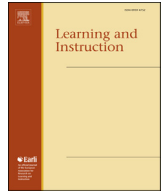




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Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents

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ABSTRACT

Numerical understanding and arithmetic skills are easier to acquire for whole numbers than fractions. The *integrated theory of numerical development* posits that, in addition to these differences, whole numbers and fractions also have important commonalities. In both, students need to learn how to interpret number symbols in terms of the magnitudes to which they refer, and this magnitude understanding is central to general mathematical competence. We investigated relations among fraction magnitude understanding, arithmetic and general mathematical abilities in countries differing in educational practices: U.S., China and Belgium. Despite country-specific differences in absolute level of fraction knowledge, 6th and 8th graders' fraction magnitude understanding was positively related to their general mathematical achievement in all countries, and this relation remained significant after controlling for fraction arithmetic knowledge in almost all combinations of country and age group. These findings suggest that instructional interventions should target learners' interpretation of fractions as magnitudes, e.g., by practicing translating fractions into positions on number lines.

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1. Introduction

Understanding fractions is crucial for mathematics learning: It not only requires a deeper understanding of numbers than is ordinarily gained through experience with whole numbers, it is also predictive for students' mathematical achievement years later (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Siegler et al., 2012). Despite increasing research interest in the domain of fractions, almost all studies of the role of fraction magnitude understanding in mathematics learning have been conducted in the U.S., limiting the generality of the findings to only U.S. students and adults. This study aims to deepen our understanding of the pivotal role of fraction magnitude understanding for students' general math achievement in three countries on three different continents that differ greatly in cultural and educational practices.

1.1. The integrated theory of numerical development

Our starting point was Siegler, Thompson, and Schneider's (2011) *integrated theory of numerical development*.¹ As discussed there, current theories of numerical development fail to integrate whole numbers and fractions within a single framework (e.g., Geary, 2006; Leslie, Gelman, & Gallistel, 2008; Wynn, 2002). Although these theories differ in many particulars, they all posit a gap between an early developing, "natural" understanding of whole numbers and a later developing, flawed, limited, or hard-won understanding of fractions. To the extent that relations between whole numbers and fractions are posited, the earlier

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¹ It should be noted that knowledge acquisition, instruction, and cognitive development are closely intertwined both in real life and in the integrated theory of numerical development. Therefore, the word *development* in the latter theory's name should be understood in its broadest meaning, i.e., as integrating – rather than excluding – knowledge acquisition and instruction as important influences on people's numerical development. However, for readability reasons, the authors decided not to include all three sources of competence growth in the theory's name and refer to the, in the learning sciences, well-known and frequently used term *development*.

developing understanding of whole numbers is said to interfere with the later developing understanding of fractions. For instance, according to conceptual change theories (DeWolf & Vosniadou, in this issue; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2010), children form an initial theory of number as counting units before they encounter fractions, and draw heavily on this initial understanding of number to make sense of fractions. Children's faulty generalization of understanding of number as counting units interferes with their learning about fractions, a phenomenon often referred to as the "whole number bias" (Ni & Zhou, 2005).

Siegler, Thompson, and Schneider's (2011) *integrated theory of numerical development* recognizes these important differences between learning of whole numbers and fractions, but also emphasizes a crucial continuity that unites their acquisition – steadily expanding understanding of the connection between numbers and their magnitudes. Within this perspective, development of understanding of rational numbers involves both a gradual expansion of the range of whole numbers whose magnitudes are understood (from smaller to larger) and a conceptual change from an initial understanding of numbers in terms of characteristic features of whole numbers to a later understanding of rational numbers in terms of a single defining feature, their magnitudes (see Wu, 2001, 2009, for a similar argument).

The integrated theory differs from conceptual change theories in two main ways. One is in its recognizing the *positive* role of whole number magnitude knowledge in learning fractions, as indicated by longitudinal relations between first graders' knowledge of whole number magnitudes and 7th and 8th graders' knowledge of fraction magnitudes and fraction arithmetic, even after statistically controlling for the IQ, working memory, socio-economic status, race, and other relevant variables (Bailey, Siegler, & Geary, 2014). The second main difference between the integrated theory and conceptual change theories of fraction knowledge is that the integrated theory views interference from whole number knowledge as only one of several sources of difficulty in learning fractions. Evidence for this view comes from findings that despite whole number errors, such as $1/2 + 2/3 = 3/5$ being common, confusion with other fraction operations, such as the confusion between fraction addition and fraction multiplication evident in $1/3 * 2/3 = 2/3$, can be even more common (Siegler & Pyke, 2013).

Within this integrated theory, the reason why fractions are more difficult to learn than whole numbers is the same reason why fractions are crucial to numerical development. A fraction is a ratio or division of two whole numbers, numerator and denominator, and is thus considerably more complex than a single whole number. Whole numbers have unique predecessors and successors, but this is not true of fractions. Multiplying a whole number always leads to a larger number and dividing a whole number always leads to a smaller number, but again this is not true of fractions. Thus, generating a mature understanding of rational numbers requires understanding both the one property that all rational numbers share – that they have magnitudes that can be located and ordered on number lines – and understanding that other properties that unite whole numbers do not unite rational numbers.

Consistent with this theory, Siegler et al. (2011) found strong relations between U.S. 6th and 8th graders' fraction magnitude understanding and their general mathematics achievement, even when their mutual relation to fraction arithmetic was statistically controlled. However, these and other data on this topic were collected almost exclusively in the U.S. It thus remains an open question whether the findings are due to the proposed general cognitive learning mechanisms of the theory or to specific properties of the U.S. cultural and educational system (e.g., cultural

beliefs about mathematics, teacher training, time spent on mathematics, mathematics curricula).

