An integrated theory of whole number and fractions development

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ABSTRACT

This article proposes an integrated theory of acquisition of knowledge about whole numbers and fractions. Although whole numbers and fractions differ in many ways that influence their development, an important commonality is the centrality of knowledge of numerical magnitudes in overall understanding. The present findings with 11- and 13-year-olds indicate that, as with whole numbers, accuracy of fraction magnitude representations is closely related to both fractions arithmetic proficiency and overall mathematics achievement test scores, that fraction magnitude representations account for substantial variance in mathematics achievement test scores beyond that explained by fraction arithmetic proficiency, and that developing effective strategies plays a key role in improved knowledge of fractions. Theoretical and instructional implications are discussed.

1. Introduction

Prominent contemporary theories of numerical development have focused on development of knowledge about whole numbers, relegating development of knowledge about other types of numbers, such as fractions and negative numbers, to secondary status (e.g., Geary, 2006; Leslie, Gelman, & Gallistel, 2008; Wynn, 2002). To the extent that these theories address development of understanding of other types of numbers, it is usually to note differences between their acquisition and that of whole numbers, and to document ways in which whole number understanding distorts understanding

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of them. Learning about whole numbers is depicted as fundamentally different from, and discontinuous with, learning about other types of numbers.

To cite one example, advocates of privileged domains theories argue that specialized learning mechanisms make it easier to learn about whole numbers than about fractions or other types of numbers (Gelman & Williams, 1998; Wynn, 2002). Indeed, some proponents of privileged domains approaches argue that constraints that facilitate learning about whole numbers interfere with learning about fractions. This perspective is reflected in Gelman and Williams' (1998, p. 618) claim that, "children's knowledge of natural numbers (a core domain) serves as a conceptual barrier to later learning about other numbers and their mathematical structures, for example, fractions." Gelman and Williams' argument was that fractions learning is hindered (and whole number learning helped) by children being predisposed to assume that each number has a unique successor, that sets can be counted by assigning numbers to objects in a 1:1 fashion, and that the final number in a count can be used to represent the cardinality of the set that was counted. Similarly, Wynn (1995, p. 176) argued "There are also limits to the kinds of numerical entities the accumulator mechanism represents. It does not represent numbers other than positive numbers...For example, children have great difficulty learning to think of fractions as numerical entities...These facts suggest that the positive integers – the very values that the accumulator model is capable of representing – are psychologically privileged numerical entities."

Evolutionary theories of numerical development take a similar stance. For example, Geary's (2006) evolutionary theory proposes that whole numbers are biologically primary and that fractions and other types of numbers are biologically secondary. Within this theory, as within the privileged domains approach, the constraints and biases that make whole numbers easy to learn (e.g., that counting objects in a set should yield a unique cardinal value, which corresponds to the last number counted) make fractions hard to grasp, because the biases are helpful with whole numbers but misleading with fractions.

Some conceptual change theories (Ni & Zhou, 2005; Vosniadou, Vamvakoussi, & Skopeiliti, 2008) place greater emphasis on development of fractions knowledge than do privileged domains and evolutionary theories, but they are similar in emphasizing differences between learning about whole numbers and fractions and in emphasizing how the "whole number bias" interferes with fractions learning. For example, Vamakoussi and Vosniadou (2010) proposed that before encountering rational numbers, children form a "coherent explanatory framework of number as *counting number* which, in terms of the theoretical framework that we propose, constitutes an initial, domain-specific *theory* of number" (italics in original). They go on to say (p. 187), "Within the framework theory approach to conceptual change, the phenomenon of students' misconceptions due to faulty natural number reasoning...(is) an indication that students draw heavily on their initial understandings of number to make sense of rational numbers."

Although these theories differ in many particulars, they share an underlying commonality. All posit qualitative differences between an early developing, "natural" understanding of whole numbers and a later developing, flawed or hard-won, understanding of fractions. To the extent that relations between the two are posited, the earlier developing understanding of whole numbers is said to interfere with the later developing understanding of rational numbers. The theory of numerical development proposed in this article differs in emphasizing a crucial continuity between acquisition of understanding of whole numbers and fractions, as well as differences between the acquisitions.

1.1. An integrated theory of numerical development

In the present article, we propose an alternative theory of numerical development that emphasizes a key developmental continuity across all types of real numbers. This theory proposes that numerical development is at its core a process of progressively broadening the class of numbers that are understood to possess magnitudes and of learning the functions that connect that increasingly broad and varied set of numbers to their magnitudes. In other words, numerical development involves coming to understand that all real numbers have magnitudes that can be ordered and assigned specific locations on number lines. The basic idea resembles Case and Okamoto's (1996) proposal that the central conceptual structure for whole numbers, a mental number line, is eventually extended to other types of numbers, including rational numbers.

Within the present theory, a complementary, and equally crucial, part of numerical development is learning that many properties that are true of whole numbers –having unique successors, being countable, including finite number of entities within any given interval, allowing expression as a single symbol, invariably increasing or staying the same with addition and multiplication, and invariably decreasing or staying the same through subtraction and division – are not true of numbers in general. The difficulty of learning this lesson, and the behavioral consequences of failing to learn it, are the focus of the theories of numerical development described above. These difficulties are real, dramatic, and important, but they are not the whole story.

One way of thinking about this pair of developments (learning the properties shared and not shared by different types of numbers) is as a gradual change from initially conceptualizing numbers in terms of characteristic features (salient properties of whole numbers that are not necessarily properties of other types of numbers) to later distinguishing between defining features (properties of all real numbers, in particular their magnitudes) and features that apply to some but not all classes of numbers. This change is analogous to the shift from characteristic to defining features in semantic development described by Keil and Batterman (1984). Keil and Batterman noted that young children often categorize objects and events on the basis of salient features that are not definitional for the category. For example, 5- and 6-year-olds tend to categorize a land area as an island if and only if it is warm and sunny, to categorize a vehicle as a taxi if and only if it is yellow and has four wheels, and to categorize an uncle if and only if he is an adult, as well as a sibling of one of the child's parents, or the husband of the child's aunt. Semantic development for these and many other categories requires learning that less salient features are definitional, regardless of the presence or absence of characteristic features. Thus, for most 9- and 10-year-olds, islands must be surrounded by water on all sides, regardless of their climate; taxis must pick up passengers who pay for being driven to chosen locations, regardless of the vehicles' appearance; uncles only must be male siblings of the child's mother or father, or the husband of the child's aunt, regardless of the uncle's age. As in numerical development, the change from emphasizing characteristic features to emphasizing defining features of semantic categories occurs gradually over a period of years and occurs at different times for different entities (different categories in one case; different ranges and types of numbers in the other).

One implication of the present theory of numerical development is that acquisition of knowledge about fractions emerges as a crucial process in numerical development, rather than being of secondary importance. Learning about fractions provides the first major opportunity for children to learn that a variety of salient and invariant properties of whole numbers are not definitional for numbers in general. This understanding does not come easily; although children receive repeated instruction on fractions starting in third or fourth grade (NCTM, 2006), even high school and community college students often confuse properties of fractions and whole numbers (Schneider & Siegler, 2010; Vosniadou, et al., 2008).

The importance of fractions within the present theory of numerical development dovetails with the importance of fractions within mathematics education. Learning of fractions has long been recognized as a serious challenge for teachers and mathematics educators. This is not only true in the US; similar difficulties with fractions have been noted in many countries, including ones with high mathematics achievement, such as Japan (Nunes & Bryant, 2008; Stafylidou & Vosniadou, 2004; Yoshida & Sawano, 2002). These difficulties in learning fractions led the National Mathematics Advisory Panel (2008 p. 18) to conclude, "The most important foundational skill not presently developed appears to be proficiency with fractions (including decimals, percent, and negative fractions). The teaching of fractions must be acknowledged as critically important and improved before an increase in student achievement in algebra can be expected."

The research presented in this article tests two main predictions (as well as several other less fundamental predictions) that arise from the present theory and that do not follow from any of the theories of numerical development described above. If fractions are crucial for overall mathematical understanding, and if understanding magnitudes is crucial for understanding fractions, then (1) Understanding of fraction magnitudes should be strongly related to proficiency at fractions arithmetic; and (2) Understanding of fraction magnitudes should be strongly related to overall mathematical knowledge.

Neither of these relations is obvious or logically necessary. Children could memorize fraction arithmetic algorithms without understanding the magnitudes of the fractions being manipulated. Indeed,

many mathematics educators have lamented that this is exactly what most students do (e.g., Cramer, Post, & del Mas, 2002; Hiebert & Wearne, 1986; Mack, 1995; Sowder et al., 1998). However, rote memorization without understanding tends to be inaccurate over short time periods and becomes even more inaccurate over longer periods (Reyna & Brainerd, 1991). Thus, fraction arithmetic procedures seemed likely to be more accurately remembered by children who understand the magnitudes of the fractions used in the computation than by children who do not understand the fraction magnitudes. One reason is that accurate fraction magnitude representations make it possible to estimate the results of fraction arithmetic operations and to reject implausible solutions. This, in turn, might lead children to reject flawed arithmetic procedures that produced implausible solutions and to continue trying to learn a procedure that generates reasonable answers. Consistent with this perspective, Hecht (Hecht, 1998; Hecht, Close, & Santisi, 2003; Hecht & Vagi, 2010) has found repeatedly that measures of conceptual understanding of fractions correlate positively with fraction addition skill. Another reason for the prediction is that knowledge of whole number arithmetic correlates positively with knowledge of whole number magnitudes (Booth & Siegler, 2008; Siegler & Ramani, 2009). If knowledge of magnitudes play the same role with fractions as it does with whole numbers, as envisioned by the present theory, similar relations between knowledge of numerical magnitudes and arithmetic would be expected with both types of numbers.

