Proximal Point Method and Elliptic Regularization

A. Kaplan *   R. Tichatschke †

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Abstract:[1]
A generalized proximal point method for solving variational inequalities with maximal monotone operators is developed. It admits a successive approximation of the feasible set and of a symmetric component of the operator as well as an inexact solving of the regularized problems. The conditions on the approximation are coordinated with the properties of finite element methods for solving problems in mathematical physics. The choice of the regularizing functional exploits a possible "reserve of monotonicity" of the operator in the variational inequality.
For the *minimal surface problem* and related variational inequalities as well as for the *convection-diffusion problem* the studied method extends the principle of elliptic regularization. A special convergence analysis shows a more qualitative convergence of the method applied to these problems than it follows from the general theory of proximal point methods. Also applications to some variational inequalities from the elasticity theory are investigated.

Keywords: proximal point algorithms, variational inequalities, maximal monotone operators, elliptic regularization, minimal surface problem, convection-diffusion problem, elasticity theory.

AMS subject classification: 47J20, 47H05, 47A52, 49Q05, 65J20, 65K10, 90C25

1 Introduction

Let $(V, \|\cdot\|)$ be a Hilbert space with the topological dual V', and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V and V'.

We consider the variational inequality

$$VI(\mathcal{Q}, \varphi, K) \qquad find\ u \in K\ and\ p \in \partial\varphi(u):$$
$$\langle \mathcal{Q}(u) + p, v - u \rangle \geq 0, \quad \forall\, v \in K,$$

where

K is a convex, closed subset of V;

*University of Trier, D-54286 Trier
†University of Trier, D-54286 Trier, e-mail: tichat@uni-trier.de
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$\mathcal{Q} : V \to V'$ is a single-valued monotone operator, its domain $D(\mathcal{Q})$ contains K and $\mathcal{Q}$ is hemicontinuous on K;

$\varphi : V \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is a convex, proper and lower semicontinuous (lsc) functional, $\partial\varphi$ denotes the subdifferential of φ and $D(\partial\varphi) \supset K$.

The proximal point method, originally introduced by MARTINET [26] to solve convex variational problems and later on investigated in a more general setting by ROCKAFELLAR [34], has initiated a lot of new algorithms for solving various classes of variational inequalities and related problems. Main directions in the development of these methods were viewed, for instance, in [15].

In the present paper a general algorithmic framework for solving $VI(\mathcal{Q}, \varphi, K)$, called *generalized proximal point method (GPP-method)*, is developed. It joins the proximal regularization and the data approximation in a manner of a diagonal process, i.e. approximation of K and $\partial\varphi$ is improved after each proximal iteration. If the operator $\mathcal{Q}$ possesses a certain reserve of monotonicity as described by the assumptions (1-ii) and (2-ii) below, conditions on the choice of the regularizing functional admit the application of *weak proximal regularization* as well as of *regularization on a subspace* (see [11], [12] for these approaches).

The paper is mainly focused upon the applications of proximal-like methods for solving variational inequalities with degenerate or singularly perturbed elliptic operators. The conditions on the data approximation in the GPP-method (cf. assumptions (2-iii)-(2-v)) are weaker that those arising from the theory of variational convergence for ill-posed problems, and at the same time they are well-coordinated with the estimates of finite element interpolation in Sobolev spaces (see the analysis of these conditions in Section 4 below).

For variational inequalities related to the minimal surface problems and for the convection-diffusion problem, which are considered in Section 4, the studied general framework covers a new (proximal based) *elliptic regularization*[2] method, in which a successive approximation of the set K is performed by means of the finite element method on a sequence of triangulations.

Applying the *proximal elliptic regularization*, one can choose the regularization parameter separated from 0, and then one obtains a sequence of uniformly elliptic auxiliary problems with a common constant of ellipticity. On this way the singularly perturbed convection-diffusion problem is approximated by a sequence of unperturbed elliptic problems. This property of the family of regularized operators provides a good stability of the auxiliary problems in case a successive approximation of K is performed by means of standard finite element techniques.

Noteworthy is that for the minimal surface problems (with and without obstacles) considered in the space $H^1(\Omega)$, the special analysis based on Theorem 2 establishes a new result on the convergence of the proximal elliptic regulariza-

[2]The idea of elliptic regularization, proposed by J.-L. LIONS [23] and O.A. OLEJNIK [30], consists in an approximation of a degenerate elliptic problem (parabolic problems are treated as a special case) by a sequence of non-degenerate elliptic problems. In the classical scheme it is carried out by adding the term $\epsilon\theta$ (θ an appropriately chosen operator) to the operator of the original problem and by considering the sequence of solutions of the regularized problems for $\epsilon \to 0$. Elliptic regularization is an efficient tool for the theoretical analysis of degenerate problems (cf. [18], [23], [24]) and serves as a basis for some numerical methods (see [6] and references therein). However, the necessity to drive the parameter ϵ to 0 enforces hard requirements on the exactness of the discretization and causes ill-conditioning of the discretized problems.

tion method in the space $W^{1,1}(\Omega)$ (see Theorem 3), whereas the general theory of proximal point methods guarantees weak convergence in $H^1(\Omega)$ only.

The proved strong convergence for the convection-diffusion problem is not so surprising, because the operator of the problem is strongly monotone although singularly perturbed, too. However, in this case the regularization permits to obtain "well-behaved" discretized problems.

The paper is organized as follows: In Section 2 we describe the mentioned algorithmic framework and discuss the assumptions concerning the variational inequality $VI(\mathcal{Q},\varphi,K)$ and its data approximation. The convergence analysis is carried out in Section 3, and in Section 4 applications of the GPP-method to several problems in mathematical physics are studied. In the final Section 5 we summarize the main peculiarities of the approach developed.

2 Generalized proximal point method

In the sequel the following assumption concerning $VI(\mathcal{Q},\varphi,K)$ will be used.

Assumption 1

(1-i) $K \cap \mathrm{int} D(\partial\varphi) \neq \emptyset$;

(1-ii) for a given linear continuous and monotone operator $\mathcal{B} : V \to V'$ with symmetry property $\langle \mathcal{B}u, v\rangle = \langle \mathcal{B}v, u\rangle$, the inequality

$$\langle \mathcal{Q}(u) - \mathcal{Q}(v), u - v\rangle \geq \langle \mathcal{B}(u - v), u - v\rangle, \quad \forall\ u, v \in D(\mathcal{Q})$$

is valid;

(1-iii) $VI(\mathcal{Q},\varphi,K)$ is solvable.

We denote the solution set of $VI(\mathcal{Q},\varphi,K)$ by $SOL(\mathcal{Q},\varphi,K)$ and write $VI(\mathcal{Q},K)$ and $SOL(\mathcal{Q},K)$ in case $\varphi \equiv 0$.

With the normality operator

$$\mathcal{N}_K : u \mapsto \begin{cases} \{z \in V' : \langle z, u - v\rangle \geq 0 \quad \forall\ v \in K\} & \text{if } u \in K, \\ \emptyset & \text{otherwise,} \end{cases}$$

the maximal monotonicity of $\mathcal{Q} + \mathcal{N}_K$ follows from Theorem 1 in [33], and then (1-i) provides that the operator $\mathcal{Q} + \mathcal{N}_K + \partial\varphi$ is maximal monotone, too (see [33], Theorem 3).

The algorithmic framework studied includes a successive approximation of the set K and of the functional φ by a sequence $\{K^k\}$, $K^k \subset K$, of convex closed sets and by a sequence $\{\varphi_k\}$, $\varphi_k : V \to \bar{\mathbb{R}}$, of convex functionals, respectively. Moreover, we suppose that φ_k is Gâteaux-differentiable on K and $\nabla\varphi_k$ is hemicontinuous on K.

Let $r : V \to \mathbb{R}$ be a convex Gâteaux-differentiable functional such that

$$v \mapsto r(v) + \langle \mathcal{B}v, v\rangle$$

is strongly convex on $D(\mathcal{Q})$, where $\mathcal{B}$ satisfies (1-ii). For an appropriate operator $\mathcal{B}$ see, for instance, Subsection 4.3 below.

We make also use of the controlling sequences $\{\delta_k\}$ and $\{\chi_k\}$ such that

$$\delta_k \geq 0, \quad \lim_{k\to\infty} \delta_k = 0, \quad 0 < \chi_k \leq \bar{\chi} < \infty. \tag{1}$$

The choice of $\{K^k\}$, $\{\varphi_k\}$, r as well as the sequences $\{\delta_k\}$ and $\{\chi_k\}$ will be specified in Assumption 2.