1.2. Previous studies on fraction understanding

Although research interest in students' acquisition of fraction knowledge and skill has increased in recent years, such studies are still far less numerous than studies of whole number understanding. However, the limited number of studies of fractions and the much larger number of studies of whole numbers have revealed highly similar relations among magnitude understanding, arithmetic and general mathematics achievement (Siegler, Fazio, Bailey, & Zhou, 2013).

The same types of behavioral methods have proved useful for investigating fraction as whole number magnitudes: magnitude comparison tasks, in which participants compare the magnitudes of two whole numbers or fractions and indicate the larger one, and number line estimation tasks, in which participants indicate the position of a given whole number or fraction on an empty number line with clearly indicated start and end point. Studies using these methods have consistently revealed that, as with whole number magnitude representations, the precision of fraction magnitude representations differs greatly between and within individuals, depending on students' (instructional) experiences with fractions and the size of the fractions (Siegler & Pyke, 2013; Siegler et al., 2011). Also as with whole numbers, fraction magnitude understanding has proved to be quite strongly correlated with other aspects of mathematics learning. On top of this correlational evidence, recent investigations provide evidence for predictive relations between earlier fraction magnitude understanding and subsequent knowledge of fraction arithmetic, algebra and overall math achievement (Bailey et al., 2012; Booth & Newton, 2012; Siegler et al., 2012). To cite one example, Siegler et al. (2012) demonstrated that 5th graders' fraction knowledge predicts their mastery of algebra and overall mathematics achievement in high school, 5 or 6 years later, even after controlling for IQ, reading achievement, working memory, family income and education, and whole number knowledge. The same relations were found in both U.K. and U.S. longitudinal samples. Moreover, Fuchs et al. (2013) demonstrated that instruction focused on fraction magnitude understanding improved not only understanding of fraction magnitudes but also fraction arithmetic proficiency among children with mathematics learning difficulties. Taken together, these results indicate that magnitude representations are as central to knowledge of fractions as to knowledge of whole numbers.

However, to the best of our knowledge, all previous behavioral studies of the role of fraction magnitude understanding in mathematics learning have been conducted in the U.S. – with the one exception of Siegler et al. (2012), which included both U.S. and U.K. samples. This raises questions about the generality of the findings and (consequently) the applicability of the integrated theory of numerical development to populations in other countries and continents. Differences in instructional methods, curricular devices, teacher expertise, and students' absolute levels of achievement might all limit the generality of the findings that have been viewed as supporting the integrated theory. Therefore, in the present study, we investigated students' fraction understanding in three countries with quite different instructional methods and teaching practices: Belgium (Flanders), China, and the U.S.

1.3. Differences in teacher knowledge and instructional practices in mathematics

International investigations of (prospective) teachers' knowledge, instructional practices and student performances in the

domain of mathematics not only document country-specific differences in (prospective) teachers' knowledge and instructional practices, but also stress the importance of these variables for students' mathematical development. Systematic studies in Europe and North America point to deficits in (prospective) teachers' content and pedagogical content knowledge of mathematics in general and rational numbers in particular (e.g., Ball, 1990; Hill, Rowan, & Ball, 2005; Krauss et al., 2008; Merenuoto & Lehtinen, 2002; Senk et al., 2012). For instance, most U.S. elementary school teachers participating in the study of Ball (1990) could correctly solve a fraction division task, but they had serious conceptual difficulties in understanding the meaning of the algorithm for division of fractions and in generating an appropriate representation for a division of fraction task. Likewise, the seminal interview study of Ma (1999) on Chinese and U.S. teachers' mathematical knowledge revealed large differences in mathematical content and pedagogical knowledge of teachers in the two countries, including their understanding and teaching of fractions. Recent studies on Belgian (Flanders) prospective teachers' content and pedagogical content knowledge of rational numbers and other areas of mathematics also demonstrated weaknesses in these students' knowledge of mathematical content and pedagogy (Depaepel et al., 2013; Verschaffel, Janssens, & Janssen, 2005). Since teachers' mathematical knowledge is crucial for both instructional quality and student achievement (Hattie, 2009), country-specific differences in the quality of the mathematics lessons and in students' math performances can be expected.

Findings from the most recent TIMSS (Trends in International Mathematics and Science Study; Mullis, Martin, & Foy, 2008; Mullis, Martin, Foy, & Arora, 2012) also revealed a number of cultural differences relevant to mathematics teaching and learning. The participating countries, including Belgium (Flanders), China and the U.S., differed not only in teacher training, experience, and career satisfaction, but also in the amount and quality of mathematics instruction at school, and in students' mathematical experiences outside of school. Moreover, these cultural and instructional differences were strongly related to 4th and 8th graders' performances in the domain of mathematics. Although the TIMSS did not specifically focus on fraction understanding, these cultural and educational differences seem likely to influence learning of fractions as well as other areas of mathematics. Many of the same national differences have been documented in smaller, experimental studies on mathematics knowledge in China and the U.S. and in other international comparisons (e.g., Stevenson & Stigler, 1992).

