. Hence, (A.5) coincides with (4.11).

Next , I show that $\lim_{\kappa\to 0} \Psi\left(\kappa, B_t\right) = 0$ and $\lim_{\kappa\to\infty} \Psi\left(\kappa, B_t\right) = \infty$. To make the argument, one has to consider both factors on the right-hand side of (A.3).

First, since $g^A\left(\kappa, \eta^A\right)$ is increasing in κ on $\mathbb{R}_{++}$ and bounded from below by zero, $\lim_{\kappa\to 0} g^A\left(\kappa, \eta^A\right)$ is finite and $\lim_{\kappa\to\infty} g^A\left(\kappa, \eta^A\right)$ is finite or infinite. Since $g^B\left(\kappa\right)$ is decreasing on $\mathbb{R}_{++}$ and bounded from below by zero, $\lim_{\kappa\to 0} g^B\left(\kappa\right)$ is either finite or infinite and $\lim_{\kappa\to\infty} g^B\left(\kappa\right)$ is finite. As a consequence, $\lim_{\kappa\to 0}\left(1 + \left(1+\eta^A\right) g^A\left(\kappa, \eta^A\right)\right) / \left(1 + g^B\left(\kappa\right)\right)$ is finite, and $\lim_{\kappa\to\infty}\left(1 + \left(1+\eta^A\right) g^A\left(\kappa, \eta^A\right)\right) / \left(1 + g^B\left(\kappa\right)\right)$ is either finite or infinite.

Second, I have to know the limits $\lim_{\kappa\to 0} \kappa/s\left(R(\kappa)\right)$ and $\lim_{\kappa\to\infty} \kappa/s\left(R(\kappa)\right)$. As to the limit $\kappa \to 0$, the following cases must be considered. First, if $\lim_{\kappa\to 0} R(\kappa) = R(0)$ is finite, then $s\left(R(0)\right) \in (0, 1)$ and $\lim_{\kappa\to 0} \kappa/s\left(R(\kappa)\right) = 0$. Second, if $\lim_{\kappa\to 0} R(\kappa) = R(0)$ is infinite, then $\lim_{\kappa\to\infty} s\left(R(\kappa)\right) = s\left(R(0)\right) \geq 0$. If $s\left(R(0)\right) > 0$, then it is immediate that $\lim_{\kappa\to 0} \kappa/s\left(R(\kappa)\right) = 0$. If $s\left(R(0)\right) = 0$, then savings goes to zero as the rental rate of capital approaches infinity. However, even in this case it holds that $\lim_{\kappa\to 0} \kappa/s\left(R(\kappa)\right) = 0$. To see this, consider the Euler equation of cohort t, $u'(c_{t+1}^o) = u'(c_t^y)/R(\kappa)$. By assumption, $\lim_{\kappa\to 0} R(\kappa) = \infty$ and $\lim_{\kappa\to 0} s\left(R(\kappa)\right) = 0$. Therefore,

$$\lim_{\kappa\to 0} \frac{u'(c_t^y)}{R(\kappa)} = \lim_{\kappa\to 0} \frac{u'\left(\left(1 - s\left(R(\kappa)\right)\right) w_t\right)}{R(\kappa)} = \frac{u'(w_t)}{\lim_{\kappa\to 0} R(\kappa)} = 0.$$

Hence, from the Euler condition $\lim_{\kappa\to 0} u'\left(c^o_{t+1}\right) = 0$. Since u satisfies the Inada condition, I find with the budget constraint of an old individual at $t+1$

$$\lim_{\kappa\to 0} c^o_{t+1} = \frac{w_t}{\nu} \lim_{\kappa\to 0} \left[R(\kappa)\, s\left(R(\kappa)\right)\right] = \infty. \tag{A.6}$$

Next, observe that $f(\kappa) > \kappa f'(\kappa) > 0$ since $F_2 > 0$ for $\kappa > 0$. Then, using (3.12) and $q^B = g^B(\kappa)$, I may express $\kappa R(\kappa)$ as

$$\begin{aligned} \kappa R(\kappa) &= B_t\,(1-\delta)\left(1+g^B(\kappa)\right)\left[\kappa f'(\kappa) - \kappa i\left(g^B(\kappa)\right)\right] \\ &< B_t\,(1-\delta)\left(1+g^B(\kappa)\right)\left[f(\kappa) - \kappa i\left(g^B(\kappa)\right)\right] \equiv \hat{R}(\kappa). \end{aligned} \tag{A.7}$$