GPP-Method:

Given $u^1 \in K$, with u^k from the $(k-1)-th$ step (u^1 if $k=1$), at the $k-th$ step we define u^{k+1} by solving the problem

$$(\mathbf{P^k}) \quad \begin{array}{l} find\ u^{k+1} \in K^k : \\ \langle \mathcal{Q}(u^{k+1}) + \nabla\varphi_k(u^{k+1}) + \chi_k(\nabla r(u^{k+1}) - \nabla r(u^k)), v - u^{k+1}\rangle \\ \qquad\qquad\qquad \geq -\delta_k \|v - u^{k+1}\|, \quad \forall\, v \in K^k. \end{array}$$

Assumption 2

(2-i) $r : V \to \mathbb{R}$ is a convex functional and the mapping ∇r is Lipschitz continuous on $D(\mathcal{Q})$;

(2-ii) with given constants $\tilde{\chi} \geq 0$, $m > 0$ and the operator $\mathcal{B}$ satisfying (1-ii), the inequality

$$\frac{\tilde{\chi}}{2}\langle \mathcal{B}(u-v), u-v\rangle + r(u) - r(v) - \langle \nabla r(v), u - v\rangle \geq m\|u-v\|^2$$

holds for all $u, v \in D(\mathcal{Q})$, and the choice of $\bar{\chi}$ in (1) provides $2\bar{\chi}\tilde{\chi} < 1$;

(2-iii) for each $w \in K$, there exists a sequence $\{w^k\}$, $w^k \in K^k$, such that

$$w^k \rightharpoonup w, \quad \mathcal{Q}(w^k) \to \mathcal{Q}(w), \quad \lim_{k\to\infty} \varphi_k(w^k) = \varphi(w),$$

(symbols "$\rightharpoonup$" and "$\to$" indicate weak- and strong convergence, respectively);

(2-iv) $\varphi_k(v) \geq \varphi(v), \quad \forall\, v \in K, \ \forall\, k;$

(2-v) with given nonnegative constants c_1, c_2, c and sequences $\{h_k\}$, $\{\sigma_k\}$ satisfying

$$\sum_{k=1}^{\infty} \frac{h_k}{\chi_k} < \infty, \quad \sum_{k=1}^{\infty} \frac{\sigma_k}{\chi_k} < \infty, \tag{2}$$

for some $u^* \in SOL(\mathcal{Q}, \varphi, K)$ there exists a sequence $\{w^k\}$, $w^k \in K^k$, such that

$$\begin{array}{rcl} \|w^k - u^*\| & \leq & c_1 h_k, \\ \|\mathcal{Q}(w^k) - \mathcal{Q}(u^*)\|_{V'} & \leq & c_2\sigma_k, \\ \varphi_k(w^k) - \varphi(u^*) & \leq & c\sigma_k. \end{array} \tag{3}$$

Remark 1 In Subsection 4.1 we meet a situation when the assumption $K^k \subset K$ can be rather restrictive, especially if the sets K^k are constructed by means of standard finite element techniques. In the problems considered there, however, we have $\varphi \equiv 0$ and the operator $\mathcal{Q}$ possesses good additional properties like strict monotonicity and Lipschitz continuity on the whole space V. This permits one to replace the assumption $K^k \subset K$ by a weaker one:

(a) with a given $c_3 > 0$ and $\{h_k\}$ as in (2), for any sequence $\{v^k\}, v^k \in K^k$, there exists a sequence $\{z^k\} \subset K$ such that

$$\|z^k - v^k\| \leq c_3(\|v^k - u^*\|^2 + 1)h_k, \quad \forall\ k,$$

or

(b) with a given $c_3 > 0$ and $\{h_k\}$, $\{v^k\}$ as in (a), there exists a sequence $\{z^k\} \subset K$ such that

$$\langle \mathcal{Q}(u^*), z^k - v^k \rangle \leq c_3(\|v^k - u^*\|^2 + 1)h_k, \quad \forall k;$$

moreover, each weak limit point of $\{v^k\}$ belongs to K.

All statements of Section 3 (except for Theorem 2 if the weaker assumption (b) is used) remain true, the technical modifications in the proofs can be carried out on the base of the convergence analysis in [15]. $\diamond$

Remark 2 If the assumptions (1-ii), (2-i) and (2-ii) are valid, then for each k, the operator

$$v \mapsto \mathcal{Q}(v) + \nabla\varphi_k(v) + \chi_k(\nabla r(v) - \nabla r(v^k)) + \mathcal{N}_{K^k}(v)$$

is maximal monotone ([33], Theorem 3). Hence, the exact problem (P^k) (with $\delta_k = 0$) has a unique solution, and the solvability of the inexact problem (with $\delta_k > 0$) is guaranteed. $\diamond$

3 Convergence analysis of the GPP-method

In the sequel we need the following modification of Minty's lemma [27].

Lemma 1 Let assumption (1-i) be valid, and for some $u \in K$ and any $v \in K$ there exists $p(v) \in \partial\varphi(v)$ such that

$$\langle \mathcal{Q}(v) + p(v), v - u \rangle \geq 0. \tag{4}$$

Then, with some $p \in \partial\varphi(u)$ the inequality

$$\langle \mathcal{Q}(u) + p, v - u \rangle \geq 0 \tag{5}$$

holds for all $v \in K$.

Proof. Denote

$$\mathcal{G} : v \mapsto \mathcal{Q}(v) + \partial\varphi(v) + \mathcal{I}(v - u),$$

where $\mathcal{I} : V \to V'$ is the canonical isometry operator. The operator $\bar{\mathcal{G}} := \mathcal{G} + \mathcal{N}_K$ is maximal monotone and strongly monotone. Therefore, there exists $w \in K$, such that $0 \in \bar{\mathcal{G}}(w)$, and using the definition of $\mathcal{N}_K$, we infer from here that

$$\langle g(w), v - w \rangle \geq 0, \quad \forall\, v \in K \tag{6}$$

holds with some $g(w) \in \mathcal{G}(w)$. In view of the convexity of the functional φ, the last inequality implies

$$\langle \mathcal{Q}(w) + \mathcal{I}(w - u), v - w \rangle + \varphi(v) - \varphi(w) \geq 0, \quad \forall\, v \in K. \tag{7}$$

If $w = u$, then of course $g(w) \in \mathcal{Q}(w) + \partial\varphi(w)$ and the conclusion of the lemma is obvious.

Suppose now that $w \neq u$. We make use of the relation

$$\langle \tilde{g}(v), v - u \rangle \geq 0, \quad \forall\, v \in K, \tag{8}$$

which follows from (4) for $\tilde{g}(v) := g(v) + \mathcal{I}(v - u)$ with an appropriate $g(v) \in \mathcal{Q}(v) + \partial\varphi(v)$. Take $w_\lambda = u + \lambda(w - u)$ for $\lambda \in [0, 1)$. Obviously, $w_\lambda \in K$, and according to (8), for each λ there exists $\tilde{g}(w_\lambda) \in \mathcal{G}(w_\lambda)$ satisfying

$$\langle \tilde{g}(w_\lambda), w - u \rangle \geq 0. \tag{9}$$

Using again the convexity of φ, one can immediately conclude from (9) that

$$\langle \mathcal{Q}(w_\lambda) + \mathcal{I}(w_\lambda - u), w - u \rangle + \frac{1}{1 - \lambda}\left[\varphi(w) - \varphi(u + \lambda(w - u))\right] \geq 0. \tag{10}$$

Passing to the limit in (10) for $\lambda \downarrow 0$ and observing that the operator $\mathcal{Q}$ is hemicontinuous on K and the functional φ is lsc, we get

$$\langle \mathcal{Q}(u), w - u \rangle + \varphi(w) - \varphi(u) \geq 0. \tag{11}$$

Inequality (7) (given with $v = u$) together with (11) leads to

$$\langle \mathcal{Q}(u) - \mathcal{Q}(w), u - w \rangle + \langle \mathcal{I}(u - w), u - w \rangle \leq 0,$$

but this contradicts to the monotonicity of $\mathcal{Q}$. □

Remark 3

(a) The reverse conclusion that (5) implies (4) (with any $p(v) \in \partial\varphi(v)$) follows immediately from the monotonicity of the operator $\mathcal{Q} + \partial\varphi$. Using this fact and Lemma 1, one can easily show that $SOL(\mathcal{Q}, \varphi, K)$ is a convex closed set.