The prediction that fraction magnitude knowledge would be related to overall mathematics achievement test scores also was somewhat counter-intuitive, in that fractions magnitude knowledge is assessed little if at all on such tests. Thus, only one of the many items on the online practice battery for the test used to assess overall mathematics achievement in the present study examined knowledge of fraction magnitudes. However, knowledge of whole number magnitudes is related to mathematics achievement test scores, despite the lack of items assessing whole number magnitudes on standardized achievement tests. Moreover, understanding fractions magnitudes was hypothesized to improve learning of fractions arithmetic and understanding of pre-algebra and algebra equations (e.g., $3/4X = 6$), types of knowledge that are assessed on the standardized achievement test given to the students examined in the present study. Consistent with these predictions, Kalchman, Moss, and Case (2001) found that an instructional program that emphasized fraction magnitudes was highly effective in promoting understanding of rational numbers, including decimals and percentages as well as fractions.

Numerical development clearly includes many important acquisitions other than knowledge of numerical magnitudes, such as learning to count and to solve arithmetic problems. However, understanding of numerical magnitudes appears to be particularly central to understanding what numbers are, and has been shown to be related to many other aspects of mathematical development, including counting (Ramani & Siegler, 2008; Whyte & Bull, 2008) arithmetic (Booth & Siegler, 2008; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Geary, Hoard, Nugent, & Byrd-Craven, 2008; Siegler & Ramani, 2009), memory for and categorization of numbers (Laski & Siegler, 2007; Opfer & Thompson, 2008; Thompson & Siegler, 2010), and mathematics achievement test scores (Booth & Siegler, 2006; Geary et al., 2007, 2008; Halberda, Mazzocco, & Feigenson, 2008; Siegler & Booth, 2004). Moreover, manipulations that improve numerical magnitude knowledge have been shown to be causally related to increased proficiency in other important skills, including arithmetic (Booth & Siegler, 2008; Siegler & Ramani, 2009) and counting (Ramani & Siegler, 2008; Whyte & Bull, 2008). These are among the reasons why the present theory of numerical development focuses on acquisition of knowledge about numerical magnitudes. In the remainder of this introductory section, we first summarize current knowledge of whole number magnitude representations and their relation to developmental and individual differences in mathematical understanding, then summarize current knowledge of the development of understanding of fractions, and then describe the present study.

1.2. Development of representations of whole number magnitudes

A wide range of theories of numerical cognition propose that knowledge of whole numbers is organized around a mental number line, in which number symbols (e.g., "7") are connected to non-verbal representations of quantity in an ordered, horizontally-oriented array. The non-verbal representations of quantity appear to be largely spatial (e.g., de Havia & Spelke, 2010; Gevers, Ratinckx, De Baene, & Fias, 2006), though other sensory modalities also seem to be included in the

representation (e.g., Jordan & Baker, 2007; Lourenco & Longo, 2010). Both behavioral and brain imaging data support the mental number line construct (Ansari, 2008). One large body of evidence comes from studies of the SNARC Effect (spatial-numerical associations of response codes), the tendency of people in cultures with left-to-right orthographies to respond faster on the left to smaller numbers and on the right to larger numbers. For example people more quickly answer the question, "Which is bigger, 7 or 4" when 7 is chosen with a right-side key press than with a left-side key press (Dehaene, Bossini, & Giraux, 1993). The SNARC effect is present not only for tasks that are related to magnitude, such as numerical magnitude comparison, but also for tasks that have no relation to magnitude, such as judging whether numbers are odd or even, suggesting automatic access to the number line representation (Hubbard, Piazza, Pinel, & Dehaene, 2005). Brain-damaged patients with left-side neglect displace upward (rightward on the number line) their bisections of numerical ranges of more than a few numbers (e.g., they estimate that the midpoint of the range 11–19 is 17), just as they do with physical lines (Zorzi, Priftis, & Umiltà, 2002). The horizontal, intra-parietal sulcus (HIPS), a brain area believed to be central to the mental number line, shows greater activation during comparison of numbers close in magnitude than during comparison of numbers further apart, presumably because finer magnitude discriminations require greater activation of relevant brain areas (Ansari, 2008; Hubbard et al., 2005).

A non-verbal foundation for this mental number line is present even in infancy. From at least age 6 months, infants can choose the more numerous of two sets of dots, as long as the ratio of the sets is at least 2:1. Over the ensuing months, infants become able to discriminate between sets with smaller ratios, such as 3:2 (Lourenco & Longo, 2010), with the development of this approximate number system continuing for many years (Halberda et al., 2008).

When preschoolers first learn about numerals (symbolically expressed whole numbers), they do not connect the numerals to the numbers' magnitudes. Whether the task involves being asked to give an experimenter N objects, to identify the larger of two numbers, or to identify the number of objects counted, 2- to 4-year-olds who can count flawlessly from 1 to 10 often do not show knowledge of the magnitudes of numbers in that range (Le Corre, Van de Walle, Brannon, & Carey, 2006; Opfer, Thompson, & Furlong, 2010; Ramani & Siegler, 2008; Schaeffer, Eggleston, & Scott, 1974).

A task that has proved particularly useful for studying the development of whole number magnitude representations, both in this early period and for years thereafter, is number line estimation. A typical number line estimation task involves presenting lines with a number at each end (e.g., 0 and 100) and no other numbers or marks in-between, and asking participants to locate a third number on the line (e.g., "Where does 74 go?"). Then a new number line is presented, and participants are asked to locate a different number on that line. The procedure continues until participants have estimated numbers from all parts of the numerical range.

The number line task has several important advantages for measuring representations of numerical magnitudes. It can be used with any real number – large or small, positive or negative, integer or fraction, rational or irrational. It transparently reflects the ratio characteristics of the number system. Just as 80 is twice as large as 40, so the estimated location of 80 should be twice as far from 0 as the estimated location of 40. The task is practiced infrequently compared to skills such as counting and arithmetic, so estimates reflect people's sense of the magnitudes of the numbers rather than memorization of procedures that have been repeatedly practiced.

Ideally, the function relating actual and estimated positions of numbers on a number line should be linear, with a slope of 1, as in the equation $y = x$. Although this relation might seem obvious, children do not understand it for a surprisingly long time. Instead, estimation patterns undergo a transition from a logarithmic to a linear function, that is, from a function in which estimated magnitudes at the low end of the scale are further apart than estimated magnitudes at the high end to a representation in which estimates are equally spaced for equal differences between numbers. The transition occurs earlier for small numerical ranges than for larger ones. For example, 3- and 4-year-olds' number line estimates for the numbers 0 to 10 follow a logarithmic pattern, whereas 5- and 6-year-olds' estimates follow a linear pattern (Berteletti, Lucangeli, Piazza, Dehaene, & Zorzi, 2010; Opfer et al., 2010). This means that 3- and 4-year-olds' estimates of 2 and 3 are farther apart than their estimates of 7 and 8, whereas 5- and 6-year-olds' estimates for 2 and 3 are the same distance apart as for 7 and 8. Similar transitions occur between kindergarten and 2nd grade for the 0–100 range (Geary et al., 2007, 2008;

Siegler & Booth, 2004), between 2nd and 4th grade for the 0–1000 range (Opfer & Siegler, 2007), and between 3rd and 6th graders for 0–10,000 and 0–100,000 number lines (Thompson & Opfer, 2010) (see Siegler, Thompson, and Opfer (2009) for a review).

As implied by the hypothesis that number line estimation reflects a general representation of numerical magnitudes, the same developmental changes from non-linear to linear estimation patterns occur at the same ages on at least two other estimation tasks: numerosity estimation ("There is 1 dot in this beaker and 1000 in this one; hold down the mouse until there are N dots in this empty beaker.") and measurement estimation ("This short line is 1 zip long; this long line is 1000 zips long; draw a line N zips long.") (Booth & Siegler, 2006). Similar developmental patterns in the same age ranges have been demonstrated on tasks other than estimation; for example, categorization of the numbers 0–100 shifts from a logarithmic to a linear distribution between kindergarten and 2nd grade (Laski & Siegler, 2007) and for the numbers 0–1000 for 2nd graders who receive corrective feedback on the placement of numbers near 150 on number lines (Opfer & Thompson, 2008).

In addition to these developmental patterns, consistent individual differences have been found on a variety of tasks hypothesized to measure numerical magnitude representations of whole numbers. For example, most 2nd and 4th graders either produced linear patterns on all three estimation tasks that they were presented (number line, numerosity, and measurement estimation) or did not produce linear patterns on any of them. Individual differences in numerical magnitude representations also have been found to be positively related to other individual differences in mathematical knowledge, including individual differences in arithmetic competence (Booth & Siegler, 2006; Gilmore, McCarthy, & Spelke, 2007; Halberda et al., 2008; Holloway & Ansari, 2008; Mundy & Gilmore, 2009; Schneider et al., 2008) and scores on standardized math achievement tests (Booth & Siegler, 2006; Halberda et al., 2008; Laski & Siegler, 2007; Siegler & Booth, 2004).