Then, for $\kappa < 1$, $\kappa R(\kappa) < \hat{R}(1)$. Therefore,

$$\frac{s\left(R(\kappa)\right)}{\kappa} = \frac{R(\kappa)s\left(R(\kappa)\right)}{\kappa R(\kappa)} > \frac{R(\kappa)s\left(R(\kappa)\right)}{\hat{R}(1)}. \tag{A.8}$$

In light of (A.6) , $\lim_{\kappa\to 0} s\left(R(\kappa)\right)/\kappa = \infty$, and $\lim_{\kappa\to 0} \kappa/s\left(R(\kappa)\right) = 0$ as desired. Therefore, $\lim_{\kappa\to 0} \Psi(\kappa, B_t) = 0$.

Finally, note that $\lim_{\kappa\to\infty} \kappa/s\left(R(\kappa)\right) = \infty$ since $\lim_{\kappa\to\infty} s\left(R(\kappa)\right) \in [0,1]$. Hence, $\lim_{\kappa\to\infty} \Psi(\kappa, B_t) = \infty$.

Hence, the right-hand side of (A.4) is increasing on $\mathbb{R}_{++}$ approaching zero as $\kappa \to 0$ and infinity as $\kappa \to \infty$. Accordingly, there is a unique κ that satisfies (4.9) given $(\kappa_t, B_t) \in \mathbb{R}^2_{++}$. With this value at hand, (4.10) delivers a unique $B_{t+1} > 0$. ■

A.4 Proof of Proposition 2

This proposition follows immediately from (5.3), (5.4), Lemma 1, and Lemma 2. ■

A.5 Proof of Lemma 3

Consider (5.5). With a_t and b_t from (3.16) and (3.9) I find,

$$\begin{aligned} \frac{\partial V_t}{\partial q^A_t} &= A_{t-1}(1-\delta)\left(1+\eta^A\right)L_t\left[F_2\left(b_tK_t, a_tL_t\right) - i\left(q^A_t\right) - \left(1+\left(1+\eta^A\right)q^A_t\right)i'\left(q^A_t\right)\right] \\ &+ \eta^A A_{t-1}(1-\delta)L_t\left(1+\left(1+\eta^A\right)q^A_t\right)i'\left(q^A_t\right) = \frac{\eta^A w_t L_t}{1+\left(1+\eta^A\right)q^A_t} > 0, \\ \frac{\partial V_t}{\partial q^B_t} &= B_{t-1}(1-\delta)K_t\left[F_1\left(b_tK_t, a_tL_t\right) - i\left(q^B_t\right) - \left(1+q^B_t\right)i'\left(q^B_t\right)\right] = 0. \end{aligned} \tag{A.9}$$

In equilibrium the expressions in brackets vanish in accordance with (A.1). Hence, (5.7) is shown. The results stated in (5.8) are immediate from Lemma 1, Lemma 2, and (5.6). ■

A.6 Proof of Proposition 3

Claim 1 Consider $\lambda' < \lambda$ sufficiently close. Simplifying the notation, I denote $V_{t+1} \equiv V(K_{t+1}, L_{t+1})$ the equilibrium income under λ. Here, it is understood that

$$q^A_{t+1} = g^A\left(\kappa\left(B_t K_{t+1}/A_t L_{t+1}\right), \eta^A\right) \quad \text{and} \quad q^B_{t+1} = g^B\left(\kappa\left(B_t K_{t+1}/A_t L_{t+1}\right)\right) \tag{A.10}$$

in accordance with Lemma 1 and Lemma 2. Similarly, under λ' the equilibrium income is $V'_{t+1} \equiv V\left(K'_{t+1}, L'_{t+1}\right)$, where $q^{A\prime}_{t+1}$ and $q^{B\prime}_{t+1}$ differ from (A.10) as they now depend on K'_{t+1}/L'_{t+1}.

Step 1: For (K_{t+1}, L_{t+1}) and $\left(K'_{t+1}, L'_{t+1}\right)$ sufficiently close, (5.13) states the first-order Taylor approximation of V'_{t+1} at $V_{t+1}(K_{t+1}, L_{t+1})$. As explained in the main text, the total effect of a small change of K_{t+1} and L_{t+1} on V_{t+1} may be written as in (5.14).