(b) Under assumption (1-i) the following statements are equivalent:

(b1) $u \in K$ and $\langle \mathcal{Q}(v), v - u \rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall\, v \in K$;

(b2) $u \in K$ and $\exists\, p \in \partial\varphi(u)$: $\quad \langle \mathcal{Q}(u) + p, v - u \rangle \geq 0, \quad \forall\, v \in K$.

Indeed, the implication (b2) ⇒ (b1) is evident. But if (b1) is fulfilled, then the monotonicity of $\mathcal{Q}$ and convexity of φ yield

$$\langle \mathcal{Q}(v), v - u \rangle + \langle p(v), v - u \rangle \geq 0, \quad \forall\, v \in K,\ \forall\, p(v) \in \partial\varphi(v),$$

and Lemma 1 provides the validity of (b2).

◇

Now, with $\tilde{\chi}$ from (2-ii) we introduce the function

$$\Gamma: \ (u,v) \mapsto \tilde{\chi}\langle \mathcal{B}(v-u), v-u\rangle + r(u) - r(v) - \langle \nabla r(v), u-v\rangle. \qquad (12)$$

For u^* chosen as in (2-v), $\Gamma(u^*, \cdot)$ plays the role of a Lyapunov function in the convergence analysis.

Lemma 2 Let the assumptions (1-ii), (1-iii), (2-i), (2-ii), (2-iv) and (2-v) be satisfied and $\sum_{k=1}^{\infty} \frac{\delta_k}{\chi_k} < \infty$. Then it holds

(a) the sequence $\{u^k\}$ generated by the GPP-method is bounded;

(b) $\lim_{k\to\infty} \Gamma(u^{k+1}, u^k) = 0$;

(c) $\lim_{k\to\infty} \|u^{k+1} - u^k\| = 0$;

(d) sequence $\{\Gamma(u^*, u^k)\}$ converges.

Sketch of the proof. Let $\{w^k\}$ be chosen as in assumption (2-v). Taking into account (2-iv), (2-v) and the convexity of φ_k, we obtain

$$\begin{aligned} \langle \nabla\varphi_k(u^{k+1}), w^k - u^{k+1}\rangle &\leq \varphi_k(w^k) - \varphi_k(u^{k+1}) \\ &= \left[\varphi_k(w^k) - \varphi(u^*)\right] + \left[\varphi(u^*) - \varphi_k(u^{k+1})\right] \\ &\leq c\sigma_k + \varphi(u^*) - \varphi(u^{k+1}). \end{aligned} \qquad (13)$$

Because $u^* \in SOL(\mathcal{Q}, \varphi, K)$ and $u^{k+1} \in K^k \subset K$, the inequality

$$\langle \mathcal{Q}(u^*), u^{k+1} - u^*\rangle + \varphi(u^{k+1}) - \varphi(u^*) \geq 0 \qquad (14)$$

is valid.

Applying (13) and (14) for the estimation of

$$\langle \nabla r(u^k) - \nabla r(u^{k+1}), w^k - u^{k+1}\rangle,$$

the rest of the proof is analogous to those of Lemma 2 in [15]. □

Lemma 3 Let the assumptions (1-i), (2-i), (2-iii) and (2-iv) be fulfilled. Moreover, suppose that the sequence $\{u^k\}$ generated by the GPP-method is bounded and

$$\lim_{k\to\infty} \|u^{k+1} - u^k\| = 0.$$

Then each weak limit point of $\{u^k\}$ is a solution of $VI(\mathcal{Q}, \varphi, K)$.

Proof. Let $\{u^k\}_{k\in\mathfrak{K}}$ converge weakly to $\bar{u}$. Because $K^k \subset K \ \forall\ k$ and K is a closed convex set, one gets $\bar{u} \in K$, whereas $\lim_{k\to\infty} \|u^{k+1} - u^k\| = 0$ implies $u^{k+1} \rightharpoonup \bar{u}$ if $k \in \mathfrak{K},\ k \to \infty$.

According to (2-iii), for each $v \in K$, one can choose a sequence $\{v^k\}$, $v^k \in K^k$, such that $v^k \rightharpoonup v$ for $k \to \infty$ and

$$\lim_{k\to\infty} \|\mathcal{Q}(v^k) - \mathcal{Q}(v)\|_{V'} = 0, \qquad \lim_{k\to\infty} \varphi_k(v^k) = \varphi(v). \qquad (15)$$

By the definition of $\{u^k\}$ (see (P^k)) and the inclusion $v^k \in K^k$, the inequality

$$\begin{aligned}\langle \mathcal{Q}(u^{k+1}) \quad + \quad & \nabla\varphi_k(u^{k+1}) + \chi_k\left(\nabla r(u^{k+1}) - \nabla r(u^k)\right), v^k - u^{k+1}\rangle \\ & \geq -\delta_k \|v^k - u^{k+1}\|\end{aligned}$$

holds for all k. Due to the monotonicity of $\mathcal{Q}$, the convexity of φ_k and (2-iv), this leads to

$$\begin{aligned}\langle \mathcal{Q}(v^k) \quad + \quad & \chi_k\left(\nabla r(u^{k+1}) - \nabla r(u^k)\right), v^k - u^{k+1}\rangle + \varphi_k(v^k) - \varphi(u^{k+1}) \\ & \geq -\delta_k \|v^k - u^{k+1}\|.\end{aligned}$$

Now, passing to the limit in the latter inequality for $k \in \mathfrak{K},\ k \to \infty$, we obtain from (1), (15), (2-i), $\lim_{k\to\infty} \|u^{k+1} - u^k\| = 0$, $v^k \rightharpoonup v$, $u^{k+1} \rightharpoonup \bar{u}$ and the lower semicontinuity of φ that

$$\langle \mathcal{Q}(v), v - \bar{u}\rangle + \varphi(v) - \varphi(\bar{u}) \geq 0, \quad \forall\ v \in K.$$

Finally, Lemma 1 and Remark 3(b) enable us to conclude that $\bar{u} \in SOL(\mathcal{Q}, \varphi, K)$. $\square$

Theorem 1 Let the Assumptions 1 and 2 be fulfilled and $\sum_{k=1}^{\infty} \frac{\delta_k}{\chi_k} < \infty$. Then it holds

(i) Problem (P^k) is solvable for each k, the sequence $\{u^k\}$ generated by the GPP-method is bounded and each weak limit point of $\{u^k\}$ is a solution of $VI(\mathcal{Q}, \varphi, K)$.

(ii) If, in addition, assumption (2-v) holds for each $u \in SOL(\mathcal{Q}, \varphi, K)$ (the constants c_1, c_2, c may depend on u) and

$$v^k \rightharpoonup v \text{ in } V,\ v^k \in K^k \quad \textit{implies} \quad \nabla r(v^k) \rightharpoonup \nabla r(v) \text{ in } V', \tag{16}$$

then the whole sequence $\{u^k\}$ converges weakly to $u^* \in SOL(\mathcal{Q}, \varphi, K)$.

(iii) If, moreover, there exists a linear compact operator $\hat{\mathcal{B}} : V \to V'$ such that $B + \hat{\mathcal{B}}$ is strongly monotone, then $\{u^k\}$ converges strongly to $u^* \in SOL(\mathcal{Q}, \varphi, K)$.