Findings regarding whole number magnitude representations also have illuminated other aspects of numerical knowledge, such as arithmetic. If learning answers to arithmetic problems were solely a matter of rote memorization, there would be no particular reason to expect a relation between the accuracy of magnitude representations and arithmetic competence. However, if learning answers to arithmetic problems is a meaningful process, accurate magnitude representations might indicate the implausibility of many answers and the plausibility of a few, producing more peaked distributions of activation around the correct answer, thus facilitating correct retrieval. Consistent with the latter perspective, experiences that improve numerical magnitude representations not only increase subsequent learning of correct answers to arithmetic problems but also lead to errors being closer to the correct answer on trials where children err (Booth & Siegler, 2008; Siegler & Ramani, 2009).

1.3. Development of fraction magnitude representations

A variety of findings call into question whether magnitudes play as great a role in knowledge of fractions as they do with whole numbers. Indeed, these findings raise questions about whether most children and adults represent fraction magnitudes at all. When a nationally representative sample of US 8th graders was asked on the National Assessment of Educational Progress (NAEP) to choose whether 12/13 + 7/8 is closest to 1, 2, 19, or 21, the answer 2 was chosen by fewer students than either 19 or 21 (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981). The lack of understanding of fraction magnitudes implied by this finding was not an ephemeral phenomenon; on a recent NAEP, only 50% of 8th graders correctly ordered three fractions (NCTM, 2007). This poor performance cannot be explained by children not trying hard on group-administered standardized achievement tests; a recent study that involved 1:1 testing of 6th graders with a mean IQ of 116 reported only 59% correct ordering of a set of fractions (Mazzocco & Devlin, 2008). The problem also is not limited to rational numbers written in common fractions notation; in an experimental study, more than half of 5th graders consistently erred when comparing the magnitudes of two decimals when the larger decimal had fewer digits (e.g., .345 versus .67) (Rittle-Johnson, Siegler, & Alibali, 2001). Nor does the problem disappear after middle school; fewer than 30% of 11th graders correctly translated .029 as 29/1000 on the 2004 NAEP (Kloosterman, 2010). Some investigators have concluded that even university students do not represent fraction magnitudes in a systematic way (Bonato, Fabbri, Umiltà, & Zorzi, 2007).

The issue of whether magnitude representations are central to understanding of fractions calls into question the generality of conclusions about developmental and individual differences in numerical knowledge that have been derived from studies of whole numbers. Consider age-related changes in numerical representations. As noted, transitions from logarithmic to linear representations of whole numbers have been documented over a wide age span for different ranges of whole numbers. However, it is unknown whether such logarithmic to linear transitions are unique to whole numbers or whether they also occur with other types of numbers, such as fractions.

Current understanding of individual differences in numerical magnitude representations is limited in the same way. Also as noted previously, accuracy of whole number magnitude representations has been found to correlate quite strongly with standardized mathematical achievement test scores, arithmetic proficiency, and other measures of mathematical knowledge. Again, whether such consistent individual differences extend beyond whole numbers is unknown.

In addition to the predicted similarities between the role of magnitude representations for fractions and whole numbers, several differences between development of fractions and whole number magnitude knowledge also were expected. One was that logarithmic representations of fraction magnitudes were not expected to be apparent. Dehaene and his colleagues (e.g., Dehaene & Mehler, 1992) have hypothesized that logarithmic representations of whole number magnitudes reflect the frequency of encountering different numbers. Because people much more frequently encounter small whole numbers than larger ones, the psychological space between the small numbers seems larger. For example, 2 and 4 are encountered more frequently than 82 and 84, and thus seem more different and further apart (Dehaene, 1997). However, frequency of encountering fractions appears unrelated to the fractions' magnitudes. Fractions such as ½, ¼, and ¾ are much more common than both smaller fractions (e.g., 1/29) and larger ones (e.g., 28/29). If relative frequency of encountering whole numbers explains logarithmic representations of whole numbers, and if relative frequency of encountering fractions is unrelated to fractions' magnitudes, logarithmic fractions magnitude representations would not be expected.

Another important way in which development of knowledge of fractions magnitudes seems likely to differ from development of whole number magnitudes is in the role of strategies. Although strategic influences have been recognized in a wide range of problem solving and reasoning contexts (Siegler, 1996), reviews of the literature on whole number magnitude representations (e.g., Ansari, 2008; Dehaene, Dehaene-Lambertz & Cohen, 1998; Fias & Fischer, 2005; Hubbard et al., 2005) typically do not even mention strategies or strategy choices (for exceptions, see Geary et al., 2007, 2008). The implicit assumption is that people invariably use a particular representation of numerical magnitudes and that the research task is to determine the characteristics of that representation. Sometimes, the assumption is explicit, as when Dehaene (1997, p. 78) described logarithmic representations of whole number magnitudes as occurring "like a reflex" that cannot be inhibited. Consistent with the assumption that magnitude representation is an automatic, non-strategic process, number line estimation with whole numbers is no less accurate under time pressure than without time pressure (Siegler & Opfer, 2003).

The lack of emphasis on strategies for choosing whole number magnitude representations might reflect such strategies not being important in processing numerical magnitudes in general, perhaps because magnitudes are automatically accessed through the approximate number system (Feigenson, Dehaene, & Spelke, 2004). Another possibility, however, is that strategies for representing numerical magnitudes might be more apparent with fractions than with whole numbers. Fractions are encountered much less often than whole numbers, and processing of their magnitudes seems likely to be slower and under greater voluntary control, qualities identified by Ericsson and Simon (1980) as characteristic of tasks on which people can accurately report strategy use. In other numerical domains that meet the Ericsson and Simon criteria, such as whole number arithmetic, trial-by-trial strategy assessments have proved useful in explaining both developmental and individual differences in accuracy and speed (Siegler, 1996). The same seemed likely to be true with fractions.

1.4. The present study

In the current study, we presented 6th and 8th graders (11- and 12-year-olds and 13- and 14-year-olds) with three assessments of fraction magnitude knowledge – 0–1 number line estimation, 0–5

number line estimation, and 0–1 magnitude comparison – as well as with fraction arithmetic problems. Verbal reports of strategy use were obtained immediately after each number line estimation and arithmetic problem, to allow examination of relations between strategy use and speed and accuracy on each task. We also obtained students' mathematics achievement test scores, to examine their relation to the three measures of fraction magnitude knowledge and to fraction arithmetic proficiency. We made six predictions; the first two were based on previous findings that knowledge of fractions is generally poor, and the last four reflected the present hypothesis that understanding of fraction magnitudes is closely related to other types of knowledge of fractions and other aspects of mathematics:

(1) Even after years of fractions instruction, fraction magnitude representations, whether measured by magnitude comparison or number line estimation, will be quite inaccurate in both 6th and 8th grade.
(2) Despite information about fractions magnitudes being taught explicitly in 3rd and 4th grade (NCTM, 2007), this understanding should still be increasing between 6th and 8th grade, due to students learning about fraction magnitudes from solving problems involving proportions and percentages in those grades.
(3) No logarithmic to linear transition should be present with fractions, because frequency of encountering fractions (and therefore knowledge of specific fractions) is correlated minimally if at all with fraction magnitudes, at least in the 0–1 range.
(4) Students should use a variety of strategies to solve fraction number line estimation and arithmetic problems, and the quality of these strategies should be related to students' accuracy and speed in solving problems, as with whole number arithmetic.
(5) Individual differences in knowledge of fractions magnitudes should correlate highly with success at solving fraction arithmetic problems.
(6) Individual differences in knowledge of fractions magnitudes should correlate highly with individual differences in overall mathematics achievement test scores.

2. Method

2.1. Participants

Participants were 24 6th graders (M CA = 11.69, SD = 0.44; 50% girls; 88% Caucasian, 8% Asian, 4% Biracial) and 24 8th graders (M CA = 13.69, SD = 0.61; 50% girls; 92% Caucasian, 4% Asian, 4% Hispanic), recruited from two, predominantly middle-income, public school districts near Pittsburgh, PA. All children who returned consent forms were included in the study. The mathematics proficiency of students in the study was slightly above the state average; 81% of students in the sample scored at or above a proficient level on the state test of mathematics achievement (the PSSA), versus 74% for sixth and eighth graders in the state as a whole and 79% for their schools as a whole (http://www.portal.state.pa.us/portal/server.pt/community/school_assessments/7442/2008-2009_pssa_and_ayp_results/60028). Testing was done by a female Caucasian research associate.

2.2. Tasks

2.2.1. Number line estimation

On each number line estimation task, participants were sequentially presented 10 number lines on a computer screen. On the 0–1 task, each number line included a left endpoint labeled "0," a right endpoint labeled "1," and no other marks or numbers. Above the midpoint of each line was a fraction whose position participants needed to estimate: 1/19, 1/7, 1/4, 3/8, 1/2, 4/7, 2/3, 7/9, 5/6, or 12/13. The 0–5 task was identical, except that "5" was written at the right end of the number line, and the fractions were 1/19, 4/7, 7/5, 13/9, 8/3, 11/4, 10/3, 7/2, 17/4, and 9/2. In both cases, one fraction was drawn from each tenth of the number line, and presentation order was random. Participants responded on each trial by moving the cursor to the desired position on the number line and clicking the mouse. After children explained their answer, an unmarked number line and a different fraction appeared on the screen.