Thus, international comparisons of (prospective) teachers' mathematics knowledge and instructional approaches demonstrate clear differences among countries, including Belgium, China, and the U.S. These educational and instructional differences appear to impact students' mathematical development. Cross-cultural performance differences in the domain of mathematics in general were observed in, for instance, the most recent TIMSS (Mullis et al., 2008, 2012): Of the 29 East Asian, European, and North American countries that participated in the 2011 Grade 4 TIMSS, Chinese students ranked 4th, Belgian students 7th, and U.S. students 11th. Of the 18 North American, East Asian, and European countries in the Grade 8 TIMSS, Chinese students ranked 3rd and U.S. students 9th (no information was available for Belgian students in grade 8). Such performance differences in the domain of mathematics in general and the domain of fractions in particular have been observed in other international comparisons as well (e.g., Wang & Lin, 2009).

1.4. The present study

To test the generality of the major assumption of the integrated theory of numerical development, namely that fraction magnitude

understanding is crucial to overall mathematics learning, we selected three countries that are known to differ in teachers' knowledge, instructional practices and students' performances in the domain of mathematics, namely Belgium (Flanders), China and the U.S. As documented above, previous work on (prospective) teachers' mathematical knowledge indicated large differences in the knowledge required to effectively teach mathematics between Chinese and U.S. teachers, favoring the former. The mathematical knowledge of Belgian (prospective) teachers appears to be rather limited, closer to that of the U.S. teachers than to that of the Chinese teachers. On the other hand, Belgian and Chinese teachers are similar to each other and different than U.S. teachers in emphasizing magnitude interpretations and using number lines often during fraction instruction.

Based on the results of international comparisons of student achievement and teacher knowledge of mathematics and approaches to teaching mathematics, we hypothesized that Chinese students would perform better than U.S. students on tasks assessing fraction magnitude understanding and fraction arithmetic (=Hypothesis 1a). We further expected Belgian students to perform less well on these tasks than Chinese students (=Hypothesis 1b); but better than U.S. students (=Hypothesis 1c).

Of greater interest, despite the expected performance differences on the fraction tasks among the participating countries, we did *not* expect differences among the three countries in the relation between students' fraction magnitude understanding and their general mathematics achievement. Indeed, the integrated theory of numerical development implies that similar relations should be present regardless of the particular educational and cultural system, due to the inherent relations among different types of numbers and the inherent centrality of magnitude understanding to subsequent mathematics learning. We thus formulated our second hypothesis as follows: Not only in the U.S. sample, but also in the Belgian and the Chinese samples, students' fraction magnitude understanding should be strongly related to their general mathematical achievement (=Hypothesis 2).

From many perspectives, our second hypothesis is counterintuitive. Mathematics teachers in China have much deeper fraction knowledge than the ones in the U.S. (and presumably than the ones in Belgium). Hence, they might be able to compensate for differences in the students' magnitude understanding by adapting their instruction to the students' prior knowledge. This would lead to increases in fraction arithmetic skills and math achievement that are largely independent of students' magnitude understanding. Moreover, country-specific differences in the relation between fraction magnitude understanding and general mathematics achievement might be expected on the basis of differences in fraction instruction. Fraction instruction in the U.S. is almost exclusively based on the part-whole interpretation, whereas students in China and Belgium also encounter substantial emphasis on the measurement interpretation of fractions. From our analyses of the textbooks, it can be concluded that U.S. students are overwhelmingly confronted with the representation and interpretation of fractions as parts of a whole, e.g., as 1 slice of a pizza cut into 4 equal slices. By contrast, mathematics textbooks used in Chinese and Belgian classes do not only emphasize part-whole representations and interpretations of fractions, but offer the students substantial experience with measurement representations and interpretations of fractions from the start of fraction instruction, e.g., by including (empty) number lines on which different fractions need to be positioned, ordered and compared in terms of their magnitude. The relation in the U.S. between knowledge of magnitudes, which is at the heart of the measurement interpretation, and mathematics achievement therefore might be present because only students who are particularly good at mathematics develop

Table 1
Number, gender and age of participants per country and grade.

Country	Grade	n	Gender ^a		Age (in years)	
			m	f	M	SD
Belgium	6	30	13	17	11.2	.5
	8	34	11	22	13.3	.5
China	6	30	22	8	12.3	.6
	8	39	19	20	14.3	.7
U.S.	6	31	19	12	11.4	.6
	8	23	11	12	13.2	.4

^a Gender information is missing for 1 Belgian 8th grader.

magnitude knowledge when it is not emphasized in instruction. By contrast, Chinese and Belgian students might not show such a relation, because children in those countries are taught the measurement interpretation, thus reducing the need to induce it.

In the present study, we tested these hypotheses in samples of 6th and 8th grade students from Belgium (Flanders), China and the U.S. Despite differences in culture and educational approaches among the three participating countries, the timing of fraction instruction is highly similar. In all three countries, students are introduced to fractions (beyond the simplest ones, such as $1/2$) in 2nd or 3rd grade; receive increasingly intensive instruction in fractions, including fraction arithmetic, in 4th through 6th grade; and receive further practice with fractions, embedded in pre-algebra and algebra problems, in later grades. Given the similar timing of fraction instruction in the participating countries, we decided to keep grade rather than chronological age constant across the countries.²

2. Method

2.1. Participants

A sample of 187 students from Belgium (Flanders), China and the U.S. participated to the study. Only students with parental consent were included in the study. Table 1 describes the number of students (boys, girls) and their age (expressed in years) per grade and country.

