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REVIEW ARTICLE

Not a One-Way Street: Bidirectional Relations Between Procedural and Conceptual Knowledge of Mathematics

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Abstract There is a long-standing and ongoing debate about the relations between conceptual and procedural knowledge (i.e., knowledge of concepts and procedures). Although there is broad consensus that conceptual knowledge supports procedural knowledge, there is controversy over whether procedural knowledge supports conceptual knowledge and how instruction on the two types of knowledge should be sequenced. A review of the empirical evidence for mathematics learning indicates that procedural knowledge supports conceptual knowledge, as well as vice versa, and thus that the relations between the two types of knowledge are bidirectional. However, alternative orderings of instruction on concepts and procedures have rarely been compared, with limited empirical support for one ordering of instruction over another. We consider possible reasons for why mathematics education researchers often believe that a conceptual-to-procedural ordering of instruction is optimal and why so little research has evaluated this claim. Future empirical research on the effectiveness of different ways to sequence instruction on concepts and procedures is greatly needed.

Keywords Conceptual knowledge · Procedural knowledge · Mathematics learning

Instruction

More than 30 years ago, Resnick and Ford (1981) noted "the relationship between computational skill and conceptual understanding is one of the oldest concerns in the psychology of mathematics" (p. 246). Seventeen years later, Sowder (1998, as quoted in Star 2005) wrote

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"whether developing skills with symbols leads to conceptual understanding, or whether the presence of basic understanding should precede symbolic representation and skill practice, is one of the basic disagreements" in mathematics education. In 2014, the question "how can we help children develop fluency with basic facts and skills while still promoting understanding of the underlying concepts?" was included in a list of the current grand challenges of mathematical cognition by an expert panel (Alcock et al. 2014). As demonstrated by these quotes, there is a long-standing and ongoing debate about the relations between two types of knowledge—conceptual and procedural. This debate is based on different beliefs about the development and teaching of conceptual and procedural knowledge.

Conceptual knowledge is defined as knowledge of concepts, which are abstract and general principles (e.g., Byrnes and Wasik 1991; Canobi 2009; Rittle-Johnson et al. 2001). For example, the National Research Council defined it as "comprehension of mathematical concepts, operations, and relations" (Kilpatrick et al. 2001, p. 5). Conceptual knowledge can be implicit or explicit, and thus does not have to be verbalizable (e.g., Goldin Meadow et al. 1993). *Procedural knowledge* is often defined as knowledge of procedures (e.g., Byrnes and Wasik 1991; Canobi 2009; Rittle-Johnson et al. 2001). A procedure is a series of steps, or actions, done to accomplish a goal. This knowledge often develops through problem-solving practice, and thus is tied to particular problem types. In mathematics education research, conceptual and procedural knowledge have sometimes been defined based on the *quality* of the knowledge rather than the *type* of knowledge, in particular whether the knowledge is richly connected (Hiebert and LeFevre 1986). However, there is now some consensus that this definition was not appropriate, with agreement that they should be defined based on the type of knowledge (Baroody et al. 2007; Star 2005).

Learners clearly need to develop both conceptual and procedural knowledge in a domain, but controversy arises over how the two types of knowledge are related. It is widely agreed that conceptual knowledge often supports and leads to procedural knowledge. Children's conceptual knowledge can help them invent and understand procedures (Gelman and Williams 1998; Halford 1993; Hiebert and LeFevre 1986). But, is it a "one-way street" from conceptual knowledge to procedural knowledge, or does procedural knowledge also support and lead to conceptual knowledge? Some claim yes, that children gradually derive conceptual knowledge from implementing procedures by abstraction processes, such as representational redescription (Karmiloff-Smith 1992; Siegler and Stern 1998). Others claim no, that procedural knowledge does not lead to conceptual knowledge and can even interfere with gaining conceptual knowledge (Kamii and Dominick 1997, 1998). This debate between a bidirectional perspective and a conceptual-to-procedural (unidirectional) perspective is the focus of this paper.