Proof. Conclusion (i) follows immediately from Remark 2 and the Lemmata 2 and 3.
The proof of conclusion (ii) is the same as in [15], Theorem 1.
Thus, it remains to prove (iii) only. Let u^* be the weak limit of $\{u^k\}$. Choosing $\{w^k\}$ according to (2-v) we obtain from (1-ii)

$$\begin{aligned}&\langle \mathcal{B}(u^{k+1} - u^*), u^{k+1} - u^*\rangle \\ &\quad = \langle \mathcal{B}(u^{k+1} - w^k), u^{k+1} - w^k\rangle - \langle \mathcal{B}(u^* - w^k), u^{k+1} - u^*\rangle \\ &\qquad - \langle \mathcal{B}(u^{k+1} - w^k), u^* - w^k\rangle \\ &\quad \leq \langle \mathcal{Q}(u^{k+1}) - \mathcal{Q}(w^k), u^{k+1} - w^k\rangle - \langle \mathcal{B}(u^{k+1} - u^*), u^* - w^k\rangle \\ &\qquad - \langle \mathcal{B}(u^{k+1} - w^k), u^* - w^k\rangle.\end{aligned} \tag{17}$$

Now, we estimate the term $\langle \mathcal{Q}(u^{k+1}), u^{k+1} - w^k \rangle$ by setting $v = w^{k+1}$ in (P^k), and then insert this estimate in (17). Together with (2-iv) this yields

$$
\begin{aligned}
&\langle \mathcal{B}(u^{k+1} - u^*), u^{k+1} - u^* \rangle \\
&\quad \le \langle \mathcal{Q}(w^k), w^k - u^{k+1} \rangle + \langle \nabla \varphi_k(u^{k+1}), w^k - u^{k+1} \rangle \\
&\qquad + \chi_k \langle \nabla r(u^{k+1}) - \nabla r(u^k), w^k - u^{k+1} \rangle + \delta_k \| w^k - u^{k+1} \| \\
&\qquad + \langle \mathcal{B}u^* + \mathcal{B}w^k - 2\mathcal{B}u^{k+1}, u^* - w^k \rangle \\
&\quad \le \langle \mathcal{Q}(w^k) - \mathcal{Q}(u^*), w^k - u^{k+1} \rangle + \langle \mathcal{Q}(u^*), w^k - u^* \rangle + \langle \mathcal{Q}(u^*), u^* - u^{k+1} \rangle \\
&\qquad + \chi_k \langle \nabla r(u^{k+1}) - \nabla r(u^k), w^k - u^{k+1} \rangle + \delta_k \| w^k - u^{k+1} \| \\
&\qquad + \langle \mathcal{B}u^* + \mathcal{B}w^k - 2\mathcal{B}u^{k+1}, u^* - w^k \rangle + \left[\varphi_k(w^k) - \varphi(u^{k+1}) \right]. \qquad (18)
\end{aligned}
$$

Taking into account that the sequences $\{w^k\}$ and $\{u^{k+1}\}$ are bounded, one can conclude that all terms in the right hand side of (18) tend to zero for $k \to \infty$. Indeed, it vanishes

the first, second and sixth term in view of (2-v);

the third term because $u^k \rightharpoonup u^*$;

the fourth term due to $\|u^{k+1} - u^k\| \to 0$ and (2-i);

the fifth term owing to $\delta_k \to 0$;

the last term in view of (2-v), $u^{k+1} \rightharpoonup u^*$ and the lower semicontinuity of φ.

Thus, (18) implies

$$\lim_{k \to \infty} \langle \mathcal{B}(u^k - u^*), u^k - u^* \rangle = 0. \qquad (19)$$

At the same time

$$\lim_{k \to \infty} \langle \hat{\mathcal{B}}(u^k - u^*), u^k - u^* \rangle = 0 \qquad (20)$$

follows from $u^k \rightharpoonup u^*$ and the compactness of $\hat{\mathcal{B}}$. Adding (19), (20) and obeying the strong monotonicity of $\mathcal{B} + \hat{\mathcal{B}}$ we conclude finally that $\{u^k\}$ converges to u^* strongly in V. □

Remark 4 From the compactness of the canonical injection $\mathcal{I} : H^1(\Omega) \to L^2(\Omega)$ (Ω is here an open domain in $\mathbb{R}^2$ with a Lipschitz continuous boundary) and the *second Korn inequality*, the existence of an operator $\hat{\mathcal{B}}$ satisfying condition (iii) of Theorem 1 can be shown, in particular, for ill-posed elliptic variational inequalities, which describe the *problem of linear elasticity with given friction* and the *two-body contact problem* (see [31] for the mathematical formulations and [12] for the proximal method with weak regularization). We deal with these problems in Subsection 4.3.

For the problem of linear elasticity, for example, the operator $\hat{\mathcal{B}}$ defined by

$$\langle \hat{\mathcal{B}}u, v \rangle = \int_\Omega (u_1 v_1 + u_2 v_2) dx, \quad \forall\, u, v \in V := [H^1(\Omega)]^2$$

is appropriate. ◇

The following assumption serves to establish a more qualitative convergence of the GPP-method in those situations when the conditions (ii) and (iii) of Theorem 1 are not guaranteed.

Let $Y \supset V$ be a Banach space with the norm $\|\cdot\|_Y$, and

$$\mathrm{dist}_Y(y, A) := \inf_{z \in A} \|y - z\|_Y.$$

According to Lemma 2, there exists $\rho > 0$ such that $\{u^k\} \subset \mathbb{B}_\rho$ and $\mathbb{B}_\rho \cap SOL(\mathcal{Q}, \varphi, K) \neq \emptyset$, where $\mathbb{B}_\rho := \{v \in V : \|v\| \leq \rho\}$.

Denote $S^* = SOL(\mathcal{Q}, \varphi, K) \cap \mathbb{B}_\rho$.

Assumption 3 There exists a continuous function $\tau : [0, \infty) \to [0, \infty)$, $\tau(0) = 0$, $\tau(s) > 0 \ \forall \ s > 0$, such that

$$\langle \mathcal{Q}(v), v - u \rangle + \varphi(v) - \varphi(u) \geq \tau\left(\mathrm{dist}_Y(v, S^*)\right), \quad \forall \ u \in S^*, \forall \ v \in K \cap \mathbb{B}_\rho.$$

For a fixed $u \in K \cap \mathbb{B}_\rho$, the function

$$\xi(\cdot, u) : \ v \to \langle \mathcal{Q}v, v - u \rangle + \varphi(v) - \varphi(u)$$

possesses the properties

$$\begin{array}{lll} \xi(v, u) \geq 0 & \forall \ v \in K \quad \Leftrightarrow \quad u \in S^*, \\ \xi(v, u) = 0 & \text{if} \ \ v \in SOL(\mathcal{Q}, \varphi, K), \ u \in S^*, \end{array}$$

and Assumption 3 describes a growth condition for the function $\inf_{u \in S^*} \xi(\cdot, u)$ on the set $(K \cap \mathbb{B}_\rho) \setminus S^*$.

In case $\mathcal{Q} = 0, K = V, Y = V$ and $\tau(s) = cs^2$, Assumption 3 is closely related to the growth condition used by Kort and Bertsekas [19] for the quadratic method of multipliers in convex programming, which in fact is the proximal point method applied to the dual program.

The result below will be applied in Subsection 4.1 to show $W^{1,1}$-convergence of the iterates of the GPP-method for the *minimal surface problem* and related variational inequalities considered in the H^1-space.

Theorem 2 Let the conditions of Lemma 2 and Assumption 3 be fulfilled. Moreover, suppose that (2-v) is valid for each $u^* \in S^*$ and the operator $\mathcal{Q}$ is bounded on $K \cap \mathbb{B}_\rho$. Then, for the sequence $\{u^k\}$ generated by the GPP-method it holds

$$\lim_{k \to \infty} \mathrm{dist}_Y(u^k, S^*) = 0. \tag{21}$$

Proof. With $u^* \in S^*$ and $\{w^k\}$ chosen as in (2-v), one gets

$$\begin{aligned} &\Gamma(u^*, u^{k+1}) - \Gamma(u^*, u^k) \\ &\quad = -\Gamma(u^{k+1}, u^k) + \langle \nabla r(u^k) - \nabla r(u^{k+1}), u^* - w^k + w^k - u^{k+1} \rangle \\ &\quad\ \ + 2\tilde{\chi} \langle \mathcal{B}(u^k - u^{k+1}), u^* - u^{k+1} \rangle, \end{aligned}$$

and applying (P^k) to estimate the term $\langle \nabla r(u^k) - \nabla r(u^{k+1}), w^k - u^{k+1}\rangle$ we obtain

$$\begin{aligned}
&\Gamma(u^*, u^{k+1}) - \Gamma(u^*, u^k) \\
&\quad \le 2\tilde{\chi}\langle \mathcal{B}(u^k - u^{k+1}), u^* - u^{k+1}\rangle \\
&\quad + \langle \nabla r(u^k) - \nabla r(u^{k+1}), u^* - w^k\rangle + \frac{\delta_k}{\chi_k}\|w^k - u^{k+1}\| \\
&\quad + \frac{1}{\chi_k}\langle \mathcal{Q}(u^{k+1}), w^k - u^{k+1}\rangle + \frac{1}{\chi_k}\langle \nabla\varphi_k(u^{k+1}), w^k - u^{k+1}\rangle. \qquad (22)
\end{aligned}$$