As shown in Table 1, the Belgian sample consisted of 30 6th graders and 34 8th graders; all were students at a predominantly middle-income school in Flanders, Belgium. The Chinese sample involved 30 6th graders and 39 8th graders, coming from a school in Shijiazhuang City, North China's Hebei Province. The U.S. sample included 31 6th graders (87% Caucasian, 7% Indian, 7% Multiracial) and 23 8th graders (91% Caucasian, 4% African-American, 4% Middle Eastern), recruited from two predominantly middle-income public school districts near Pittsburgh, PA. We observed no differences in the number of students and in the number of boys and girls in the different samples, all $ps > .05$, but Chinese students were approximately 1 year older than Belgian and U.S. students, $F(2, 186) = 24.68, p < .01$.

2.2. Materials

2.2.1. Fraction number line estimation

We used two versions of the number line estimation task that differed only in the numerical range of the line and the presented

numbers. The range of the number line was either zero-to-one or zero-to-five. As in past studies, the number line was 12 inches (about 31 cm) long. It had labels and hatch marks only at the start point (i.e., 0) and the end point (i.e., 1 or 5). On each trial, a different fraction was presented above the middle of the number line. Children were asked to click the mouse on the correct location of the value of the fraction on the number line.

We measured accuracy by computing percentage absolute error (PAE) for each child, computed as $100 \cdot \text{abs}(\text{estimated value} - \text{correct value}) / \text{numerical range of the number line}$. For example, if a student was asked to locate $5/2$ on a 0–5 number line and marked the location corresponding to $3/2$, the PAE for that trial would be $100 \cdot \text{abs}(1.5 - 2.5) / 5$ or 20%. (Note that PAE varies inversely with accuracy; the higher the PAE, the less accurate the estimate). In the 0–1 range, we presented the 29 fractions: $0/1, 1/9, 1/8, 1/7, 1/6, 1/5, 2/9, 1/4, 2/7, 1/3, 3/8, 2/5, 3/7, 4/9, 1/2, 5/9, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 7/9, 4/5, 5/6, 6/7, 7/8, 8/9$, and $1/1$. In the 0–5 range, we presented the 15 fractions: $2/9, 1/2, 4/7, 5/3, 17/9, 2/1, 9/4, 19/8, 8/3, 13/4, 7/2, 23/6, 4/1, 22/5$, and $9/2$.

2.2.2. Fraction magnitude comparison

We presented 12 numerical magnitude comparison trials. On each trial, children saw two fractions next to each other in the middle of an otherwise empty screen. The fraction on the right side was always $3/5$. Children had to decide whether the other fraction had a higher or a lower value. They were instructed to enter their answer by clicking at the smaller of the two numbers. We used the fractions $1/9, 1/4, 2/7, 1/3, 5/9, 4/7, 2/3, 4/6, 7/6, 4/3, 7/4$ and $2/1$, and scored the percentage of correctly solved trials.

2.2.3. Fraction arithmetic

We assessed fraction arithmetic competence by four addition and four division trials. The addition problems were $3/6 + 1/6, 5/2 + 5/3, 4/7 + 3/8, 10/3 + 5/11$. The division problems had the same numbers and differed only in the operator. Children were allowed to do calculations on a sheet of paper. They were told that they did not have to reduce their answer to its lowest form, but could do so if they wanted. We scored the percentage of correctly solved trials.

2.2.4. General math achievement

We assessed the general math achievement competencies of the participating students using country-specific standardized mathematical achievement tests. For the Belgian sample, we applied two grade-specific Flemish standardized mathematical achievement tests (for 6th graders, LVS Wiskunde Midden 6; Deloof, 2005; for 8th graders, Peiling Wiskunde Eerste Graad A-stroom secundair onderwijs; Gielen et al., 2010); the participating Flemish 6th and 8th graders received above-average scores on these tests, with group mean scores of $M = 38, SD = 12$ (maximum score = 59) and $M = 30.50, SD = 3.5$ (maximum score = 39), respectively. We assessed Chinese students' mathematics proficiency on the basis of their mid-term mathematics examination score. Average mathematics proficiency scores were $M = 90.08 (SD = 6.40)$ for the 6th graders and $M = 86.95 (SD = 10.91)$ for the 8th graders (maximum score = 100). The mathematics proficiency of the U.S. students was administered on the basis of the school-reported PSSA (Pennsylvania System of School Assessment), the standardized test used in the part of the U.S. where the study was performed. Most students, i.e., 71% of the 6th and 61% of the 8th graders, scored at or above a proficient level on this state test of mathematics achievement (for 6th graders, $M = 1440, SD = 179$; for 8th graders, $M = 1459, SD = 258$).

² We selected three countries that do not differ in the timing of fraction instruction. However, as suggested by one reviewer, it would be very interesting to explore the effectiveness of different timing of fraction and pre-algebra instruction on students' further math development in future investigations.

2.3. Design and procedure

In each country, children were tested individually for about 30 min in a quiet room in their school. The procedure followed was identical in the three participating countries. All measures were presented on a laptop computer with a screen resolution of 1024 × 768 px. The assessment program saved answers in log files, which we used for our analyses. The order of the tasks and the order of the items in each task were randomized for each student. Participants completed all items on one task before being presented any items on the next.

3. Results

Results are presented in two parts. We first discuss the results relevant to the expected performance differences among Belgian, Chinese and U.S. students (cf. Hypotheses 1a, 1b and 1c). Next, we present the results relevant to the major goal of this study, i.e., we test the basic assumption of the integrated theory of numerical development that fraction magnitude understanding is closely related to students' general mathematics achievement in all three countries (cf. Hypothesis 2).