Reform efforts in US mathematics education have tended to focus on the unidirectional, conceptual-to-procedural knowledge perspective. The belief is that conceptual knowledge should be developed over an extended period of time prior to instruction and practice with procedures (see Baroody (2003), Kilpatrick et al. (2001), and Resnick and Ford (1981) for historical reviews). Most recently, the National Council of Teachers of Mathematics (NCTM 2014) explicitly asserted a conceptual-to-procedural perspective in their principle that "procedural fluency follows and builds on a foundation of conceptual understanding" (p. 42)."Conceptual understanding (i.e., the comprehension and connection of concepts, operations, and relations) establishes the foundation, and is necessary, for developing procedural fluency (i.e., the meaningful and flexible use of procedures to solve problems)" (NCTM p. 7). This principle indicates that students should initially develop a foundation of conceptual understanding and that procedural knowledge should not be developed prior to the extended development of conceptual knowledge. We confirmed that the language used was deliberate, reflecting the "strong belief" of the authors of the report that developing procedural fluency "should not come first" (J. Wanko, personal communication, September 24, 2014). This stance minimizes bidirectional relations between conceptual and procedural knowledge, wherein procedural knowledge supports conceptual knowledge, as well as vice versa.

The goal of the current paper is to review empirical evidence on (1) the relations between conceptual and procedural knowledge over time and (2) the impact of interventions that manipulate the type and order of instruction. Are the relations between conceptual and procedural knowledge unidirectional or bidirectional? Does one ordering of instruction lead to better learning than another ordering of instruction?

Evaluation Criteria

When evaluating evidence on the relations between conceptual and procedural knowledge, both the measures used to assess the two types of knowledge and the studies' research designs are critical to consider. First, consider how the two types of knowledge are measured. The two types of knowledge must be assessed independently in order to study the relations between them, preferably using multiple measures for each type of knowledge (Schneider and Stern 2010). Conceptual knowledge has been measured using a large variety of tasks, ranging from evaluating the correctness of an example or procedure to providing definitions and explanations of concepts (see Crooks and Alibali 2014; Rittle-Johnson and Schneider 2015). A critical feature of conceptual tasks is that they be relatively unfamiliar to participants, so that participants have to derive an answer from their conceptual knowledge, rather than implement a known procedure for solving the task. Measures of procedural knowledge almost always involve solving problems, and the outcome measure is usually accuracy of the answers or procedures. Procedural tasks are familiar—they involve problem types people have solved before, and thus should know procedures for solving. Sometimes, the tasks include near transfer problems—problems with an unfamiliar feature that require either recognition that a known procedure is relevant or small adaptations of a known procedure to accommodate the unfamiliar problem feature (e.g., Renkl et al. 1998; Rittle-Johnson 2006). For both types of knowledge, continuous knowledge measures are more appropriate than categorical measures because they capture gradual changes in knowledge, including changes in the depth and breadth of knowledge. Earlier debates over whether conceptual or procedural knowledge developed first rested on categorical measures to justify that children had or did not have a particular type of knowledge, and making categorical claims became untenable (see Rittle-Johnson and Schneider (2015) and Rittle-Johnson and Siegler (1998) for reviews).

Second, consider appropriate research designs. One source of evidence for potential bidirectional relations comes from longitudinal studies that evaluate whether prior procedural knowledge is related to subsequent conceptual knowledge, as well as vice versa. Causal evidence for bidirectional relations comes from experimental studies that manipulate each type of knowledge and evaluate whether it leads to increases in the other type of knowledge. Evidence for optimal ordering of instruction comes from comparisons of interventions that differ in the inclusion and/or ordering of instruction on concepts and procedures. The most direct evidence would come from studies that compared a conceptual-to-procedural instructional order relative to a procedural-to-conceptual order. Experimental designs with control groups that vary only in the ordering of instruction and randomization of the participants into the experimental groups would provide the strongest evidence.

Evaluation

Bidirectional Relations: Does Procedural Knowledge Support Conceptual Knowledge As Well As Vice Versa?