2.2.2. Magnitude comparison

Participants were asked to compare to 3/5 a fraction shown on the computer screen: 3/8, 5/8, 2/9, 4/5, 4/7, 5/9, 8/9, or 2/3. If the fraction was smaller than 3/5, the participant was to press the "a" key; if the fraction was larger than 3/5, the participant was to press the "l" key.

2.2.3. Fraction arithmetic

Participants were presented 8 problems, 2 for each of the 4 arithmetic operations: 3/5 + 1/2, 3/5 + 2/5, 3/5 – 1/2, 3/5 – 2/5, 3/5 * 1/2, 3/5 * 2/5, 3/5 ÷ 1/2, and 3/5 ÷ 2/5. One of the two problems for each arithmetic operation had operands with equal denominators. Problems appeared one at a time on the computer screen, and participants typed their answers to the problems on the computer keyboard. Participants were able to use scrap paper to help them solve the arithmetic problems.

2.2.4. Standardized math achievement test

The mathematics section of the Pennsylvania System of School Assessment (PSSA), the standardized state achievement test used in Pennsylvania, provided a measure of students' overall knowledge of mathematics toward the end of 5th and 7th grades, roughly half a year before the study was conducted. At both grade levels, the PSSA samples a wide range of skills including knowledge of whole number and fraction arithmetic; probability and statistics; interpretation of tables, graphs, and figures; pre-algebra; geometry; and series extrapolation.

2.3. Design and procedure

Participants were tested individually in a quiet room in their school and completed all tasks on a laptop computer during a single session that lasted roughly 30 min. The tasks and the problems on each task were presented in random order. Participants completed all problems on one task before solving any problems on the next. After participants answered each problem, the program prompted them to explain their strategy by speaking into a microphone. Participants were asked to describe what they thought in their head, what they wrote on scrap paper, and how they generated their answer. These explanations were audio recorded for later coding.

3. Results

3.1. Number line estimation

We examined three aspects of the 6th and 8th graders' number line estimates: accuracy, linearity, and strategies.

3.1.1. Accuracy

Accuracy of number line estimation was indexed by percent absolute error (PAE), defined as:

$$PAE = (|\text{Child's Answer} - \text{Correct Answer}|)/\text{Numerical Range}$$

For example, if a child was asked to locate 5/2 on a 0–5 number line, and marked the location corresponding to 3/2, the PAE would be 20% $((|1.5 - 2.5|)/5)$. Note that PAE varies inversely with accuracy; the higher the PAE, the *less* accurate the estimate.

A 2 (Grade: 6th or 8th) × 2 (Number line: 0–1 or 0–5) ANOVA revealed that estimates were more accurate on 0–1 than on 0–5 number lines, PAEs = 13% and 24%, $F(1, 46) = 46.99$, $p < .001$, partial $\eta^2 = .51$, and that 8th graders' estimates were more accurate than those of 6th graders, PAE's = 15% and 22%, $F(1, 46) = 4.87$, $p < .05$, partial $\eta^2 = .10$. Although the interaction was not statistically significant, 8th graders were more accurate than 6th graders on the 0–5 number lines, PAE = 19% versus 29%, $t(46) = 2.46$, $p < .05$, $d = .77$, but the two groups did not differ on the 0–1 number lines, PAE = 11% versus 15%, $t(46) = 1.40$, n.s.

Accuracy of number line estimation on the 0–1 and 0–5 scales was quite strongly related for both older and younger children. Correlations between PAE on the 0–1 and 0–5 number line estimation tasks were $r(22) = .56$, $p < .01$ for 6th graders and $r(22) = .61$, $p < .01$ for 8th graders.

3.1.2. Linearity

To describe the pattern of number line estimates, we computed the best fitting logarithmic and linear functions relating the fraction that was presented to participants' estimates of its position on the number line. Most 6th and 8th graders' number line estimates fit a linear function better than a logarithmic one. On 0–1 estimates, the best fitting linear function for the group mean estimates on each fraction accounted for 98% of the variance in the 6th graders' estimates and 100% of the variance in 8th graders' estimates, whereas the best fitting logarithmic function accounted for 86% of the variance in both 6th and 8th graders' estimates. The relatively good absolute fits of the logarithmic functions appeared to be due in large part to the substantial correlations inherently present between logarithmic and linear functions. Analyses of individual performance indicated that on 0–1 number lines, the linear function fit better than the logarithmic function for 79% of 6th graders, mean $R^2_{lin} = .57$ versus mean $R^2_{log} = .49$, $t(23) = 3.95$, $p < .001$, $d = .98$, and for 88% of 8th graders, mean $R^2_{lin} = .81$ versus mean $R^2_{log} = .69$, $t(23) = 6.30$, $p < .001$, $d = 1.66$.

A similar pattern emerged with 0–5 number lines for 8th graders. The best fitting linear function for the group mean estimates of each fraction accounted for 96% of the variance in the 8th graders' 0–5 number line estimates, whereas the best fitting logarithmic function accounted for 66% of the variance. Analyses of individual performance indicated that the best fitting linear function accounted for greater variance in estimates on 0–5 number lines than the best fitting logarithmic function for 75% of 8th graders, mean $R^2_{lin} = .62$ versus mean $R^2_{log} = .47$, $t(23) = 3.36$, $p < .01$, $d = .78$. In contrast, no simple function fit most 6th graders' estimates on 0–5 number lines at all well. For the group mean estimates, the best fitting linear function accounted for 13% of variance, and the best fitting logarithmic function accounted for 42%. For individual 6th graders, the best fitting linear function accounted for 26% of the variance in estimates on 0–5 number lines, and the best fitting logarithmic function accounted for 28% (n.s.).

As these results suggest, on 0–1 number lines, percent variance in individual children's estimates accounted for by the best fitting linear function increased from 6th to 8th grade, mean $R^2_{lin} = 57\%$ versus 81%, $t(46) = 2.47$, $p < .05$, $d = .70$. Increases from 6th to 8th grade also were evident in the slope of individual children's best fitting linear function, mean slope = .58 versus .87, $t(46) = 2.55$, $p = .01$, $d = .73$. Similarly, on 0–5 number lines, percent variance in individual children's estimates accounted for by the best fitting linear function increased between 6th and 8th grade from 26% to 62%, $t(46) = 3.76$, $p < .001$, $d = 1.09$, and the mean slope of individual children's best fitting linear function increased from .16 to .61, $t(46) = 3.41$, $p < .001$, $d = .99$. Thus, the 8th graders' estimates fit the ideal function $y = x$ (estimated value = presented value) considerably better than did the 6th graders' estimates for both 0–1 and 0–5 number lines.

3.2. Strategies

Considerable strategy use was apparent on number line estimation with fractions, particularly on 0–5 number lines. This frequency of strategy use was related to the accuracy of estimates.

3.2.1. Classification of strategies

Strategies were classified on the basis of overt behavior and immediately retrospective self-reports. When overt behavior clearly indicated the child's approach, that behavior was the basis of the strategy assessment; otherwise, the child's self-report was used.

The two main types of number line estimation strategies were *numerical transformation strategies*, in which participants transformed the presented fraction to a more convenient number, and *number line segmentation strategies*, in which participants generated subjective landmarks on the number line to help locate the fractions. Both types of strategies could be, and often were, used on a single trial; children could transform the fraction to a more convenient numerical form and segment the number line in a way that helped them locate the fraction on it.

In coding numerical transformation strategies, we distinguished only between using a numerical transformation and not using one. The most common numerical transformations were rounding the fraction ("5/9 is a bit more than ½"), simplifying it ("9/5 = 1 and 4/5, which is a little less than 2"), or translating it into a different form ("12/13 is about 90%). The reason for not distinguishing among these numerical transformations was that they overlapped and could not be reliably distinguished. For example, the statement "12/13 is about 90%" involves a translation from a common fraction to a percentage, but it also involves rounding (12/13 actually is more than 90%).

The main number line segmentation strategies were division into *halves*; division into *fifths* or whole number units (e.g., placing marks on a 0–5 number line at the estimated positions of 1, 2, 3, and 4); division into units corresponding to the *denominator* (e. g., dividing a 0–1 number line into sevenths to locate 4/7); *flawed* approaches (e.g., on a 0–1 number line, reporting, "I put 3/7 near 0 because 3 rounds down"); and *none/unknown* (e.g., saying, "I don't know"). The lengthy solution times on none/unknown trials, roughly 10 s, suggested that children might have used some strategy, but neither self-reports nor overt behavior indicated what it was. Other numerical transformation and number line segmentation strategies were used occasionally, but as indicated in Table 1, both 6th and 8th graders appeared to use the specified approaches on the large majority of trials.