Now, taking into account the convexity of φ_k and (2-iv), inequality (22) yields

$$\begin{aligned}
&\Gamma(u^*, u^{k+1}) - \Gamma(u^*, u^k) \\
&\quad \le 2\tilde{\chi}\langle \mathcal{B}(u^k - u^{k+1}), u^* - u^{k+1}\rangle \\
&\quad + \langle \nabla r(u^k) - \nabla r(u^{k+1}), u^* - w^k\rangle + \frac{\delta_k}{\chi_k}\|w^k - u^{k+1}\| \\
&\quad + \frac{1}{\chi_k}\langle \mathcal{Q}(u^{k+1}), w^k - u^*\rangle + \frac{1}{\chi_k}\left(\varphi_k(w^k) - \varphi(u^*)\right) \\
&\quad + \frac{1}{\chi_k}\left[\langle \mathcal{Q}(u^{k+1}), u^* - u^{k+1}\rangle + \varphi(u^*) - \varphi(w^k)\right].
\end{aligned}$$

In view of Assumption 3 and $\chi_k \le \bar{\chi}$, the last term can be replaced by

$$-\frac{1}{\bar{\chi}}\,\tau\left(\operatorname{dist}_Y(u^{k+1}, S^*)\right).$$

Finally, passing to the limit in the so modified inequality, and owing to Lemma 2, the assumptions (2-i), (2-v), the boundedness of the operator $\mathcal{Q}$ on $K \cap I\!\!B_\rho$, and $\sum_{k=1}^{\infty}\frac{\delta_k}{\chi_k} < \infty$, we immediately obtain

$$\lim_{\tau\to\infty}\tau(\operatorname{dist}_Y(u^{k+1}, S^*)) = 0.$$

Now the properties of τ imply the validity of (21). □

We conclude this section with a statement which can be useful for checking Assumption 3 in case $Y = V$. It allows us to analyze a growth property of the function

$$v \mapsto \inf_{u\in S^*}\{\langle \mathcal{Q}(v), v - u\rangle + \varphi(v) - \varphi(u)\}$$

in a neighborhood of S^* only. By the way, a growth condition like (23) with $\tau(s) = cs^2$ and $\tau(s) = cs$ was introduced in [11] to investigate the rate of convergence of multi-step proximal regularization methods.
With ρ as above and a given $\delta \in (0, \rho)$, we consider the set

$$K_\delta = \{v \in K \cap I\!\!B_\rho : \operatorname{dist}_V(v, S^*) \le \delta\}.$$

Denote $u^*(v) = \arg\min_{w\in S^*}\|v - w\|$.

Lemma 4 Suppose that $\tau : [0,\infty) \to [0,\infty)$ is a nondecreasing function such that

$$\langle \mathcal{Q}(v), v - u\rangle + \varphi(v) - \varphi(u) \ge \tau(\|v - u^*(v)\|) \qquad (23)$$

is valid for any $v \in K_\delta$, $u \in S^*$. Then the inequality

$$\inf_{u \in S^*} [\langle \mathcal{Q}(v), v-u\rangle + \varphi(v) - \varphi(u)] \geq \tau\left(\frac{\delta}{2\rho}\|v - u^*(v)\|\right) \tag{24}$$

holds for any $v \in K \cap \mathbb{B}_\rho$.

Proof. Obviously, we need to check (23) for $v \in (K \cap \mathbb{B}_\rho) \setminus K_\delta$ only. Take an arbitrary $u \in S^*$. Due to the convexity of $K \cap \mathbb{B}_\rho$, the set $\{\lambda v + (1-\lambda)u : 0 < \lambda < 1\}$ belongs to $K \cap \mathrm{int}\mathbb{B}_\rho$. Thus, there exists $\bar\lambda = \lambda(v,u) \in (0,1)$ such that

$$\bar v = \bar\lambda v + (1-\bar\lambda)u \in \mathrm{bd}K_\delta.$$

Using

$$\bar v - u = \bar\lambda(v-u), \qquad \frac{1-\bar\lambda}{\bar\lambda}(\bar v - u) = v - \bar v,$$

from the monotonicity of $\mathcal{Q}$ and the convexity of φ we obtain

$$\frac{1-\bar\lambda}{\bar\lambda}\langle \mathcal{Q}(v) - \mathcal{Q}(\bar v), \bar v - u\rangle = \langle \mathcal{Q}(v) - \mathcal{Q}(\bar v), v - \bar v\rangle \geq 0$$

and

$$\varphi(v) - \varphi(u) \geq \frac{1}{\bar\lambda}(\varphi(\bar v) - \varphi(u)).$$

Therefore,

$$\langle \mathcal{Q}(v), v-u\rangle = \frac{1}{\bar\lambda}\langle \mathcal{Q}(v), \bar v - u\rangle \geq \frac{1}{\bar\lambda}\langle \mathcal{Q}(\bar v), \bar v - u\rangle$$

and

$$\langle \mathcal{Q}(v), v-u\rangle + \varphi(v) - \varphi(u) \geq \frac{1}{\bar\lambda}[\langle \mathcal{Q}(\bar v), \bar v - u\rangle + \varphi(\bar v) - \varphi(u)]$$

hold, and inequality (23) yields

$$\begin{aligned}\langle \mathcal{Q}(v), v-u\rangle + \varphi(v) - \varphi(u) &\geq \frac{1}{\bar\lambda}\,\tau(\|\bar v - u^*(\bar v)\|)\\ &\geq \tau(\|\bar v - u^*(\bar v)\|).\end{aligned}$$

But, $\|\bar v - u^*(\bar v)\| = \delta$ and $\|v - u^*(v)\| \leq 2\rho$, hence

$$\|\bar v - u^*(\bar v)\| \geq \frac{\delta}{2\rho}\|v - u^*(v)\|,$$

and taking into account the nondecreasing of τ, we conclude that

$$\langle \mathcal{Q}(v), v-u\rangle + \varphi(v) - \varphi(u) \geq \tau\left(\frac{\delta}{2\rho}\|v - u^*(v)\|\right).$$

Because $u \in S^*$ is arbitrarily chosen, this leads to (24). □

4 Applications

In Subsections 4.1 and 4.2 below we deal with elliptic variational problems in the space $V = H^1(\Omega)$, where Ω is an open domain in $\mathbb{R}^2$ with a Lipschitz continuous boundary Γ. In this context the convex closed set K is defined as

$$K = \{v \in V : v = g \text{ on } \Gamma_1\} \tag{25}$$

or

$$K = \{v \in V : v = g \text{ on } \Gamma,\ v \geq \psi \text{ a.e. on } \Omega\}, \tag{26}$$

where $\Gamma_1 \subseteq \Gamma$, *mes* $\Gamma_1 > 0$; g and ψ are sufficiently smooth functions on $\bar{\Omega} := \Omega \cup \Gamma$ and $\psi \leq g$ on Γ.

Applying the GPP-method to these problems, a successive approximation of K by means of the finite element method on a sequence of triangulations $\{\mathcal{T}_k\}$ is performed.

When we check the conditions on approximation, formulated in Assumption 2, we will suppose that

Ω is a polygonal domain;

the solution of the problem is sufficiently smooth;

a standard finite element method with piece-wise linear basis functions on the regular sequence of triangulations $\{\mathcal{T}_k\}$ of Ω is applied (see [2] for notions and terminology of finite element methods).

By h_k the characteristic triangulation parameter of $\mathcal{T}_k$ is denoted, i.e. the length of the largest edge of the triangles in $\mathcal{T}_k$; Σ_k indicates the set of vertices of all triangles in $\mathcal{T}_k$; $\Sigma_k(\Gamma_1)$, $\Sigma_k(\Gamma)$ are the sets of all vertices lying on Γ_1 and Γ, respectively; $\mathfrak{P}_1$ denotes the space of polynomials in two variables of degree ≤ 1.

Then, on the functional space

$$V^k := \{v \in C(\bar{\Omega}) : v|_T \in \mathfrak{P}_1(T)\ \forall\ T \in \mathcal{T}_k\} \tag{27}$$

the sets (25) and (26) are approximated by

$$K^k := \{v \in V^k : v(a_i) = g(a_i)\ \forall\ a_i \in \Sigma_k(\Gamma_1)\} \tag{28}$$

and

$$\begin{aligned} K^k := \{v \in V^k : v(a_i) &= g(a_i)\ \forall\ a_i \in \Sigma_k(\Gamma), \\ v(a_i) &\geq \psi(a_i)\ \forall\ a_i \in \Sigma_k\}, \end{aligned} \tag{29}$$

respectively.