Step 2: To quantify the difference $K'_{t+1} - K_{t+1}$, consider the Taylor approximation $K'_{t+1} \approx K_{t+1} + \partial K_{t+1}/\partial\lambda(\lambda' - \lambda)$ at λ. The derivative involved is obtained from

$$s\left(R\left(\kappa\left(\frac{B_t K_{t+1}}{A_t L_t(1+\lambda)}\right), B_t\right), \nu\right) w_t L_t = K_{t+1}, \tag{A.11}$$

which restates (4.8) using Lemma 1, Lemma 2, and $L_{t+1} = L_t(1+\lambda)$ to uncover the dependency of K_{t+1} on λ. Implicit differentiation delivers

$$\frac{\partial K_{t+1}}{\partial \lambda} = \frac{K_{t+1}}{1+\lambda}\frac{-\varepsilon}{1-\varepsilon} \gtreqless 0 \quad \Leftrightarrow \quad \varepsilon^s_R \gtreqless 0, \tag{A.12}$$

where all elasticities are evaluated at κ_{t+1}. Hence,

$$\frac{\partial K_{t+1}}{\partial \lambda} \gtreqless 0 \quad \Leftrightarrow \quad \varepsilon^s_R \gtreqless 0 \quad \text{and} \quad K'_{t+1} \gtreqless K_{t+1} \quad \Leftrightarrow \quad \varepsilon^s_R \lesseqgtr 0. \tag{A.13}$$

Finally, noting that $L'_{t+1} - L_{t+1} = L_t(\lambda' - \lambda)$ I have

$$dV_{t+1} \equiv V'_{t+1} - V_{t+1} \approx \frac{dV_{t+1}}{dK_{t+1}}\frac{\partial K_{t+1}}{\partial \lambda}\left(\lambda' - \lambda\right) + \frac{dV_{t+1}}{dL_{t+1}} L_t\left(\lambda' - \lambda\right). \tag{A.14}$$

Then, straightforward manipulations deliver (5.9).

To prove (5.10) consider $dv_{t+1} \equiv v'_{t+1} - v_{t+1}$, where $v_{t+1} = V_{t+1}/\left(L_t(\nu + 1 + \lambda)\right)$ and $v'_{t+1} = V'_{t+1}/\left(L_t(\nu + 1 + \lambda')\right)$. With dV_{t+1} of (A.14) I have

$$dv_{t+1} \approx dV_{t+1}/\left(L_t(\nu + 1 + \lambda)\right) - V_{t+1}\left(\lambda' - \lambda\right)/\left(L_t(\nu + 1 + \lambda)^2\right).$$

Hence,

$$dv_{t+1} \lesseqgtr 0 \quad \Leftrightarrow \quad dV_{t+1} - v_{t+1}L_t(\lambda' - \lambda) \lesseqgtr 0. \tag{A.15}$$

Straightforward manipulations using the results derived above reveal that (A.15) coincides with (5.10). Hence, Claim 1 of Proposition 3 is proven.

Claim 2 Consider $\nu' > \nu$ sufficiently close.

Step 1: The first-order Taylor approximation at (K_{t+1}, L_{t+1}) is now

$$dV_{t+1} \equiv V'_{t+1} - V_{t+1} \approx \frac{dV_{t+1}}{dK_{t+1}}(K'_{t+1} - K_{t+1}), \tag{A.16}$$

taking into account that $L'_{t+1} = L_{t+1}$.

Step 2: To quantify the difference $K'_{t+1} - K_{t+1}$, consider the Taylor approximation $K'_{t+1} - K_{t+1} \approx \partial K_{t+1}/\partial \nu(\nu' - \nu)$ at ν. The derivative involved may be directly derived from (5.4) noting that $L_{t+1}\partial\left(K_{t+1}/L_{t+1}\right)/\partial\nu = \partial K_{t+1}/\partial\nu$. Hence,

$$\frac{\partial K_{t+1}}{\partial \nu} = \frac{s_\nu\left(R_{t+1},\nu\right) w_t L_t}{1-\varepsilon} > 0, \tag{A.17}$$

where the elasticities are evaluated at κ_{t+1}. The sign follows from $s_\nu > 0$ in conjunction with (4.11). Hence,

$$K'_{t+1} \gtreqless K_{t+1} \quad \Leftrightarrow \quad \nu' \gtreqless \nu. \tag{A.18}$$

To prove (5.12) consider again $dv_{t+1} \equiv v'_{t+1} - v_{t+1}$. The result follows from the same steps that led to (A.15) with $\nu' - \nu$ replacing $\lambda' - \lambda$. Then, straightforward manipulations using the results derived above deliver inequality (5.12). This completes the proof of Proposition 3. ■