These strategies took a substantial amount of time to execute. The mean of the median solution times across number lines and age groups was 9.5 s. On 0–1 number lines, the means of the medians of individual children's solution times were 8.4 s for 6th graders and 7.3 s for 8th graders; on 0–5 number lines, they were 11.9 s for 6th graders and 10.3 s for 8th graders. These times were far longer than the times that children of these ages take on number line estimation with whole numbers. For example, Siegler and Opfer (2003) found that imposing a 4 s limit on number line responses had no effect on 2nd, 4th, and 6th graders' accuracy relative to allowing unlimited response time. The response times suggest that fractions number line estimation is far from automatic; rather, it appears to be a controlled, strategic process.

3.2.2. Frequency of strategy use

Transformation and segmentation strategies were used on at least some trials by the large majority of both older and younger children. Among 8th graders, 75% of children were classified as using a numerical transformation strategy at least once on 0–1 number lines, as were 79% on 0–5 number lines. The corresponding percentages for the segmentation strategy were 83% and 92%. Among 6th graders, 50% were classified as using a numerical transformation strategy at least once on 0–1 number lines, as were 58% on 0–5 number lines. The corresponding percentages for segmentation strategies were 83% and 79%.

A 2 (grade) $\times$ 2 (Number line) ANOVA on the number of trials on which a numerical transformation strategy was used indicated that participants used numerical transformation strategies more often on 0–5 than on 0–1 number lines, 44% of trials versus 14%, $F(1, 46) = 38.83$, $p < .001$, partial $\eta^2 = .46$. The analysis also indicated that 8th graders used numerical transformation strategies more often than 6th graders did, 39% versus 19% of trials, $F(1, 46) = 10.68$, $p < .005$, partial $\eta^2 = .19$. As shown in the leftmost two columns of data in Table 1, a grade $\times$ number line interaction was also present, $F(1, 46) = 4.43$, $p < .05$, partial $\eta^2 = .09$. Both 8th and 6th graders used numerical transformation strategies more often on 0–5 than on 0–1 number lines, but the difference was larger for 8th graders, 59% versus 18%, $t(23) = 6.09$, $p < .001$, $d = 1.49$, than for 6th graders, 25% versus 7%, $t(23) = 2.89$, $p < .01$, $d = .74$.

Table 1
Percent use of numerical transformation and number line segmentation strategies for number line estimation.

Grade/number line	Numerical transformations		Number line segmentation				
	Round, simplify, translate	None/unknown	Halves	Fifths	Denominators	Flawed	None/unknown
6th 0–1	7	91	22	0	16	0	61
6th 0–5	25	71	10	18	2	0	68
8th 0–1	18	81	26	0	12	3	59
8th 0–5	59	41	11	38	1	5	46

Note: Percentages do not always add to 100% due to occasional use of other strategies.

A parallel grade × number line ANOVA was conducted on the number of trials on which number line segmentation strategies were used. It revealed a grade by number line interaction, $F(1, 46) = 6.95$, $p = .01$, partial $\eta^2 = .13$. On 0–1 number lines, 6th and 8th graders used segmentation strategies equally often (39% and 41% of trials), but on 0–5 number lines, 8th graders used segmentation strategies considerably more often than 6th graders did, 54% versus 32%, $t(46) = 2.42$, $p < .05$, $d = .70$. Thus, for both numerical transformation and number line segmentation strategies, developmental differences in strategy use were greater on the more demanding 0–5 number lines.

3.2.3. Strategy use and estimation accuracy

Frequency of use of numerical transformation strategies was highly predictive of PAE on 0–5 number lines: for 6th graders, $R^2 = .51$, $F(1, 22) = 22.69$, $p < .001$; for 8th graders, $R^2 = .79$, $F(1, 22) = 80.45$, $p < .001$. On 0–1 number lines, frequency of numerical transformations was somewhat predictive of PAE for 8th graders, $R^2 = .18$, $F(1, 22) = 4.80$, $p < .05$, though not for 6th graders (n.s.).

The corresponding analysis of the relation between frequency of use of segmentation strategies and number line accuracy used as predictors the frequency of (1) halves division, (2) denominator-based division, (3) whole number division (fifths), (4) flawed strategies, and (5) no strategy or unknown strategy. On 0–5 number lines, the 6th graders' estimation accuracy (PAE) was predicted by the frequency of division of the number line into whole number segments, $R^2 = .32$, $F(1, 22) = 10.20$, $p < .005$. The 8th graders' estimation accuracy was predicted by frequency of the halves strategy and infrequency of no strategy, the two variables together accounting for 38% of the variance in PAE, $F(2, 21) = 6.44$, $p < .01$. On 0–1 number lines, 6th graders' accuracy was predicted by frequency of division into the number of units indicated by the denominator, $R^2 = .18$, $F(1, 22) = 4.90$, $p < .05$; 8th graders' accuracy was predicted by infrequency of flawed strategies, $R^2 = .27$, $F(1, 22) = 8.14$, $p < .01$. Thus, both numerical transformation and number line segmentation strategies were related to estimation accuracy.

3.3. Magnitude comparison

3.3.1. Accuracy

Number of correct magnitude comparisons tended to improve from 6th to 8th grade, 68% versus 79% correct, $t(46) = 1.78$, $p < .10$. In both grades, distance of the comparison fraction from 3/5 was related to number of correct answers for that fraction: $r(6) = .76$, $p < .05$ for 6th graders, and $r(6) = .66$, $p < .10$, for 8th graders. This distance effect with fractions parallels that often found with whole numbers (e.g., Moyer & Landauer, 1967).

Because magnitude comparison and number line estimation are both believed to reflect understanding of numerical magnitudes, individual differences on the two tasks were expected to be related. Consistent with this expectation, number of correct magnitude comparisons and PAE on 0–1 number lines correlated significantly both for 6th graders, $r(22) = -.48$, $p < .05$, and for 8th graders, $r(22) = -.70$, $p < .05$ (Tables 5 and 6). Number of correct magnitude comparisons and PAE for 0–5 number line estimates correlated $r(22) = -.33$, n. s. for 6th graders and $r(22) = -.67$, $p < .01$ for 8th graders. (The negative correlations reflect greater magnitude knowledge being indexed by higher percent correct on magnitude comparison and lower PAE on number line estimation).

3.4. Arithmetic

We examined accuracy and strategy use for fractions arithmetic, as well as the relation between the two.

3.4.1. Accuracy

A 2 (Grade) × 4 (Operation: addition, subtraction, multiplication, or division) × 2 (Denominator: same or different) ANOVA on number of correct fraction arithmetic answers yielded main effects for all three variables: grade (32% correct for 6th graders versus 60% for 8th graders, $F(1, 46) = 11.65$, $p < .001$, partial $\eta^2 = .20$); arithmetic operation (48% correct for addition, 51% for subtraction, 60% for multiplication, and 25% for division, $F(3, 138) = 9.28$, $p < .001$, partial $\eta^2 = .17$); and

denominator (49% correct for equal denominators versus 43% correct for different denominators, $F(1, 46) = 4.19$, $p < .05$, partial $\eta^2 = .08$).

As shown in Table 2, accuracy improved for all four fraction arithmetic operations between 6th and 8th grade. The very low percent correct on 6th graders' division of fractions reflected the fact that, contrary to our expectation, these students had not been taught division of fractions at the time the study was conducted. Even ignoring the data on 6th graders' division of fractions, the results indicated strikingly poor knowledge of fraction arithmetic at both grade levels. The 6th graders erred on the majority of items on all four arithmetic operations, and the 8th graders erred on 40% of items.

The ANOVA also revealed an interaction between arithmetic operation and equality of denominators, $F(3, 138) = 7.24$, $p = .01$, partial $\eta^2 = .14$. This interaction seemed likely to be due to differences among the four fraction arithmetic algorithms. Adding and subtracting fractions require common denominators, whereas multiplying and dividing them do not. Thus, accuracy on fraction addition and subtraction problems with common denominators should be higher than on problems without them, because being presented with common denominators eliminates the need to generate them without changing the value of the fraction. Consistent with this logic, children answered fraction addition and subtraction problems with common denominators more accurately than fraction addition and subtraction problems without them (for addition, 58% versus 38% correct, $t(47) = 2.87$, $p < .01$, $d = .40$; for subtraction, 60% versus 42%, $t(47) = 3.29$, $p < .01$, $d = .46$). Also consistent with the logic, accuracy was no higher on fraction multiplication and division problems with common denominators than on ones without them (for multiplication, 52% versus 67%, $t(47) = 1.85$, $p < .10$; for division, 25% versus 25%, n.s.). The marginally significant effect for fraction multiplication problems with common denominators being solved less often was due to 34% of 6th and 8th graders treating such items like addition problems, and incorrectly maintaining the denominators despite multiplying the numerators (e.g., claiming $3/5 * 2/5 = 6/5$).

3.5. Strategies

The 6th and 8th graders were classified as using four main fraction arithmetic strategies. *Correct strategies* involved use of a procedure that yielded the correct answer if executed correctly. *Independent whole numbers strategies* involved performing the arithmetic operation on the numerators and denominators separately, as if they were independent whole numbers (e.g., $3/5 + 1/2 = 4/7$). *Wrong fractions operation strategies* involved treating the numerator or denominator incorrectly in a way that would be correct for that component in a different fractions arithmetic operation (e.g., maintaining the common denominator on a multiplication problem, as in $3/5 * 2/5 = 6/5$). *None/unknown strategy* usually involved saying, "I guessed" or "I don't know." Correct strategies were used on 49% of trials, independent whole number strategies on 27%, wrong fractions operation strategies on 15%, no strategy or unknown strategy on 6%, and idiosyncratic incorrect strategies on 3%.