3.1. Differences among countries on fraction tasks

3.1.1. Fraction number line estimation

A 3 (Country: Belgium, China or U.S.) × 2 (Grade: 6th or 8th) ANOVA revealed grade- and country-specific differences in students' mean PAE on the 0–1 number line estimation task. As expected, 8th graders more accurately estimated fraction magnitudes than 6th graders, PAE 's = 7% and 12%, $F(1, 186) = 11.79$, $p < .01$, $partial\ eta^2 = .06$. Also as expected, Belgian, Chinese, and U.S. students' estimation accuracy differed, $F(2, 186) = 22.67$, $p < .01$, $partial\ eta^2 = .20$. Chinese students were much more accurate than U.S. students, mean $PAE = 6\%$ and 16% . Belgian students' accuracy (mean $PAE = 7\%$) was as high as that of Chinese students and much higher than that of U.S. students. The non-significant Country × Grade interaction, $F(2, 186) < 1$, revealed that these country-specific differences in accuracy of estimates were present in both 6th and 8th grade (Table 2).

We observed similar results on 0–5 number lines. Sixth graders provided less accurate estimates than 8th graders, mean $PAE = 20\%$ and 11% , $F(1, 186) = 44.41$, $p < .01$, $partial\ eta^2 = .20$. On this more difficult number line task, Chinese students estimated more accurately than Belgian students, who, in turn, estimated more accurately than U.S. students, $PAE = 9\%$, 16% and 23% , $F(2, 186) = 34.46$, $p < .01$, $partial\ eta^2 = .28$. Again, the Country × Grade interaction was not significant, $F(2, 186) < 1$.

3.1.2. Fraction magnitude comparison

A 3 (Country: Belgium, China or U.S.) × 2 (Grade: 6th or 8th) ANOVA indicated that Chinese and Belgian students compared fraction magnitudes more accurately than U.S. students, M 's = 82%, 82%, and 71%, $F(2, 186) = 3.58$, $p = .03$, $partial\ eta^2 = .04$; and that

8th graders compared them more accurately than 6th graders, M 's = 85% and 72%, $F(1, 186) = 11.59$, $p < .01$, $partial\ eta^2 = .06$ (Table 2). The Country × Grade interaction was not significant, $F(2, 186) < 1$.

3.1.3. Fraction arithmetic

A parallel Country × Grade ANOVA on fraction addition revealed main effects of Country, $F(2, 186) = 86.74$, $p < .01$, $partial\ eta^2 = .49$, and Grade, $F(1, 186) = 13.95$, $p < .01$, $partial\ eta^2 = .07$, and also a Country × Grade interaction, $F(2, 186) = 7.07$, $p < .01$, $partial\ eta^2 = .07$. The interaction reflected fraction addition in Belgium and China being similarly (highly) accurate in 6th and 8th grade, but performance in the U.S. starting out far lower and improving considerably between 6th and 8th grade (Table 2).

For the fraction division task, a parallel Country × Grade ANOVA showed main effects of Country, $F(2, 186) = 82.01$, $p < .01$, $partial\ eta^2 = .48$, and Grade, $F(1, 186) = 13.95$, $p < .01$, $partial\ eta^2 = .07$. Chinese students divided fractions more accurately than Belgian students, M 's = 91% and 71%; Belgian students divided more accurately than U.S. students, $M = 22\%$ correct; and 8th graders divided more accurately than 6th graders, M 's = 75% and 53%. The Country × Grade interaction just failed to reach significance, $F(2, 186) = 2.79$, $p = .06$, $partial\ eta^2 = .03$. Division accuracy increased between 6th and 8th grade for both Belgian and U.S. samples, whereas Chinese 6th and 8th graders were similarly accurate.

3.1.4. Conclusion

As expected, we observed clear differences among Belgian, Chinese and U.S. students in absolute levels of fraction knowledge. Consistent with Hypothesis 1a, Chinese students obtained higher scores on all fraction tasks than U.S. students; these results were observed in both 6th and 8th grade. Moreover, as indicated in Hypothesis 1c, Belgian students were consistently more accurate than their U.S. peers. The expected performance differences between Belgian and Chinese students, however, were only observed on the most difficult fraction tasks. Our findings revealed performance differences between the latter two groups of students on 0–5 number lines and fraction division, but contrary to Hypothesis 1b, similar differences were not present on number line and magnitude comparison with fractions from 0 to 1 or on fraction addition.

3.2. Relations among magnitude knowledge, arithmetic, and math achievement

To test our major hypothesis – namely, that not only in the U.S. sample, but also in the Belgian and Chinese samples, fraction magnitude understanding is strongly related to general mathematics achievement (Hypothesis 2) – we conducted correlational and hierarchical regression analyses. To increase the reliability of our measures and the statistical power of our tests, we averaged PAE 's on the 0–1 and 0–5 number lines into a composite PAE , and we combined the addition and division percentages correct into a single arithmetic composite.

Table 2

Performances on fraction tasks per country and grade (SD).

Country	Grade	Number line 0-1 PAE	Number line 0-5 PAE	Comparison accuracy	Addition accuracy	Division accuracy
Belgium	6	9.38 (8.01)	20.08 (11.28)	76.67 (21.04)	79.17 (24.64)	56.67 (42.00)
	8	5.05 (4.34)	12.20 (6.37)	86.52 (21.02)	86.76 (20.63)	84.56 (26.12)
China	6	9.82 (8.05)	13.73 (7.68)	74.17 (33.93)	90.83 (15.37)	89.17 (26.00)
	8	3.88 (2.05)	5.97 (2.97)	88.46 (19.37)	91.03 (14.61)	92.95 (20.64)
U.S.	6	17.38 (12.88)	26.23 (9.99)	65.86 (22.81)	17.74 (30.41)	14.52 (21.19)
	8	14.93 (11.79)	17.97 (8.85)	76.81 (17.58)	51.09 (40.93)	31.52 (38.60)

Table 3
Correlations between fraction tasks and general math achievement score per country and grade.