A.7 Proof of Proposition 4

1. "$\Rightarrow$": If condition (5.17) holds, then, at $q_t^B = \delta/(1-\delta)$, (3.10) and (3.12) deliver a unique $\kappa^* \in (0,\infty)$ since $f'(\kappa) < 0$ on $\mathbb{R}_{++}$. To see that there also exists a unique $B^* \in \mathbb{R}^2_{++}$, rewrite (5.15) as $B^* s\left(R(\kappa^*, B^*), \nu\right) = const. > 0$. Since $s_R(\cdot) \geq 0$, the left-hand side is strictly increasing in B^* taking on values form zero to infinity as B^* varies on this interval. Hence, there is a unique value $B^* > 0$ that satisfies (5.15).

   "$\Leftarrow$": If $i'\left(\delta/(1-\delta)\right)/(1-\delta) + i\left(\delta/(1-\delta)\right) \geq \lim_{\kappa\to 0} f'(\kappa)$, then (3.12) delivers $m_t = 0$, hence $\kappa_t = 0$. If $\lim_{\kappa\to\infty} f'(\kappa) \geq i'\left(\delta/(1-\delta)\right)/(1-\delta) + i\left(\delta/(1-\delta)\right)$ then (3.12) delivers $m_t = \infty$, hence $\kappa_t = \infty$.

2. Since (5.16) determines κ^*, I obtain g^* from Lemma 1 and (3.16). The stated findings about the steady-state growth factors of a_t, w_t, c_t^y, c_t^o, and s_t are immediate from $a_t = A_t$, (A.2), (3.3), and the budget constraints. Steady-state growth of K_t follows from (4.8). All other growth factors under b) result from (3.4) in conjunction with (E4), and (4.3). Finally, (3.16) gives $b_t = B^*$, (A.2) delivers $R^* = B^* i'(\delta/(1-\delta))/(1-\delta)$, and σ_t evaluated at the steady state gives σ^*.

3. From the proof of Proposition 1, I know that (4.11) implies $\Psi_\kappa\left(\kappa_{t+1}, B_t\right) > 0$. Similarly, one readily verifies that $\Psi_B\left(\kappa_{t+1}, B_t\right) \leq 0$ if $s_R(\cdot) \geq 0$. It is then straightforward to see that (4.9) defines a continuously differentiable function $\phi^\kappa : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$, i.e., $\kappa_{t+1} = \phi^\kappa(\kappa_t, B_t)$. Consider (4.10) for $t+1$ and substitute $\kappa_{t+1} = \phi^\kappa\left(\kappa_t, B_t\right)$. This gives

   $$B_{t+1} = B_t(1-\delta)\left(1 + g^B\left(\kappa_{t+1}\right)\right) = B_t(1-\delta)\left(1 + g^B\left(\phi^\kappa\left(\kappa_t, B_t\right)\right)\right) \equiv \phi^B\left(\kappa_t, B_t\right), \tag{A.19}$$

   which is also a continuously differentiable function $\phi^B : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$. Hence, with (κ_1, B_1) given by (4.10) and (4.12), the dynamical system can be stated as

   $$(\kappa_{t+1}, B_{t+1}) = \phi\left(\kappa_t, B_t\right) \equiv \left(\phi^\kappa(\kappa_t, B_t), \phi^B(\kappa_t, B_t)\right),\ t = 1, 2, ..., \infty. \tag{A.20}$$

   The steady state is a fixed point of (A.20). To study the local behavior of the system around the steady state, I need the eigenvalues of the Jacobian $D\phi(\kappa^*, B^*)$. I study each of its four elements in turn.

(a) First, consider $\phi^\kappa(\kappa_t, B_t)$. Implicit differentiation of (A.4) and evaluation at the steady state gives

$$\phi^\kappa_\kappa(\kappa^*, B^*) \quad = \quad \frac{B^* \, \tilde{w}_\kappa\left(\kappa^*\right)}{\Psi_\kappa\left(\kappa^*, B^*\right)} > 0, \tag{A.21}$$

where use is made of (A.2) and $\Psi_\kappa\left(\kappa^*, B^*\right) > 0$. Assuming that the set $\Delta\kappa_t = 0$ is stable in the vicinity of the steady state is equivalent to the assumption that $\phi^\kappa_\kappa(\kappa^*, B^*) \in (0, 1)$. Similarly, I obtain

$$\phi^\kappa_B(\kappa^*, B^*) \quad = \quad \frac{\Psi\left(\kappa^*, B^*\right) - B^* \Psi_B\left(\kappa^*, B^*\right)}{B^* \, \Psi_\kappa\left(\kappa^*, B^*\right)} > 0, \tag{A.22}$$

since $\Psi_B\left(\kappa^*, B^*\right) \leq 0$. Finally, note that (A.21) and (A.22) imply that the slope of the set $\Delta\kappa_t = 0$ is positive, i. e., $dB_t/d\kappa_t|_{\Delta\kappa_t = 0} = (1 - \phi^\kappa_\kappa(\kappa^*, B^*))/\phi^\kappa_B(\kappa^*, B^*) > 0$.