3.5.1. Correct strategies

Because children erred on 14% of attempted uses of correct arithmetic strategies (28 errors in 198 attempted uses), we analyzed the frequency of attempted use of correct strategies, independent of

Table 2
Percent correct on fractions arithmetic problems with equal and unequal denominators.

Operation	Denominator	Sixth grade	Eighth grade
Addition	Equal	54	63
Addition	Unequal	21	54
Subtraction	Equal	54	67
Subtraction	Unequal	25	58
Multiplication	Equal	38	67
Multiplication	Unequal	54	79
Division	Equal	4	46
Division	Unequal	4	46

whether they generated a correct answer. A 2 (Grade) × 2 (Arithmetic Operation: Addition/subtraction or multiplication/division) × 2 (Denominator: equal or unequal) ANOVA showed greater frequency of correct fraction arithmetic strategies for 8th than for 6th graders, 64% versus 34%, $F(1, 46) = 11.12$, $p < .005$, partial $\eta^2 = .20$, and greater use of correct strategies for problems with equal than with unequal denominators, 53% versus 45%, $F(1, 46) = 11.50$, $p < .001$, partial $\eta^2 = .20$ (Tables 3 and 4).

An interaction between equality of denominators and arithmetic operation was also present, $F(1, 46) = 10.68$, $p < .005$, partial $\eta^2 = .19$. Consistent with the need for equal denominators when adding and subtracting fractions but not when multiplying and dividing them, this interaction emerged because correct strategies were used more often on addition and subtraction problems with equal denominators than on problems with unequal denominators, 63% versus 42%, $t(47) = 3.91$, $p < .001$, $d = .57$, whereas correct strategies were used equally often on multiplication and division problems with equal and unequal denominators (44% and 48%, respectively).

3.5.2. Independent whole numbers strategy

Treating numerators and denominators as if they were independent whole numbers led to errors on three of the four arithmetic operations (e.g., 3/5 – 1/2 = 2/3), though it led to correct answers on multiplication problems (e.g., 3/5 ∗ 1/2 = 3/10). Because it was impossible to tell for multiplication problems whether children used the correct approach or the independent whole numbers approach, we assumed that correct answers on multiplication problems reflected a correct strategy and omitted multiplication from analyses of the independent whole numbers strategy. Therefore, statistical analyses of use of the incorrect whole numbers strategy contrasted its use on addition and subtraction problems to its use on division problems.

A grade × arithmetic operation × equality of denominators ANOVA indicated that the independent whole numbers strategy was applied more often to problems with unequal denominators than to problems with equal denominators, 40% versus 32%, $F(1, 46) = 5.78$, $p < .05$, partial $\eta^2 = .11$, and that the strategy tended to be used more often by 6th than by 8th graders, 44% versus 27%, $F(1, 46) = 2.99$, $p < .10$, partial $\eta^2 = .06$. An interaction between arithmetic operation and equality of denominators was also present, $F(1, 46) = 5.63$, $p < .05$, partial $\eta^2 = .11$. On addition and subtraction problems, the independent whole numbers strategy was more common on problems where denominators were unequal than on ones where they were equal, 46% versus 30%, $t(47) = 3.92$, $p < .001$, $d = .59$. In contrast, the independent whole numbers strategy was applied equally often on division problems, regardless of whether denominators were equal or unequal, 33% for both. Apparently, the instructional message that addition and subtraction of fractions require equal denominators is strong enough to prevent some children from applying the independent whole numbers strategy to addition and subtraction problems with equal denominators. However, either because these children do not know how to create equal denominators or because they forget to create them on unequal denominator problems, they treat numerators and denominators on other problems as independent whole numbers.

Table 3
Percent use of fractions arithmetic strategies, 6th grade.

Operation/ denominator	Correct	Independent whole numbers	Wrong fractions operation	None/ unknown
Addition/equal	54	33	4	4
Addition/unequal	25	54	4	13
Subtraction/equal	50	38	13	0
Subtraction/unequal	25	54	8	4
Multiplication/equal	50	–	42	4
Multiplication/unequal	54	–	25	13
Division/equal	8	42	29	13
Division/unequal	8	46	21	21

Notes: The independent whole number strategy is the correct strategy for multiplication problems; to avoid double counting, trials that fit both strategies were counted as correct. Percentages do not always add to 100% due to occasional use of other strategies.

Table 4
Percent use of fractions arithmetic strategies, 8th grade.

Operation/ denominator	Correct	Independent whole numbers	Wrong fractions operation	None/ unknown
Addition/equal	75	25	0	0
Addition/unequal	58	38	0	4
Subtraction/equal	71	25	4	0
Subtraction/unequal	58	38	0	0
Multiplication/equal	71	–	25	4
Multiplication/ unequal	79	–	17	0
Division/equal	46	25	17	8
Division/unequal	50	21	25	4

Notes: The independent whole number strategy is the correct strategy for multiplication problems; to avoid double counting, trials that fit both strategies were counted as correct. Percentages do not always add to 100% due to occasional use of other strategies.

Table 5
Correlations among fractions tasks and achievement test scores, 6th grade.

	Number line 0–5 PAE	Magnitude comparison accuracy	Arithmetic accuracy	PSSA math
Number line 0–1 PAE	.56**	−.48*	−.56**	−.67**
Number Line 0–5 PAE		−.33	−.55**	−.54**
Magnitude comparison accuracy			.55**	.60**
Arithmetic accuracy				.48*

* $p < .05$.
** $p < .01$.

Table 6
Correlations among fractions tasks and achievement test scores, 8th grade.

	Number line 0–5 PAE	Magnitude comparison accuracy	Arithmetic accuracy	PSSA math
Number line 0–1 PAE	.61**	−.70*	−.64**	−.63**
Number line 0–5 PAE		−.67**	−.70**	−.86**
Magnitude comparison accuracy			.64**	.62**
Arithmetic accuracy				.79**

* $p < .05$.
** $p < .01$.

3.5.3. Wrong fractions operation strategy

Children fairly often imported procedures that would have been correct for other fractions arithmetic operations but were incorrect for the operation that was requested. Most wrong fractions operation procedures occurred on multiplication and division problems. These procedures involved establishing a common denominator, as if solving an addition or subtraction problem, and then multiplying the numerators (on multiplication problems) or dividing them (on division problems) and leaving the denominator unchanged. For example, when presented 3/5 * 1/2, some children responded "30/10." They generated this answer by first transforming the fractions to 6/10 and 5/10 and then correctly multiplying the numerators but leaving the common denominator unchanged, as in fraction addition and subtraction.

As indicated by this description, participants used wrong fractions operation strategies far more often on multiplication and division problems than on addition and subtraction problems, 25% versus 4%, $F(1, 46) = 25.31$, $p < .001$, partial $\eta^2 = .36$. Participants also tended to use wrong fractions operation strategies more often on problems with the same denominator than on problems with different denominators, 17% versus 13%, $F(1, 46) = 3.08$, $p < .10$, partial $\eta^2 = .06$. The presence of a common denominator might have activated the procedure used in addition and subtraction, in which numerators are operated on in the way indicated by the addition or subtraction sign and denominators are left unchanged.

3.5.4. Variability of strategy use

As with whole number arithmetic, fractions arithmetic was highly variable within as well as between participants. Among 6th graders, 4% of children used a single strategy on all eight problems, 46% used two strategies, and 50% used three or four strategies. Among 8th graders, 33% used a single strategy (in all cases, the correct strategy), 29% used two strategies, and 38% used three or four strategies. This variability was partially attributable to participants knowing how to solve problems for some arithmetic operations but not others, but considerable variability also was present within arithmetic operations. Most 6th graders (54%), and a substantial minority of 8th graders (38%) used different strategies for the two problems on at least one of the four arithmetic operations. In 81% of such cases, the child used a correct and an incorrect strategy; in the other 19% of cases, the child used two different incorrect strategies.

3.6. Violations of principles

As suggested by the frequency of use of inappropriate strategies, children's answers to the fraction arithmetic problems often violated basic principles of arithmetic: that adding two positive numbers must produce a sum greater than either addend; that subtracting a positive number must produce a difference smaller than the number being subtracted from; that multiplying a positive number by a number between 0 and 1 must reduce the size of the other number; and that dividing a positive number by a number between 0 and 1 must produce a quotient greater than the number being divided. Overall, 39% of answers violated these principles: 40% for addition, 42% for subtraction, 39% for multiplication, and 33% for division. A 2 (grade) × 4 (arithmetic operation) ANOVA indicated that 6th graders' answers violated the principles more often than 8th graders' answers, 46% versus 30%, $F(1, 46) = 4.45$, $p < .05$, partial $\eta^2 = .09$.

3.7. Individual differences

Individual differences in performance across tasks were highly consistent. As shown in Tables 5 and 6, 11 of the 12 correlations among the experimental tasks were significant, with the correlations for 6th graders generally being around $r = .50$ and those for 8th graders ranging from $r = .60$ to $r = .70$. These strong relations are consistent with the theoretical assumption that understanding of numerical magnitudes underlies performance on all of these tasks, even on fraction arithmetic, which in principle could reflect rote memorization of correct algorithms and therefore could be unrelated to performance on the tasks used to assess magnitude knowledge.