Country	Grade	Task	Comparison accuracy	Arithmetic accuracy	Math achievement
Belgium	6	Number line PAE	-.08	-.44**	-.65**
		Comparison accuracy		.17	.28
		Arithmetic accuracy			.25
	8	Number line PAE	-.46**	-.32*	-.55**
		Comparison accuracy		.23	.09
		Arithmetic accuracy			.35*
China	6	Number line PAE	-.00	-.35*	-.76**
		Comparison accuracy		-.18	-.01
		Arithmetic accuracy			.15
	8	Number line PAE	-.04	-.23	-.20
		Comparison accuracy		-.07	-.12
		Arithmetic accuracy			.25
U.S.	6	Number line PAE	-.62**	-.36*	-.65**
		Comparison accuracy		.36*	.57**
		Arithmetic accuracy			.64**
	8	Number line PAE	-.78**	-.31	-.76**
		Comparison accuracy		.24	.60**
		Arithmetic accuracy			.53**

* $p < .05$; ** $p < .01$.

Table 3 presents correlations between number line estimation, magnitude comparison, arithmetic, and mathematics achievement scores. Although there were differences in the pattern of correlations across countries and grades, we found consistent strong correlations between number line estimation and mathematics achievement scores in all three countries. These correlations are consistent with the integrated theory's assumption that understanding of fraction magnitudes underlies general mathematics competence in this age range. Only for Chinese 8th graders was this relation not present; its absence might be due to these students' very high performance on, and consequently small variance in, number line estimation.

The correlations shown in Table 3 point to two other important results that require further attention. First, although number line estimation and magnitude comparison are generally assumed to measure the same underlying construct, i.e., students' magnitude understanding, strong correlations between the two was only present for U.S. 6th and 8th graders and, to a lesser extent, Belgian 6th graders. Moreover, magnitude comparison was correlated with general math achievement only in the U.S. samples. As discussed in more detail below, these variable relations raise questions about the reliability and validity of the magnitude comparison task with fractions.

Second, we found relatively weak, though sometimes significant, relations between number line estimation performances and fraction arithmetic scores (ranging from $r = -.23$ to $r = -.44$). The relatively low correlations suggest that the two variables measure different competencies and that a common relation to IQ cannot explain the relation between the two variables and students' mathematics achievement test scores.

We next conducted hierarchical regression analyses to test whether a common relation to fraction arithmetic explained the relation between fraction magnitude understanding, as measured by number line PAE, and mathematics achievement scores. We evaluated the adequacy of four models for predicting mathematics achievement, namely (a) arithmetic accuracy (Model 1); (b) arithmetic accuracy + number line PAE (Model 2); (c) number line PAE (Model 3); (d) number line PAE + arithmetic accuracy (Model 4). Table 4 summarizes the results per model, country and grade.

When fraction arithmetic accuracy was first entered as a predictor of mathematics achievement, adding number line estimation considerably increased the explained variance in overall mathematics achievement in 5 of the 6 Country \times Grade combinations (all

except among Chinese 8th graders). In contrast, when number line estimation PAE was first entered as a predictor of mathematics achievement scores, adding fraction arithmetic accuracy increased the explained variance in the U.S. sample only.

Taken together, these results support the major assumption of the integrated theory of numerical development, as expressed in Hypothesis 2: Fraction magnitude understanding, as measured with number line estimation tasks, predicts mathematics achievement, even when fraction arithmetic skill is statistically controlled. The data from the Belgian 6th and 8th graders, Chinese 6th graders, and U.S. 8th graders provided clear evidence for this assumption. The data from the U.S. 6th graders were less straightforward, as fraction arithmetic competency and number line estimation were equally important for mathematics achievement. For Chinese 8th graders, neither fraction magnitude understanding nor fraction arithmetic skill predicted mathematics achievement. This might be due to Chinese 8th grade students' generally high performances, and (thus) small variance in performances, on the fraction tasks. Finally, the moderate correlations between fraction magnitude understanding and arithmetic among Belgian, Chinese and U.S. 6th and 8th graders, as well as the non-significant contribution of Belgian and Chinese students' fraction arithmetic skill when magnitude understanding was statistically controlled, argue *against* the possibility that the relations among fraction arithmetic, fraction magnitude knowledge and math achievement are only due to shared relations to IQ, which presumably would affect fraction arithmetic as well as fraction magnitude estimation. Hence, fraction magnitude understanding plays a pivotal role in mathematics development.

4. Discussion

The integrated theory of numerical development emphasizes that fractions play a key role in learning mathematics. Individual differences in the precision of fraction magnitude representations in later elementary school and middle school are hypothesized to play the same central role in mathematics achievement that differences in the precision of whole number magnitude representations play in earlier grades, when whole number magnitude representations are more variable. To test this assumption, we conducted a cross-cultural study in three countries that differ in the quality of teachers' mathematical knowledge, the instructional tools used for fraction instruction, and students' general math

Table 4
Hierarchical regression analyses: adjusted r^2 , F- and β - values per model, country and grade.