4.1 Non-parametric minimal surface problem and related variational inequalities

Formulations and properties of the problem

The classical (non-parametric) minimal surface problem, considered here in the space $V := H^1(\Omega)$, can be formulated as follows:

$$\min\{J(u) : u \in g + H_0^1(\Omega)\}, \quad J(u) := \int_\Omega \sqrt{1 + |\nabla u|^2}dx \tag{30}$$

($g \in V$ is given), i.e. among all functions $u \in H^1(\Omega)$, $u = g$ on Γ, we are looking for a function, which defines a surface $z = u(x_1, x_2)$ with the smallest area.

Introducing the operator $\mathcal{Q} : V \to V'$ defined by[3]

$$\langle \mathcal{Q}(u), v\rangle = \int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1+|\nabla u|^2}} dx \quad \forall\, u, v \in V \tag{31}$$

and the affine set

$$K := \{v \in V : v - g \in H_0^1(\Omega)\}, \tag{32}$$

problem (30) can be rewritten as variational equality

$$find\ u \in K : \quad \langle \mathcal{Q}(u), v\rangle = 0 \quad \forall\, v \in H_0^1(\Omega),$$

which in turn represents a weak formulation of the boundary value problem

$$\begin{aligned} -\text{div}\, \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} &= 0 \ \text{ on } \Omega, \\ u &= g \ \text{ on } \Gamma. \end{aligned} \tag{33}$$

Equation (33) is nothing else but the Euler equation for the classical minimal surface problem.

The non-homogeneous problem

$$\begin{aligned} -\,\text{div}\, \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} &= p \ \text{ on } \Omega, \\ u &= g \ \text{ on } \Gamma \end{aligned}$$

is known as the *Dirichlet problem for the equation of prescribed mean curvature.* For the long history and a survey of numerous investigations connected with these two problems we refer to the monographs [29] and [7].

The variational inequality $VI(\mathcal{Q}, K)$ with K given by (26) corresponds to the *minimal surface problem with an obstacle.* Problems of such type were mainly investigated in non-reflexive Banach spaces, where the functional $J(u) = \int_\Omega \sqrt{1+|\nabla u|^2} dx$ possesses better coercivity properties (see [18], Chapt. 3.4).

The problems considered here are not uniformly elliptic (see [20], Chapt. VI for the corresponding definition). Indeed, using the identity

$$\sum_{i,j=1}^{2} \frac{\partial^2 f(t)}{\partial t_i \partial t_j} \xi_i \xi_j = \frac{|\xi|^2 + (t_2\xi_1 - t_1\xi_2)^2}{(1+|t|^2)^{\frac{3}{2}}}$$

for $f(t) = \sqrt{1 + t_1^2 + t_2^2}$, we obtain

$$\beta(u)|\xi|^2 \le \sum_{i,j}^{2} \frac{\partial^2 f(t)}{\partial t_i \partial t_j}\Big|_{t=\nabla u} \xi_i \xi_j \le \frac{1}{\sqrt{1+|\nabla u|^2}} |\xi|^2, \quad \forall\, \xi \in \mathbb{R}^2,$$

where $\beta(u) > 0$, for instance $\beta(u) = (1+|\nabla u|^2)^{-\frac{3}{2}}$ is appropriate. But the right inequality shows that $\beta(u)$ cannot be separated from 0 uniformly in u.

The violation of the uniform ellipticity causes serious difficulties in the theoretical and numerical analysis of these problems, including the investigation of their solvability.

[3]Symbols $a \cdot b$ and $|a|$ stand for the inner product and the Euclidean norm of vectors in $\mathbb{R}^2$, respectively.

Remark 5 The following facts point implicitly to the nature of these difficulties:

- For the minimal surface problem with

$$\Omega := \{x \in \mathbb{R}^2 : 1 < \|x\| < 2\}, \qquad g = \begin{cases} 0 & \text{if } \|x\| = 2 \\ \gamma & \text{if } \|x\| = 1 \end{cases}$$

  there exists γ^* such that the problem is solvable for $\gamma \in [0, \gamma^*]$ and has no solution if $\gamma > \gamma^*$ (for this well-known example see, for instance, [4], Chapt. V);

- A necessary condition for the solvability of the Dirichlet problem for the equation of prescribed mean curvature is that

$$\left| \int_\omega p(x)dx \right| < mes\ \partial\omega$$

  holds for all proper subsets $\omega \subset \Omega$ (with Lipschitz continuous boundaries, $mes\ \partial\omega$ denotes the perimeter of ω), cf. [7]. ◇

The existence of a classical solution of problem (33) with continuous data was proved by T. Radó [32] in the case that Ω is a convex set. Conditions ensuring that the solution of (33) belongs to $C^{2,1}(\bar{\Omega})$ can be found in [20], Theorem IV.10.9.

For the minimal surface problem with an obstacle, but in the space $V = H_0^{1,\infty}(\Omega)$, Lewy and Stampacchia [22] have shown that the solution is in $W^{2,s}(\Omega) \cap C^1(\bar{\Omega})$, $1 \le s < \infty$, if $\psi \in C^2(\bar{\Omega})$ and Ω is a convex set with a smooth boundary.

The uniqueness of a solution (if it exists) in case K is given by (32) or (26) is a rather evident corollary of the strict convexity of the functional

$$J(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}dx \quad \text{on } K.$$

In turn, the strict convexity of J on K can be concluded by integration (over Ω) of the left inequality in (35) below given with $a = |\nabla u|$, $b = |\nabla v|$, where $u, v \in K$ (hence, $u - v \in H_0^1(\Omega)$).

Proposition 1 *The operator $\mathcal{Q}$ in (31) is Lipschitz continuous on V.*

Proof. For any $u, v, w \in V$ one gets

$$\begin{aligned}
&\left| \int_\Omega \left(\frac{1}{\sqrt{1+|\nabla u|^2}} - \frac{1}{\sqrt{1+|\nabla v|^2}} \right) \nabla u \cdot \nabla w dx \right| \\
&\quad \le \int_\Omega \left| \frac{(\nabla u - \nabla v)\cdot(\nabla u + \nabla v)}{\sqrt{1+|\nabla u|^2}\sqrt{1+|\nabla v|^2}\left(\sqrt{1+|\nabla u|^2} + \sqrt{1+|\nabla v|^2}\right)} \right| |\nabla u \cdot \nabla w| dx \\
&\quad \le \int_\Omega |\nabla u - \nabla v||\nabla w| dx \le \|u - v\|\|w\|,
\end{aligned}$$

whereas the Cauchy-Schwartz inequality implies

$$\left| \int_\Omega \frac{\nabla u - \nabla v}{\sqrt{1+|\nabla v|^2}} \cdot \nabla w dx \right| \le \left(\int_\Omega \frac{|\nabla u - \nabla v|^2}{1+|\nabla v|^2} dx \right)^{1/2} \|w\| \le \|u-v\| \|w\|.$$

Thus, we have

$$|\langle \mathcal{Q}(u) - \mathcal{Q}(v), w \rangle| \le 2\|u-v\|\|w\|, \quad \forall\, w \in V,$$

hence

$$\|\mathcal{Q}(u) - \mathcal{Q}(v)\|_{V'} \le 2\|u-v\|.$$

□

Proposition 2 *Suppose that a solution u of $VI(\mathcal{Q},K)$, with $\mathcal{Q}$ in (31) and K in (32) or (26), belongs to $W^{1,\infty}(\Omega)$. Then Assumption 3 is valid with $Y = W^{1,1}(\Omega)$, arbitrary $\rho > \|u\|$ and $\tau(s) := c(\rho)s^2$.*

Proof. Let us recall that u is the unique solution of $VI(\mathcal{Q},K)$, hence $S^* = \{u\}$.
The convexity of J implies

$$J(v) - J(u) \le \langle \mathcal{Q}(v), v-u \rangle, \quad \forall\, v \in V$$

and because $\langle \mathcal{Q}(u), v-u \rangle \ge 0$ holds true for $v \in K$, we have for all $v \in K$

$$\begin{aligned} \langle \mathcal{Q}(v), v-u \rangle &\ge J(v) - J(u) - \langle \mathcal{Q}(u), v-u \rangle \\ &= \int_\Omega \left[\sqrt{1+|\nabla v|^2} - \sqrt{1+|\nabla u|^2} - \frac{\nabla u \cdot (\nabla v - \nabla u)}{\sqrt{1+|\nabla u|^2}} \right] dx. \end{aligned} \tag{34}$$