(b) Second, consider $\phi^B\left(\kappa_t, B_t\right)$. From (A.19) I obtain

$$\phi^B_\kappa\left(\kappa^*, B^*\right) \quad = \quad B^* g^B_\kappa\left(\kappa^*\right) \phi^\kappa_\kappa\left(\kappa^*, B^*\right) < 0 \tag{A.23}$$

and

$$\phi^B_B\left(\kappa^*, B^*\right) \quad = \quad 1 + B^* g^B_\kappa\left(\kappa^*\right) \phi^\kappa_B\left(\kappa^*, B^*\right) < 1. \tag{A.24}$$

Using (A.22) and the definition of Ψ given in (A.3), I find $\phi^B_B\left(\kappa^*, B^*\right) \in (0, 1)$ if $\varepsilon^s_\kappa(\kappa^*) < 1 + \varepsilon^A_\kappa(\kappa^*)$. The latter holds since $\varepsilon^s_\kappa(\kappa^*) \leq 0$. Hence, the set $\Delta B_t = 0$ is stable in the vicinity of the steady state with monotonic convergence.

Finally, note that (A.23) and (A.24) imply that the slope of the set $\Delta B_t = 0$ is negative, i. e., $dB_t/d\kappa_t|_{\Delta B_t = 0} = \phi^B_\kappa(\kappa^*, B^*)/(1 - \phi^B_B(\kappa^*, B^*)) < 0$.

Using (A.21) - (A.24), the required Jacobian may be written as

$$D\,\phi(\kappa^*, B^*) = \begin{pmatrix} \phi^\kappa_\kappa(\kappa^*, B^*) & \phi^\kappa_B(\kappa^*, B^*) \\ B^* g^B_\kappa\left(\kappa^*\right) \, \phi^\kappa_\kappa(\kappa^*, B^*) & 1 + B^* \, g^B_\kappa\left(\kappa^*\right) \, \phi^\kappa_B(\kappa^*, B^*) \end{pmatrix}.$$

Its eigenvalues, μ_1, μ_2, are given by

$$\mu_{1,2} \quad = \quad \frac{\phi^\kappa_\kappa + \phi^B_B}{2} \pm \sqrt{\left(\frac{\phi^\kappa_\kappa + \phi^B_B}{2}\right)^2 - \phi^\kappa_\kappa}.$$

Both eigenvalues are real if $\left(\phi^\kappa_\kappa + \phi^B_B\right)^2 \geq 4\phi^\kappa_\kappa$. Since $2 > \phi^\kappa_\kappa + \phi^B_B > 0$, I have $1 > \mu_1 \geq \mu_2 > 0$. Hence, the steady state is either a stable node if the weak inequality is strict or a focus if not. If $\left(\phi^\kappa_\kappa + \phi^B_B\right)^2 < 4\phi^\kappa_\kappa$, then $D\phi$ has two distinct complex eigenvalues, and the steady state is a spiral sink since $det\left(D\phi\right) = \phi^\kappa_\kappa < 1$ (see, e. g., Galor (2007), Proposition 3.8). The stability of the sets $\Delta\kappa_t = 0$ and $\Delta B_t = 0$ imply a clockwise orientation of the spiral sink. ■

A.8 Proof of Proposition 5

The steady-state efficient capital intensity is given by (5.16). Since $g^B\left(\kappa\right)$ is independent of λ and ν, I have $\kappa^* = \kappa^{*\prime}$. Moreover, since $s_\nu\left(R(\kappa^*, B^*), \nu\right) > 0$, $s_R\left(R(\kappa^*, B^*), \nu\right) \geq 0$, and $R_B(\kappa^*, B^*) > 0$, (5.15) also delivers $\lambda > \lambda' \Rightarrow B^* > B^{*\prime}$ and $\nu' > \nu \Rightarrow B^{*\prime} < B^*$. ■

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