Longitudinal Relations Evidence for bidirectional relations between procedural and conceptual knowledge comes from longitudinal studies on the predictive relations between the two types of knowledge over time. For example, in two samples differing in prior knowledge, middle-school students' conceptual and procedural knowledge for equation solving was measured before and after a 3-day classroom intervention in which students studied and explained worked examples with a partner (Schneider et al. 2011). Conceptual and procedural knowledge were modeled as latent variables, and a cross-lagged panel design was used to directly test and compare the predictive relations from conceptual knowledge to procedural knowledge and vice versa. Each type of knowledge predicted gains in the other type of knowledge. Further, the relations were symmetrical—procedural knowledge was as predictive of conceptual knowledge as vice versa. Similar bidirectional relations have been found for preschool children learning about counting (e.g., Baroody 1992; Fuson 1988; Muldoon et al. 2007) and elementary-school children learning addition and subtraction (Baroody and Ginsburg 1986; Canobi 2009) and about decimals (Rittle-Johnson and Koedinger 2009; Rittle-Johnson et al. 2001).

The bidirectional relations between conceptual and procedural knowledge are also present over several years. For example, elementary-school children's knowledge of fractions was assessed in the winter of grade 4 and again in the spring of grade 5 (Hecht and Vagi 2010). Procedural knowledge in grade 4 predicted conceptual knowledge in grade 5 after controlling for other factors, and conceptual knowledge in grade 4 predicted procedural knowledge in grade 5. In a separate study, whole-number procedural knowledge in the second grade predicted conceptual knowledge of fractions in the fourth grade (Vukovic et al. 2014). Similar bidirectional relations across grade levels have been found for elementary-school children knowledge of whole number concepts and procedures (Cowan et al. 2011).

This general pattern of one type of knowledge predicting future knowledge of the other type does not mean that the two types of knowledge are equally well developed at any given time. There are individual differences in the relative strength of the two types of knowledge (Gilmore and Papadatou-Pastou 2009; Hallett et al. 2010, 2012; Hecht and Vagi 2012), with changes over time in their relative strength (Hecht and Vagi 2012). Nevertheless, procedural knowledge is a good and reliable predictor of improvements in conceptual knowledge.

Causal Evidence Causal evidence for bidirectional relations comes from studies that experimentally manipulate at least one type of knowledge and then measure both types of knowledge. For example, elementary-school children were given a very brief lesson on a procedure for solving mathematical equivalence problems (e.g., 6+3+4=6+__), the concept of mathematical equivalence, or were given no lesson (Rittle-Johnson and Alibali 1999). Children who received the procedure lesson gained a better understanding of the concept than the control group, and children who received the concept lesson gained greater procedural knowledge than the control group. The relations were bidirectional.

Additional research indicates that improving procedural knowledge can support improvements in conceptual knowledge. Evidence comes from studies on carefully constructed practice problems (Canobi 2009; McNeil et al. 2011, 2012, 2014). For example, elementary-school children solved packets of problems for 10 min on nine occasions during their school mathematics lessons (Canobi 2009). The problems were arithmetic problems sequenced based on conceptual principles, the same arithmetic problems sequenced randomly, or nonmathematical problems (control group). Solving conceptually sequenced practice problems supported gains in procedural as well as conceptual knowledge relative to the control condition. Solving practice problems in random order supported only modest gains in procedural knowledge and did not support gains in conceptual knowledge. Thus, improving procedural knowledge can lead to improvements in conceptual knowledge, but not all types of procedural practice support substantial improvements in either type of knowledge.

Overall, both longitudinal and experimental studies indicate that procedural knowledge leads to improvements in conceptual knowledge, in addition to vice versa. The relations between the two types of knowledge are bidirectional. It is a myth that it is a “one-way street” from conceptual knowledge to procedural knowledge.

Instructional Order: Is Conceptual-to-Procedural Best?