Perhaps the most striking result obtained in this study was the strength of the relations between number line estimation accuracy (PAE) and math achievement test scores (Tables 5 and 6). The four correlations ranged from $r = -.54$ to $r = -.86$.

To test whether a common relation to fractions arithmetic explained the relation between the measures of fractions magnitude knowledge and mathematics achievement test scores, we conducted hierarchical regression analyses, in which participants' number of correct fractions arithmetic answers was first entered as a predictor of their PSSA scores, and then the participants' PAE scores on 0–1 and 0–5 number line estimation were entered as additional predictors.

Accuracy of number line estimation proved to be an independent predictor of mathematics achievement test scores for both 6th and 8th graders. For the 23 of 24 6th graders for whom test data were available, arithmetic performance accounted for 23% of the variance in PSSA scores,

$F(1, 21) = 6.15$, $p < .05$, adding 0–1 number line estimation PAE added 24% further variance, $F(1, 20) = 8.76$, $p < .01$), and the two variables together accounted for 46% of the variance in achievement test scores. Entering 0–5 number line estimation as a predictor accounted for 4% further independent variance (n.s.).

Entering number line estimation before arithmetic resulted in a very different outcome. The 6th graders' 0–1 number line estimation PAE accounted for 45% of the variance in their achievement test scores, $F(1, 21) = 16.97$, $p < .001$; entering fraction arithmetic accuracy added only 1% to the explained variance (n.s.).

Achievement test data were unavailable for five 8th graders, due to some parents not giving permission for experimental personnel to see their child's achievement test data and some participants not having attended the district the previous year. However, the relations for the 8th graders for whom achievement test data were available were similar to the data for the 6th graders. The analyses again showed that accuracy of number line estimation explained substantial variance in achievement test performance, above and beyond the variance explained by fraction arithmetic performance. The only difference was that for the 8th graders, PSSA performance was better predicted by performance on 0–5 number lines than on 0–1 number lines. When entered first, arithmetic performance accounted for 62% of variance in PSSA scores, $F(1, 17) = 27.22$, $p < .001$. Entering 0–1 number line estimation next accounted for an additional 4% (n.s.), and entering 0–5 number line PAE third accounted for an additional 15% of variance, $F(1, 15) = 11.15$, $p < .005$. The three variables together accounted for 80% of the variance in PSSA scores.

Entering 0–1 number line estimation first accounted for 41% of the variance in the 8th graders' PSSA scores, $F(1, 17) = 11.57$, $p < .005$, entering 0–5 number line estimation next added a further 34% of variance, $F(1, 16) = 21.93$, $p < .001$, and adding arithmetic last added a marginally significant 5% of variance, $F(1, 15) = 4.07$, $p < .10$. Thus, for both 6th and 8th graders, number line estimation added substantially to the variance in overall achievement test performance accounted for by arithmetic performance, but the converse was not the case.

These results argue against the possibility that the relations among arithmetic, magnitude knowledge and overall achievement might be due to shared relations to IQ. The substantial relations between fraction magnitude knowledge and mathematics achievement that remained after arithmetic knowledge was statistically controlled, and the lack of the converse relation when magnitude knowledge was controlled, indicated that a common relation of the three variables to IQ did not explain the relation of magnitude fractions knowledge and mathematics achievement.

4. Discussion

The present integrated theory of numerical development differs from privileged domains, evolutionary, and conceptual change approaches in its emphasis on acquisition of knowledge about numerical magnitudes as a basic process uniting the development of understanding of all real numbers. This theoretical perspective led to a number of accurate predictions that would not have followed from the other theories. In particular, these alternative theories would not have predicted that individual differences in knowledge of fractions magnitudes would correlate highly with success at solving fractions arithmetic problems; that differences in fractions magnitude knowledge would correlate highly with individual differences in mathematics achievement test scores; or that performance across different tasks that measure fractions magnitude knowledge would be highly correlated. Moreover, the alternative theories also would not have predicted that development of fractions would have involved increasingly accurate linear magnitude representations and that it would not involve a transition from logarithmic to linear representations.

The lack of such predictions does not indicate that these alternative theories of numerical development are wrong; however, it does suggest that they are incomplete in an important way. Development in general is marked by continuities as well as discontinuities; by focusing exclusively on discontinuities between whole numbers and fractions knowledge, these alternative theories provide a one-sided depiction of numerical development. Recognizing and documenting commonalities and continuities, as well as differences and discontinuities, makes possible a more balanced theory. It also makes possible a more comprehensive theory of numerical development, including detailed analyses of

acquisition of understanding of all types of real numbers, acquisitions that are made from infancy through adolescence.

The present theoretical approach and empirical data also argue for the feasibility of applying to fractions (and quite likely other types of numbers) powerful and revealing methods and analytic techniques that have greatly increased understanding of whole numbers. Further, it raises numerous questions whose answers are currently unknown. In the remainder of this article, we examine implications of the present research for theories of magnitude representation, for the development of fractions and arithmetic, and for improving mathematics education.

4.1. Implications for understanding numerical magnitude representations

4.1.1. Relations between whole number and fraction magnitude representations

The mental number line has proved to be a useful construct for thinking about whole number magnitude representations and for summarizing and predicting patterns of speed and accuracy, developmental and individual differences, behavioral data and brain imaging data. The present findings indicate that the mental number line is also a useful way of thinking about fractions magnitudes, and that representations of fraction and whole number magnitudes have a number of features in common. Consider five commonalities between the role of magnitude representations with whole numbers and with fractions that are documented in the present study. As with whole numbers:

(1) Alternative measures of fraction magnitude knowledge are highly correlated.
(2) Numerical magnitude comparisons with fractions yield distance effects.
(3) Knowledge of different ranges of fractions develops at different times (earlier for fractions from 0 to 1 than from 0 to 5).
(4) Knowledge of fraction magnitudes varies greatly among individuals and correlates with both arithmetic proficiency and mathematics achievement test scores;
(5) Relations between fraction magnitude representations and mathematics achievement test scores extend beyond their common relation to arithmetic knowledge.

These commonalities attest to the value of viewing whole numbers and fractions development within a single, integrated theory.

Findings from this study also indicate important ways in which development of fractions magnitude knowledge differs from that of whole numbers.

(1) Knowledge of fraction magnitudes is acquired at much older ages. At least through 8th grade, it is far less precise than knowledge of whole number magnitudes.
(2) Unlike whole number magnitude representations, fractions magnitude representations are not accessed automatically;
(3) Explicit, reportable strategies play a much larger role in estimation of fractions magnitudes than in estimation of whole number magnitudes.
(4) Logarithmic representations are uncommon or absent altogether with fractions, at least for fractions where both numerators and denominators vary (unlike the task presented by Opfer and DeVries (2008), who demonstrated that children could be induced to extend their logarithmic representation of whole numbers to fractions on a task where numerators were constant and only denominators varied).

4.1.2. Processes contributing to the development of accurate fraction magnitude representations

How do children come to extend the mental number line representation from whole numbers to fractions? Research on the development of whole number magnitude representations suggests that analogies to better understood numbers might play an essential role. Thompson and Opfer (2010) found that encouraging second graders to draw analogies from their representations of numbers in the 0–100 range to numbers in the 0–1000, 0–10,000, and 0–100,000 ranges led the children to extend their linear representation of the smaller numbers to the larger numerical ranges.

In the present study, the strategies that 6th and 8th graders reported indicate that analogies to whole numbers also are used to generate magnitude representations for fractions. One common, and effective, strategy was to translate the fraction being estimated into a percentage of the distance between the two endpoints and then to use the percentage as if it were a whole number on a 0–100 number line. For example a child might reason that 4/5 was 80% of the distance between 0 and 1 and proceed as if locating 80 on a 0–100 number line. Improvements in number line estimation accuracy between 6th and 8th grades seems partially attributable to the 8th graders, but not the 6th graders, having been taught about percentages.

This analysis of the likely role of analogy in learning about fraction magnitudes places previous claims about the "whole number bias" (Ni & Zhou, 2005) in a different light. The basic idea of the whole number bias is that analogies to whole numbers interfere with learning about fractions. This bias has been cited to explain a variety of difficulties in fractions learning, including difficulty understanding that fractions do not have unique successors (Vamakoussi & Vosniadou, 2010); faulty fraction arithmetic rules, such as adding numerator to numerator and denominator to denominator to add fractions (Gelman, 1991; Mack, 1995); inaccurate magnitude comparison rules, for example judging that when fractions have equal numerators, the fraction with the larger denominator is greater (Stafylidou & Vosniadou, 2004); and inaccurate rules for comparing decimal fractions, for example the rule that the more digits, the larger the number (Resnick & Omanson, 1987).

Another way of viewing the same data, however, is that difficulty learning about fractions stems from drawing _inaccurate_ analogies to whole numbers, rather than from drawing analogies between whole numbers and fractions per se. Encouraging children who are learning about fractions to draw correct analogies to whole numbers, for example that fractions, like whole numbers, can express a proportion of another number (3/5: 1:: 60: 100:: 60% of 100) or that fractions, like whole numbers, can provide absolute measures of quantity (6 in. = ½ foot = 1/6 yard) might improve understanding of fractions. Fractions are far from the only domain that poses the challenge of resisting superficially appealing analogies in favor of less obvious but deeper ones (Gentner & Markman, 1997; Hummel & Holyoak, 2003). Drawing the explicit analogy that fractions are like whole numbers in having magnitudes that can be ordered and represented on number lines might also be helpful.