With $a \in \mathbb{R}^2, b \in \mathbb{R}^2$ the identity

$$\begin{aligned} \sqrt{1+|a|^2} - \sqrt{1+|b|^2} &- \frac{b \cdot (a-b)}{\sqrt{1+|b|^2}} \\ &= \frac{|a-b|^2}{\sqrt{1+|b|^2}\left(\sqrt{1+|a|^2}\sqrt{1+|b|^2} + 1 + b\cdot a\right)}. \end{aligned}$$

is evident, and using the inequality

$$\sqrt{1+|a|^2}\sqrt{1+|b|^2} \ge 1 + b \cdot a,$$

this yields

$$\begin{aligned} \sqrt{1+|a|^2} - \sqrt{1+|b|^2} - \frac{b \cdot (a-b)}{\sqrt{1+|b|^2}} &\ge \frac{|a-b|^2}{2\,(1+|b|^2)\,\sqrt{1+|a|^2}} \\ &\ge \frac{1}{2(1+|b|^2)} \frac{|a-b|^2}{1+|a|^2}. \end{aligned} \tag{35}$$

From (35), given with $a := \nabla v$, $b := \nabla u$, and (34) we conclude that

$$\langle \mathcal{Q}(v), v-u \rangle \ge \frac{1}{2(1+M^2)} \int_\Omega \frac{|\nabla v - \nabla u|^2}{1+|\nabla v|^2} dx, \quad \forall\, v \in K, \tag{36}$$

where $M := \|u\|_{W^{1,\infty}(\Omega)}$. But the Cauchy-Schwarz inequality

$$\left|\int_\Omega zw dx\right| \le \left(\int_\Omega z^2 dx\right)^{1/2} \left(\int_\Omega w^2 dx\right)^{1/2},$$

applied with $z := \frac{|\nabla u - \nabla v|}{\sqrt{1+|\nabla v|^2}}$, $w := \sqrt{1+|\nabla v|^2}$, implies

$$\left(\int_\Omega |\nabla u - \nabla v| dx\right)^2 \le \int_\Omega \frac{|\nabla u - \nabla v|^2}{1+|\nabla v|^2} dx \cdot \int_\Omega (1+|\nabla v|^2) dx.$$

Together with (36), the latter inequality leads to

$$\langle \mathcal{Q}(v), v-u\rangle \ge \frac{1}{2(1+M^2)} \cdot \frac{\left(\int_\Omega |\nabla u - \nabla v| dx\right)^2}{\int_\Omega (1+|\nabla v|^2) dx}, \quad \forall\, v \in K. \tag{37}$$

For $v \in K$, $\|v\| \le \rho$, inequality (37) gives

$$\langle \mathcal{Q}(v), v-u\rangle \ge \frac{1}{2(1+M^2)(mes\ \Omega + \rho^2)} \left(\int_\Omega |\nabla u - \nabla v| dx\right)^2. \tag{38}$$

Now, we use the fact that the standard norm and seminorm of the space $W^{1,1}(\Omega)$ are equivalent on the subspace $W_0^{1,1}(\Omega)$. Because $(u-v)|_\Gamma = 0$ holds for any $v \in K$, this implies

$$\exists\, c > 0 : \int_\Omega |\nabla u - \nabla v| dx \ge c\|u-v\|_{W^{1,1}(\Omega)}, \quad \forall\, v \in K, \tag{39}$$

and the conclusion of Proposition 2 follows from (38) and (39) with

$$c(\rho) = \frac{c^2}{2(1+M^2)(mes\ \Omega + \rho^2)}.$$

□

It should be noticed that the embedding $H^1 \subset W^{1,1}$ is not compact.

Application of GPP-method to minimal surface problems

Considering the application of the GPP-method to $VI(\mathcal{Q}, K)$ (with $\mathcal{Q}$ in (31) and K in (32) or (26)), we suppose, as already mentioned, that Ω is a convex polygonal domain, $VI(\mathcal{Q}, K)$ is solvable and its solution u^* belongs to $H^2(\Omega)$.

As regularizing functional $r : v \mapsto \|v\|^2$ is used (in case $K^k \subset K\ \forall\, k$, the choice $r(v) = \|v\|^2_{H_0^1(\Omega)}$ may be preferable). Obviously, these functionals possess the property (16).

One can easily show that the operators

$$v \mapsto \mathcal{Q}(v) + \chi_k(\nabla r(v) - \nabla r(u^k))$$

in the subproblems (P^k) of GPP-method, in distinction to the operator $\mathcal{Q}$, are uniformly elliptic. This is the reason to speak about an elliptic regularization. Moreover, choosing a positive sequence $\{\chi_k\}$ separated from 0 (this is allowed by the conditions on the regularization parameter), the uniform ellipticity of these operators with a common constant of ellipticity is guaranteed. This is an important advantage in comparison with the classical elliptic regularization approach.

Applying the finite element method as described at the beginning of Section 4, we deal here with sets K^k given by (28) (but with $\Gamma_1 = \Gamma$) or (29). The inclusion $K^k \subset K$ is not very realistic in this case, therefore we have to check Assumption 2 modified as described in Remark 1.

The validity of (2-i) and (2-ii) (with $\mathcal{B} = 0$, $\tilde{\chi} = 0$) is obvious. To show (2-iii) for an arbitrary $w \in K$, if K is given by (26), one can rewrite K in the form

$$K = g + \{v \in H_0^1(\Omega) : v \geq \psi - g\}$$

and then follow the proof of Theorem 3.2 in [10], Sect. 1.2. This provides

$$\exists\ w^k \in K^k : \quad \lim_{k\to\infty} \|w^k - w\| = 0. \tag{40}$$

If K is given by (32), the relation (40) is well-known. The application of Proposition 1 and (40) yields

$$\lim_{k\to\infty} \|\mathcal{Q}(w^k) - \mathcal{Q}(w)\|_{V'} = 0.$$

Now we check the fulfillment of condition (a) in Remark 1 in the case that the set K is given by (32).

Denote ϕ_{I_k} the linear interpolant of a function ϕ on the triangulation $\mathcal{T}_k$. For an arbitrary $v^k \in K^k$ take $z^k(v^k) = v^k + g - g^k$, where $g^k := g_{I_k}$. Obviously, $z^k(v^k) \in K$. From Theorem 3.2.1 in [2], already for $g \in H^2(\Omega)$, the estimate

$$\|g - g^k\| \leq \bar{c}\|g\|_{H^2(\Omega)} h_k$$

holds with $\bar{c}$ independent of g, h_k and $\mathcal{T}_k$. Hence,

$$\|z^k(v^k) - v^k\| \leq \bar{c}\|g\|_{H^2(\Omega)} h_k,$$

i.e. condition (a) is guaranteed with $c_3 = \bar{c}\|g\|_{H^2(\Omega)}$.

However, if the set K is given by (26), we are able to prove only the weaker condition (b) in Remark 1. In this case, for an arbitrary $v^k \in K^k$, take

$$z^k(v^k) = \max\{v^k + g - g_{I_k}, \psi\}.$$

Then $z^k(v^k) \in K$, and with $g^k = g_{I_k}$, $\psi^k = \psi_{I_k}$ the relation

$$g - g^k \leq z^k(v^k) - v^k \leq \max\{g - g^k, \psi - \psi^k\} \tag{41}$$

holds on $\bar{\Omega}$.

Green's formula yields

$$\begin{aligned} \langle \mathcal{Q}(u^*), z^k(v^k) - v^k \rangle &= -\int_\Omega \operatorname{div} \frac{\nabla u^*}{\sqrt{1 + |\nabla u^*|^2}} (z^k(v^k) - v^k) dx \\ &\quad + \int_\Gamma \frac{\partial}{\partial n} \frac{\nabla u^*}{\sqrt{1 + |\nabla u^*|^2}} (z^k(v^k) - v^k) d\Gamma, \end{aligned} \tag{42}$$

where $\frac{\partial}{\partial n}$ denotes the normal derivative on Γ.

Assuming that $g \in C^2(\bar{\Omega})$, $\psi \in C^2(\bar{\Omega})$, Theorem 3.1 in [38] provides the estimates

$$\|g - g^k\|_{C(\bar{\Omega})} \leq c(g) h_k^2, \qquad \|\psi - \psi^k\|_{C(\bar{\Omega})} \leq c(\psi) h_k^2. \tag{43}$$

From (41)-(43) we conclude the fulfillment of the first part of condition (b) in Remark 1. The second part follows from the proof of Theorem II.2.3 in [8].