Although the relations between the two types of knowledge are bidirectional, it still may be optimal for instruction to follow a particular ordering. A *conceptual-to-procedural knowledge* perspective asserts that instruction should extensively develop conceptual knowledge prior to focusing on procedural knowledge (Grouws and Cebulla 2000; NCTM 1989, 2000, 2014). Is this the most successful route to mathematical competence? Or are there multiple routes to mathematical competence?

Unfortunately, we could not find empirical evidence to directly evaluate claims for an optimal ordering of instruction. For example, we could not find a study that compared the effectiveness of instruction on concepts-then-procedures to instruction on procedures-then-concepts. Thus, we review the evidence that is given in support of the claim that a conceptual-to-procedural sequence is best, as well as additional evidence we identified that compared the impact of different types of instruction on conceptual and procedural knowledge outcomes.

The claim that instruction on concepts must proceed instruction on procedures is based in part on comparing examples of reform-oriented teaching to traditional instruction. When teacher-researchers spent considerable time developing conceptual knowledge prior to introducing and practicing conventional procedures, students gained greater conceptual knowledge and comparable procedural knowledge compared to typical classroom instruction that focused on procedural knowledge and often included little instruction on concepts (Blöte et al. 2001; Cobb et al. 1991; Fuson and Briars 1990; Hiebert and Grouws 2007; Hiebert and Wearne 1996). Further, children who were taught standard solution procedures with little attention to conceptual knowledge had very limited conceptual knowledge about the domain (Kamii and Dominick 1997, 1998; Mack 1990). While studies such as these highlight that some types of instruction on procedures do not support conceptual knowledge, these studies do not provide evidence that a concept-to-procedure sequence is better than a procedures-to-concepts sequence—only that learning procedures in conjunction with concepts appeared to be better than learning procedures with little or no attention to concepts. Multiple ways to support both types of knowledge were not considered. Furthermore, in all of these studies, the two instructional conditions differed on many dimensions (e.g., the amount of instruction on concepts; the amount and quality of student discussion).

Next, consider studies that have manipulated the type(s) of instruction. Advocates for a conceptual-to-procedural sequence cite a study that compared procedural-then-conceptual instruction to only conceptual instruction (Pesek and Kirshner 2000). In a classroom-based study, fifth-grade students were randomly selected to receive instruction on conventional procedures for finding area and perimeter prior to conceptually focused instruction (i.e., procedural-then-conceptual condition) or to receive no relevant instruction prior to the conceptually focused instruction (i.e., only-conceptual condition). Students in the two conditions had similar performance on a posttest and retention test on finding the areas and perimeters of shapes. However, the only-conceptual condition had slightly (but not statistically significantly) better performance than the procedural-then-conceptual condition (e.g., 48 % vs 42 % correct at posttest) that the authors argued may have been reliable with a substantially larger sample. In addition, qualitative descriptions of interviews with 12 students suggested subtle benefits for the only-conceptual condition on a few of the interview questions. Overall, differences between the conditions were small and not reliable.

We identified a second study that compared instruction on concepts and procedures to only instruction on concepts, although this study is rarely cited in the mathematics education research literature. In Perry (1991), fourth- and fifth-grade students were randomly assigned to receive instruction on a concept and procedure, only on the concept, or only on the procedure. In this lab study, children received a few minutes of instruction in the context of two mathematical equivalence problems. For the concept-and-procedure condition, the instruction was provided one after the other on each problem and the order of the two types of instruction on each problem was counterbalanced across children and did not impact learning outcomes. Children who received only instruction on concepts showed the greatest procedural transfer, with children in the two other conditions performing similarly. Thus, this study provides some support for the claim that instruction on procedures should not be included concurrently with instruction on concepts, at least early in instruction. Additional research is needed to evaluate whether this finding would generalize to other mathematics topics and to typical classroom contexts with much more extensive instruction and problem-solving experience.

Two additional laboratory-based experimental studies contrasted instruction on a concept versus a procedure. Children received brief one-on-one instruction on mathematical equivalence. As in Perry (1991), children who received brief instruction on a concept showed greater knowledge at posttest than children who received brief instruction on a procedure (Matthews and Rittle-Johnson 2009; Rittle-Johnson and Alibali 1999). These findings support the importance of including instruction on concepts and suggest that instruction on a procedure may not be necessary for some topics and for some learning goals. Yet, note that neither of these studies provides direct evidence on the optimal sequencing of instruction on concepts and procedures.