Country	Grade	Model	Predictor	Adjusted r^2	F	β	
Belgium	6	1	Arithmetic accuracy	.03	1.79	.25	
		2	1_Arithmetic accuracy	.38	9.89**	-.05	
			2_Number line PAE			-.67**	
		3	Number line PAE	.40	20.34**	-.65**	
	8	4	1_Number line PAE	.38	9.89**	-.67**	
			2_Arithmetic accuracy			-.05	
		1	Arithmetic accuracy	.10	4.45*	.35*	
		2	1_Arithmetic accuracy	.29	7.64**	.21	
			2_Number line PAE			-.48**	
		3	Number line PAE	.28	13.18**	-.55**	
		4	1_Number line PAE	.29	7.64**	-.48**	
			2_Arithmetic accuracy			.21	
China	6	1	Arithmetic accuracy	-.01	<1	.15	
		2	1_Arithmetic accuracy	.56	19.20**	-.14	
			2_Number line PAE			-.80**	
		3	Number line PAE	.56	37.27**	-.76**	
		8	4	1_Number line PAE	.56	19.20**	-.80**
				2_Arithmetic accuracy			-.14
			1	Arithmetic accuracy	.04	2.40	.25
			2	1_Arithmetic accuracy	.03	1.65	.21
		2_Number line PAE			-.16		
	3	Number line PAE	.02	1.60	-.20		
	4	1_Number line PAE	.03	1.65	-.16		
	U.S.	6	1	Arithmetic accuracy	.39	20.18**	.64**
			2	1_Arithmetic accuracy	.59	22.16**	.47**
				2_Number line PAE			-.48**
			3	Number line PAE	.40	21.20**	-.65**
		8	4	1_Number line PAE	.59	22.16**	-.48**
2_Arithmetic accuracy						.47**	
1			Arithmetic accuracy	.25	7.89*	.53*	
2			1_Arithmetic accuracy	.66	21.23**	.35*	
	2_Number line PAE			-.66**			
3	Number line PAE	.55	27.08**	-.76**			
4	1_Number line PAE	.66	21.23**	-.66**			
	2_Arithmetic accuracy			.35*			

* $p < .05$; ** $p < .01$.

performances. Taking into account the results of previous work, we expected differences among countries in performance on fraction number line estimation, magnitude comparison, and arithmetic (cf. Hypotheses 1a, 1b, and 1c). By contrast, we did *not* expect country-specific differences in the relation between students' fraction magnitude understanding and their general math achievement scores, as this relation is assumed to reflect a general cognitive characteristic rather than a cultural or educational artefact (cf. Hypothesis 2).

4.1. Performance differences in fraction magnitude understanding and arithmetic

In line with previous studies in the domain of fractions (e.g., Hill et al., 2005; Krauss et al., 2008; Mullis et al., 2008, 2012; Senk et al., 2012), the present study provided suggestive evidence for the influence of teachers' mathematical knowledge and instructional tools on fraction learning. Chinese students, instructed by teachers with rich mathematical knowledge and emphasizing number line interpretations of fractions, performed better on fraction magnitude understanding and fraction arithmetic tasks than U.S. students, who are often taught by teachers with shallower mathematical understanding (cf. Hypothesis 1a). Belgian students, taught by teachers whose mathematics knowledge seems to fall in between that of teachers in China and the U.S., made less errors than their U.S. peers on all fraction tasks (cf. Hypothesis 1c), but had more difficulty than their Chinese peers on the most difficult fraction tasks, i.e., 0-5 number line estimation and fraction division

(cf. Hypothesis 1b). These findings suggest that instruction in both Belgium and China is strong enough for children to master the easier fraction tasks, but only the instruction in China is strong enough for children to learn the more difficult ones. However, since we were not able to conduct fine-grained analyses on the quality of fraction instruction provided to the students in our study, it is not clear whether this pattern is primarily due to Chinese teachers providing consistently superior instruction and Belgian teachers providing instruction that was good enough for students to master easier problems but not harder ones, or whether it is largely attributable to other cultural and educational differences, the amount of time spent on more difficult fraction problems, student motivation, or mastery of related mathematical concepts. Future studies are needed to test the viability of these potential explanations.

4.2. Fraction magnitude understanding as a key for general math development

Of particular importance, the present study provided evidence for the pivotal role of fraction understanding in students' mathematics achievement across countries differing in cultural and educational practices (cf. Hypothesis 2). These results are in line with the major findings of previous studies with U.S. samples (see Bailey et al., 2012; Booth & Newton, 2012; Siegler et al., 2012; Siegler et al., 2013; Siegler & Pyke, 2013; Siegler et al., 2011) and thus indicate that prior findings on this topic are not limited to the U.S.

In Belgium, China, and the U.S., we observed consistent relations between students' fraction understanding and overall mathematics achievement. Moreover, in all three countries, fraction magnitude understanding was the best predictor of mathematics achievement scores, even after controlling for variation in fraction arithmetic skill. We did not observe this pattern of results in the oldest group of Chinese students, presumably due to ceiling effects in their knowledge of fraction magnitudes. Taken together, these results indicate that the relation between fraction magnitude understanding and mathematics achievement is not attributable to specifics of the U.S. educational system. Instead, the strong relation between fraction magnitude understanding and mathematics achievement can be interpreted as a general cognitive characteristic of all students, regardless of their specific cultural and educational background.