Another likely contributor to acquisition of fraction magnitude representations is frequency of exposure to, and knowledge about, specific fractions. The two most common fractions that were presented on the number line estimation task – 1/2 and 1/4 – elicited the most accurate estimates of any of the 20 fractions that were presented. Children might learn about the magnitude of ½ through experiences sharing and dividing candy bars, cookies, drinks, and other desired goods. They also might learn that a quarter of a dollar is the same as ¼ of a dollar, and use this knowledge to further their understanding of 1/4. Learning the magnitudes of a relatively small set of commonly encountered fractions – ½, 1/3, 2/3, ¼, and ¾ – might provide a basis for representing fraction magnitudes more generally, at least in the 0–1 range.

A third likely contributor to the acquisition process is knowledge of rules, such as that fraction magnitudes increase with increases in numerators and decrease with increases in denominators. Such rules might be involved in extending knowledge of the magnitudes of commonly encountered fractions to fractions in general (1/2 > 1/3 > ¼ > 1/12 > 1/37...).

The present integrated theory of numerical development also raises a number of questions about the development of numerical magnitudes that the present research did not address:

(1) How are representations of whole number and fraction magnitudes related at a given point in time?
(2) Do accurate whole number magnitude representations at younger ages predict accurate fraction magnitude representations later?
(3) Does earlier fraction magnitude knowledge predict later understanding of algebra?
(4) How is knowledge of whole number division, an operation that is logically related to fractions, empirically related to fraction magnitude representations?
(5) How is knowledge of other whole number arithmetic operations and whole number magnitude representations related to knowledge of fraction arithmetic and magnitude representations?

(6) How are fraction magnitude representations related to understanding of fractions principles, such as infinite divisibility; ability to translate among decimals, percentages, and fractions; and other aspects of fractions knowledge?

(7) Do magnitude representations play a similar role in acquisition of knowledge of negative numbers as with positive whole numbers and fractions?

(8) Are the brain areas that are activated by fraction magnitude processing the same as those activated by whole number magnitude processing?

As illustrated by these questions, pursuing an integrated theory promises to broaden and deepen our understanding of numerical development.

4.2. Implications for understanding arithmetic

The present findings indicate that understanding of fraction magnitudes and fractions arithmetic are closely related. If learning fraction arithmetic algorithms reflected rote memorization, as has often been claimed (e.g., Cramer & Bezuk, 1991; Hiebert, 1986; Kerslake, 1986), there would be no reason to expect such a relation. However, the strong correlations between fractions arithmetic and all three measures of magnitude knowledge in both 6th and 8th grades indicate that conceptual and procedural knowledge of fractions are intertwined. (See Hecht (1998), Hecht, Close, and Santisi (2003), Hecht and Vagi (2010), and Schneider and Stern (2010) for similar findings.)

One plausible interpretation of these results is that magnitude knowledge makes it easier to learn and remember fraction arithmetic algorithms. This might occur through children with good magnitude knowledge of fractions rejecting procedures that produce unreasonable answers, such as operating independently on numerator and denominator often does, and searching longer for procedures that produce reasonable answers. For example, children might reject the procedure that produces arithmetic errors of the form 3/5 – 1/2 = 2/3 if they recognized that subtracting a positive number cannot lead to an answer larger than the number being subtracted from. This could lead them to try other procedures and test whether they yielded plausible answers.

Alternatively, some children might better remember the arithmetic procedures that they are taught, and those children's correct execution of the algorithms might enable them to learn about fraction magnitudes through observing the answers to fraction arithmetic problems. Rittle-Johnson et al.'s (2001) finding of initial knowledge of arithmetic procedures predicting gains in conceptual understanding, including magnitudes, is consistent with this view. The present data do not allow discrimination between these paths, but the data do indicate a strong relation between knowledge of fraction magnitudes and fraction arithmetic skill that is in need of explanation.

Another implication for understanding fraction arithmetic is methodological: The same strategy assessment techniques that have proved useful with whole number arithmetic also are useful for investigating fractions arithmetic. As with whole number arithmetic, individual children used a variety of fractions arithmetic strategies. Even on a single fractions arithmetic operation, strategy use varied with problem characteristics, notably with the equality or inequality of denominators. The quality of fraction arithmetic strategy use was related to both knowledge of numerical magnitudes and to overall mathematics achievement test scores. In whole number arithmetic, these strategy assessment techniques have provided a base for computer simulation models of arithmetic learning that accounted for numerous findings regarding variations in accuracy, solution times, and strategy use across problems; discovery of useful new strategies; individual differences in arithmetic proficiency; and changes in speed, accuracy, and strategy use with problem-solving experience (Shrager & Siegler, 1998; Siegler & Shipley, 1995). The prominence of strategy use in fractions arithmetic suggests that similar models might be applicable to that area, and that it might be possible to formulate a common model of development of whole number and fractions arithmetic.

A third implication for understanding fractions arithmetic is that fraction arithmetic errors often reflect confusion about the right strategy, together with a lack of constraints on the magnitudes of answers, rather than a consistent whole number bias or other systematic misunderstanding. Previous descriptions of children's poor understanding of fractions arithmetic usually have attributed the

children's difficulty to systematic misconceptions, in particular the whole number bias. This bias has been said to lead to children treating numerators and denominators as independent whole numbers and operating on them independently, for example by subtracting numerator from numerator and denominator from denominator (Carpenter et al., 1981; Gelman, 1991; Kilpatrick, Swafford, & Findell, 2001). However, the present findings revealed greater variability in fraction procedures than implied by this attribution of errors to a systematic misconception. Roughly half of arithmetic errors stemmed from applying whole number algorithms independently to numerators and denominators, but a similar percentage reflected using parts of algorithms that would have been correct for a different fraction arithmetic operation or trying other erroneous procedures. The inconsistency of strategies even within a single arithmetic operation was striking; 40% of children correctly solved one of the pair of problems for a single arithmetic operation and erred on the other.

This variability suggests that the whole number bias is only part of the problem in understanding fractions arithmetic. Rather than reflecting a systematic misconception, fractions arithmetic knowledge seems piecemeal; understanding of whole numbers is one source of ideas about how to solve fractions arithmetic problems, but other types of numerical knowledge are also incorporated. It also is unclear whether children who use erroneous fractions arithmetic procedures believe that those procedures are correct. They might well be skeptical about their correctness but have learned that saying "I don't know" is not an acceptable alternative in school mathematics. This issue could be addressed through studies that examine children's confidence in their fraction arithmetic answers.

4.3. Instructional implications

In the United States, instruction in fractions emphasizes part-whole interpretations far more than other interpretations of fractions (Ni & Zhou, 2005; Sophian, 2007; Thompson & Saldanha, 2003). For example, students are taught to interpret 1/5 as one of five slices of pizza, but less often to think of 1/5 as one fifth of the distance from zero to one on a number line (Moseley, Okamoto, & Ishida, 2007). This is quite different than the approach to teaching fractions in Japan, China, and other countries where students understand fractions better. Indeed, many teachers in the US can only explain fractions in terms of the part-whole interpretation, unlike teachers in China and Japan who also emphasize number line and other interpretations (Ma, 1999; Moseley et al., 2007).

Part-whole interpretations have the advantages of concreteness and accessibility. When numerators and denominators are small and positive and the numerator is less than the denominator, it is easy to think about N parts of a whole that includes M parts. For example, children have little difficulty understanding that if a pizza is cut into four pieces, then each piece is 1/4 of the pizza (Mix, Levine, & Huttenlocher, 1999). However, the part-whole interpretation of fractions also has some serious limitations. Negative fractions cannot be represented in this way, it is very difficult to imagine fractions with large numerators and denominators (e.g., 734/878), and improper fractions can be confusing within the part-whole interpretation, as illustrated by one learner's reaction to being presented 4/3, "You cannot have four parts of an object that is divided into three parts" (Mack, 1993). Moreover, there is nothing in the operation of dividing an object into N parts that says that the size of the parts must be equal; many students fail to understand that the parts must be equal for fractions to have any consistent meaning (Sophian, 2007).

The data from the present study, along with data from previous studies showing beneficial effects on whole number knowledge of instruction that emphasizes numerical magnitudes (e.g., Siegler & Ramani, 2009), indicate that emphasizing that fractions are measurements of quantity might improve learning about fractions. Indeed, a common feature of instructional studies that have yielded especially promising results in teaching rational numbers, such as work by Robbie Case and his associates, is that they emphasize that fractions are measures of quantity (e.g., Cramer et al., 2002; Fujimura, 2001; Keijzer & Terwel, 2003; Moss & Case, 1999; Rittle-Johnson & Koedinger, 2002, 2009). The present integrated theory of numerical development helps to explain the prevalence of this common feature of successful instruction: If magnitudes are central to understanding fractions as well as whole numbers, then instruction that emphasizes magnitude understanding is more likely to succeed than instruction that does not emphasize magnitude understanding.

Acknowledgments

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