Finally, one study suggests that a small dosage of instruction on concepts first is preferable to a large dosage of instruction on concepts first. In two classroom experiments, sixth-grade students completed six lessons on decimal using an intelligent-tutoring system in one of two randomly assigned conditions (Rittle-Johnson and Koedinger 2009). In the conceptual-then-procedural condition, all conceptual knowledge lessons on place value were presented before the procedural knowledge lessons on adding and subtracting decimals. In an iterative condition, lessons iterated between a focus on concepts and a focus on procedures, beginning with a conceptual knowledge lesson. The iterative order supported equivalent conceptual knowledge and greater procedural knowledge relative to the conceptual-then-procedural sequence. This finding suggests that conceptual knowledge does not need to be well developed before beginning instruction on procedures. This was the only study we identified that contrasted the ordering of different types of instruction and in which all students received the same instruction and only the order of instruction differed. However, because there was no procedural-then-conceptual condition or iterative condition that began with a procedural lesson, this study does not directly inform the debate on a conceptual-then-procedural versus procedural-then-conceptual approach.

In summary, prior research has not directly evaluated whether a conceptual-then-procedural instructional sequence leads to greater learning than alternative orderings of instruction that also focus on developing both types of knowledge. A few laboratory-based studies suggest that it may be more effective to begin instruction with a brief conceptual lesson than with a brief procedural lesson. An experimental, classroom-based study suggests that iterating between lessons on concepts and procedures is more effective than providing extensive instruction on concepts before introducing instruction on procedures. However, much more research is needed on the most effective dosage and timing of instruction on concepts and procedures. We suspect that there are multiple routes to mathematical competence, such as instruction that iterates between a focus on concepts and procedures throughout instruction.

Reexamining the Beliefs: Both Empirical and Nonempirical Reasons

There is clear evidence for bidirectional relations between conceptual and procedural knowledge. It is a myth that procedural knowledge does not support conceptual knowledge. Government-sponsored consensus reports reflect the evidence for bidirectional relations between conceptual and procedural knowledge. In particular, an influential report from the National Research Council identified conceptual understanding and procedural fluency as equally important and interdependent strands of mathematical proficiency (Kilpatrick et al. 2001). Procedural fluency was defined as "skill in carrying out procedures flexibly, accurately, efficiently and appropriately" (p. 116) and thus encompasses procedural knowledge. "Procedural fluency and conceptual understanding are often seen as competing for attention in school mathematics. But pitting skills against understanding creates a false dichotomy. As we noted earlier, the two are interwoven" (p. 122). The National Mathematics Advisory Panel (2008) came to a similar conclusion, noting the mutually reinforcing benefits of both types of knowledge. The terms "interwoven" and "mutually reinforcing" imply bidirectional relations between the two types of knowledge. However, the reports do not explicitly address how instruction should support both types of knowledge.

Although the relations between the two types of knowledge are bidirectional, it may be optimal for instruction to follow a particular ordering. The prevalent *conceptual-to-procedural knowledge* perspective asserts that instruction should extensively develop conceptual knowledge prior to focusing on procedural knowledge (Grouws and Cebulla 2000; NCTM 1989, 2000, 2014). Unfortunately, we could not find empirical evidence to directly evaluate claims for an optimal ordering of instruction. Given the bidirectional relations between conceptual and procedural knowledge, we suspect that there are multiple routes to mathematical competence and that a conceptual-to-procedural ordering is not the only effective route to mathematical competence.