This conclusion does not imply that students' mathematics instructional histories have no influence on this relation. As indicated by the differences in the observed patterns for U.S. 6th graders and Chinese 8th graders, quality of instruction might strengthen or weaken the role of fraction understanding in mathematics learning. For instance, the importance of individual differences in fraction magnitude understanding might be diminished if all students receive strong mathematics instruction in which teachers maximize the learning opportunities for all students, including the weakest ones. Focused and well-designed intervention studies are needed to unravel the important but complex relation between the quality of fraction instruction and students' progress in the domain of fractions and beyond.

Finally, it might be argued that the observed relations between fraction magnitude understanding and general mathematical achievement are merely reflections of students' general intellectual abilities, as we did not control for this variable in the present study. However, both our results and the findings from other recent studies argue against this interpretation. First, the moderate correlations between fraction magnitude understanding and fraction arithmetic skill, and also the non-significant contribution of fraction arithmetic skill to general math achievement scores when

magnitude understanding was statistically controlled, argue against the possibility that the relations among arithmetic, magnitude knowledge, and overall achievement that we observed in the present study were only due to shared relations to IQ. Moreover, recent empirical investigations on this topic have demonstrated that fraction understanding is strongly related to fraction arithmetic and mathematical achievement, even after statistically controlling for IQ, reading level, executive functioning, and knowledge of all four whole number operations (Bailey et al., 2014; Siegler et al., 2012; Siegler & Pyke, 2013). Notwithstanding these findings and arguments, future studies should take into account students' general intellectual capacities when analyzing the pivotal role of fraction magnitude understanding in their further math development.

Future studies are also needed to test whether the integrated theory of numerical development applies not only to rational numbers but also to irrational numbers (i.e., numbers that cannot be expressed as common fractions, for example the square root of 2 and the number π). All real numbers, not just the rational numbers, can be compared, ordered, and interpreted as points on a number line (Bloch, 2011, p. 1). This suggests that a key step in understanding real numbers is the realization that not only whole numbers or rational numbers but in fact all real numbers have magnitudes that can be represented along a number line. To the best of our knowledge, this hypothesis has not been tested empirically so far.

4.3. Number line estimation and number comparison as assessments of numerical understanding

The present study also adds to recent discussions of relations between the tasks that are used to empirically investigate magnitude understanding (cf. Ebersbach, Luwel, & Verschaffel, 2013; Sasanguie, Defever, Van den Bussche, & Reynvoet, 2011). With whole numbers, both number line estimation and number comparison tasks correlate substantially with mathematical achievement and more specific mathematical skills in various countries and age groups (e.g., Booth & Siegler, 2006; De Smedt, Verschaffel, & Ghesquière, 2009; Holloway & Ansari, 2008; Sasanguie, Göbel, Moll, Smets, & Reynvoet, 2013; Schneider, Grabner, & Paetsch, 2009; Schneider et al., 2008; Siegler & Ramani, 2009). This demonstrates that both tasks reliably assess central aspects of mathematical competence. However, it has also been shown that versions of these tasks differing in surface structures and cognitive demands tap into partly different aspects of magnitude understanding and can yield, for example, different solution rates or different correlations with achievement measures (cf. De Smedt, Noël, Gilmore, & Ansari, 2013; Gilmore et al., 2013; Huber, Moeller, & Nuerk, 2013; Lyons, Ansari, & Beilock, 2012; Sasanguie & Reynvoet, 2013; Schneider et al., 2008; Vogel, Grabner, Schneider, Siegler, & Ansari, 2013). Our results demonstrate that number line estimation and number comparison tap into partly different aspects of magnitude understanding, not only with whole numbers but also with fractions. Future empirical research is needed to show in greater detail how task demands moderate the outcomes and the reliabilities of different assessments of magnitude knowledge for both types of numbers.

4.4. Instructional implications

The centrality of magnitude understanding for mathematical achievement has profound implications for teaching fractions. For whole numbers, it has been shown that simple feedback interventions (Opfer & Siegler, 2007) and even numerical board games (Ramani & Siegler, 2008), in which children map numbers

onto space, can substantially improve children's magnitude understanding. These interventions can be implemented in classrooms easily and effectively (Ramani, Siegler, & Hitti, 2012). The intervention effects are particularly strong for children from low-income households, who have fewer experiences in playing number games and have less numerical knowledge than their peers (Siegler, 2009). Improvements in magnitude understanding have positive causal influences on arithmetic learning (Booth & Siegler, 2008; Siegler & Ramani, 2009) and memory for numbers (Thompson & Siegler, 2010).

Our correlational analyses suggest that similar interventions might be effective in the domain of fractions. Evidence in support of this hypothesis comes from two different intervention studies with elementary school children in the domain of fractions. First, the study of Gabriel et al. (2012) with 292 4th and 5th graders who received differential instruction on fraction understanding for 10 weeks, two times a week, 30 min per session, demonstrated that learning activities based on the use of card games focusing on the representation and manipulation of fractions in terms of their magnitudes, considerably increased students' fraction magnitude understanding. Similar results were obtained in the study conducted by Fuchs et al. (2013), involving 259 at-risk 4th graders receiving different interventions about fraction understanding and arithmetic for 12 weeks, three times a week, 30 min per session. The intervention, which put a greater emphasis on measurement and number line interpretations of fractions, was more effective than an alternative instructional approach, which emphasized part-whole interpretations of fractions. Children's increases in magnitude understanding mediated their increases in other outcome measures. However, the different interventions applied in the reported studies combined a variety of instructional techniques, learner activities, and fraction representations. Subsequent research will have to clarify if, when, how, and for whom interventions can improve magnitude understanding of fractions and whether these improvements transfer to fraction arithmetic and mathematical achievement.

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