Future research needs to directly evaluate alternative instructional sequences for supporting both types of knowledge over time. Comparisons to traditional instruction that focuses primarily on learning procedures is not an adequate control condition given broad consensus on the importance of developing conceptual knowledge. Rather, a conceptual-then-procedural sequencing should be compared to alternative sequencing that includes a strong focus on conceptual as well as procedural knowledge. An iterative sequencing of instruction appears promising, and future research is needed to evaluate this sequencing, including whether beginning with one type of instruction or the other is preferable. At least for some topics, it might be optimal for an initial lesson to focus on concepts rather than procedures (Perry 1991). Simultaneous instruction on both types of knowledge is another important alternative to study (Ball et al. 2005; Baroody 1992). Some instructional activities may simultaneously support fluent use of procedures and understanding of underlying concepts, such as solving carefully sequenced practice problems with reflection prompts.

Yet, given the pervasiveness of the belief in a conceptual-then-procedural sequence despite the lack of empirical evidence, would additional research convince those who hold the belief? In fact, widespread endorsement of this belief among mathematics education researchers may help to explain why so little research has directly evaluated it. Thus, it seems important to briefly consider nonempirical reasons that might support this belief and which could impede progress in addressing it.

First, language around conceptual and procedural knowledge may also be related to the prevalence and persistence of this belief. For example, conceptual knowledge has sometimes been used to refer to knowledge that is richly connected while procedural knowledge was used to refer to knowledge that is sparsely connected. This makes it impossible to talk about knowledge of procedures that is richly connected (Star 2005, 2007; Star and Stylianides 2013). A new imprecision in language may be arising with the term *procedural fluency*. For some, procedural fluency seems to refer only to an end state of well-developed knowledge, while conceptual knowledge can refer to a variable amount of knowledge. For example, the NCTM (2014) report may have been recommending that students learn some conceptual knowledge prior to complete mastery and fluency with procedures. It is problematic to use the term procedural fluency to refer to an end state of well-developed knowledge and the term conceptual knowledge to refer to a variable, continuously developing level of knowledge. Rather, both terms should refer to knowledge that is variable and continuously developing. Using the terms procedural knowledge or fluency in more narrow ways than conceptual knowledge may promote misunderstandings and myths.

Second, acknowledgment that procedural knowledge may support the development of conceptual knowledge could be interpreted as an argument in support of old instructional ways. Histories of mathematics education in the USA frequently posit that mathematics instruction in the past was overly focused on procedures (Baroody 2003; Resnick and Ford 1981). Attention to the value of procedural knowledge early in instruction might be interpreted as a rejection of prior instructional shortcomings and failure to support modern reforms. However, it should not be.

Finally, culture may play into the persistence of this belief. The directionality of developing conceptual and procedural knowledge seems to only be debated in the USA. This may be because in the USA and some other Western cultures, practice is not believed to aid the development of understanding. In many Asian countries, by contrast, practice is viewed as a route toward understanding, where there is a public perception that only through a great deal of practice can true understanding be developed. Our anecdotal interactions with mathematics education researchers in non-Western countries suggests that they are confused by the debate in the USA. Elsewhere, it is taken as obvious that procedural knowledge can lead to conceptual knowledge (and vice versa).

Conclusion

Mathematical competence rests on developing both conceptual and procedural knowledge, and it is widely agreed that conceptual knowledge often supports and leads to procedural knowledge. Controversy arises over whether the relations are bidirectional or unidirectional (i.e., a conceptual-to-procedural perspective). Evidence indicates that the relations between conceptual and procedural knowledge are often bidirectional, with improvements in procedural knowledge often supporting improvements in conceptual knowledge as well as vice versa. It is not a one-way street from conceptual knowledge to procedural knowledge; the belief that procedural knowledge does not support conceptual knowledge is a myth.

Nevertheless, there may be an optimal ordering of instruction, such as a *conceptual-to-procedural sequence* (Grouws and Cebulla 2000; NCTM 2014). Unfortunately, claims for an optimal ordering of instruction have rarely been evaluated. Given the bidirectional relations between conceptual and procedural knowledge, we suspect that there is not an optimal ordering of instruction, but rather multiple routes to mathematical competence. Future empirical research on the effectiveness of different ways to sequence instruction on concepts and procedures is greatly needed.

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