Now, we are ready to apply the convergence results from Section 3. Let us recall that Ω is a convex polygonal domain, the sets K^k are described by (28) (with $\Gamma_1 = \Gamma$) or (29), the functions g, ψ are sufficiently smooth and for the optimal solution u^* it is supposed that $u^* \in H^2(\Omega)$.

The following statement is an immediate corollary of the Theorems 1 and 2.

Theorem 3

(i) The problems $\{(P^k)\}$, corresponding to $VI(\mathcal{Q}, K)$ (with $\mathcal{Q}$ defined by (31) and K by (32) or (26)) are solvable.

(ii) Let the controlling sequences of GPP-method satisfy the conditions (1), (2) and $\sum_{k=1}^{\infty} \frac{\delta_k}{\chi_k} < \infty$.

a) If u^* belongs to $H^2(\Omega)$, then $u^k \rightharpoonup u^*$ in V.

b) If u^* belongs to $H^2(\Omega) \cap W^{1,\infty}(\Omega)$, then $\lim_{k\to\infty} \|u^k - u^*\|_{W^{1,1}(\Omega)} = 0$.

Remark 6 A quite similar analysis can be performed for GPP-method applied to the variational formulation of the Dirichlet problem for the equation of prescribed mean curvature. Here the operator $\mathcal{Q}$ is defined by

$$\langle \mathcal{Q}(u, v) \rangle = \int_\Omega \left(\frac{\nabla u \cdot \nabla v}{\sqrt{1 + |\nabla u|^2}} - pv \right) dx, \quad \forall\, u, v \in V$$

(cf. with (31)). Under the assumption that $u^* \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$, we also obtain

$$\lim_{k\to\infty} \|u^k - u^*\|_{W^{1,1}(\Omega)} = 0.$$

◇

4.2 Convection- diffusion problem

Formulations and properties of the problem

This problem arises in many areas such as the transport and diffusion of pollutants, simulation of oil extraction from underground reservoirs, heat transport problems in the convection-dominated case, etc.

Again, let Ω be an open domain in $\mathbb{R}^2$ with a Lipschitz continuous boundary Γ, which now is divided into disjoint connected pieces Γ_1 and Γ_2, $mes\ \Gamma_1 > 0$ ($\Gamma_2 = \emptyset$ is not excluded). We consider the convection-diffusion equation

$$-\epsilon \Delta u + b \cdot \nabla u + cu = f \quad \text{on } \Omega \tag{44}$$

with boundary conditions

$$u = g_1 \ \text{ on } \Gamma_1, \qquad \frac{\partial u}{\partial n} = g_2 \ \text{ on } \Gamma_2. \tag{45}$$

The functions $b = (b_1, b_2)$, c and f are supposed to be sufficiently smooth on Ω, $g_1, g_2 \in H^2(\Omega)$; $c \geq 0$ holds on Ω and ϵ is a small positive constant such that

$$0 < \epsilon << \|b\|_{[L^\infty(\Omega)]^2}. \tag{46}$$

The unknown function u may represent the concentration of a pollutant being transported along a stream moving at velocity b and also subject to diffusive effects. Alternatively, u may represent the temperature of a fluid moving along a heated wall. The relation (46) corresponds to the situation that the diffusion is a less significant physical effect than the convection. For instance, on a windy day a pollutant moves fast in the direction of the wind, whereas a spreading due to molecular diffusion remains small.

The relation (46) causes a so-called boundary layer: a fast variation of the gradient of the solution near a part of the boundary. Such problems are called singularly perturbed. This peculiarity is illustrated in the following slightly modified example from [5].

Example 1 The equation

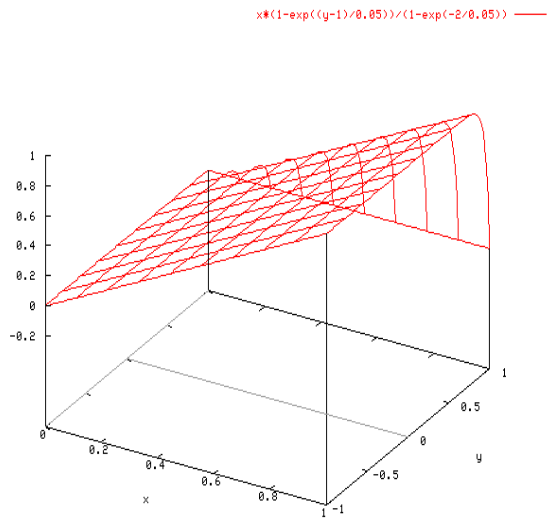
$$-\epsilon\Delta u + \frac{\partial u}{\partial x_2} = 0 \quad \text{on } \Omega := (0,1) \times (-1,1)$$

is considered subject to Dirichlet boundary conditions

$$\begin{aligned} u(x_1,-1) &= x_1, & u(x_1,1) &= 0 \\ u(0,x_2) &= 0, & u(1,x_2) &= \frac{1-\exp((x_2-1)/\epsilon)}{1-\exp(-2/\epsilon)} \\ & & &\text{(i.e. } u(1,x_2) \in [0,1] \ \forall\ x_2 \in (-1,1)). \end{aligned}$$

The unique solution of this problem is (cf. Figure)

$$u(x_1,x_2) = x_1 \frac{1-\exp((x_2-1)/\epsilon)}{1-\exp(-2/\epsilon)}.$$



One can easily see that, for small ϵ, this solution is very close to the function x_1 except near the boundary part $x_2 = 1$. But,

$$\frac{\partial u(x)}{\partial x_2} = -x_1 \frac{\exp((x_2-1)/\epsilon)}{1-\exp(-2/\epsilon)}\ \epsilon^{-1},$$

and for any $x_1 > 0,\ d > 0$

$$\lim \frac{\partial u(x)}{\partial x_2} = -\infty \quad \text{if } x_2 = 1 - d\epsilon,\ \epsilon \to 0.$$

$\diamond$

In general, in the most part of the domain the solution of problem (44), (45) is close to the solution of the reduced (hyperbolic) equation

$$b \cdot \nabla u + cu = f \tag{47}$$

with appropriate boundary conditions. If $\Gamma_- \subset \Gamma_1$, where $\Gamma_- = \{x \in \Gamma : b \cdot n < 0\}$ is a so-called inflow boundary (n denotes the outward-pointed unit vector normal to Γ), then this boundary condition is

$$u = g \quad \text{on} \quad \Gamma_-. \tag{48}$$

Boundary layers arise near an outflow boundary $\Gamma_+ = \{x \in \Gamma : b \cdot n > 0\}$ and a characteristic boundary $\Gamma_0 = \{x \in \Gamma : b \cdot n = 0\}$, where the solutions of the problems (44), (45) and (47), (48) can differ significantly, and the boundary layer functions ($x_1 \exp((x_2-1)/\epsilon)$ in our example) characterize approximately the difference between these solutions.

The presence of boundary layers causes serious difficulties for the applications of discretization techniques (finite-difference- and finite element methods) to convection-diffusion problems. There are numerous publications dealing with special discretization procedures and special algorithms for solving discretized convection-diffusion problems (see [5], [35] and references therein).

Introducing the space $V = \{v \in H^1(\Omega) : u|_{\Gamma_1} = 0\}$ with the norm $\|v\| := \|\nabla v\|_{[L^2(\Omega)]^2}$ (see [2], Theorem 1.2.1 concerning the equivalence of this norm and the standard norm of $H^1(\Omega)$ in case $mes\ \Gamma_1 > 0$), one can describe a weak formulation of problem (44), (45) as follows:

find $u \in V$ *such that*

$$\epsilon \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega [(b \cdot \nabla u)v + cuv]dx = \int_\Omega \bar{f} v dx + \epsilon \int_{\Gamma_2} \bar{g} v d\Gamma,\ \forall\ v \in V, \tag{49}$$

where $\bar{f} = f + \epsilon \Delta g_1 - b \cdot \nabla g_1 - cg_1$, $\bar{g} = g_2 - \frac{\partial g_1}{\partial n}$.

Applying the trace inequality

$$\|v\|_{L^2(\Gamma)} \le c_1 \|v\|, \quad \forall\ v \in H^1(\Omega) \tag{50}$$

(see, for instance, [2], Section 1.2) to estimate the term $\epsilon \int_{\Gamma_2} \bar{g} v d\Gamma$, the continuity of the functional

$$v \mapsto \int_\Omega \bar{f} v dx + \epsilon \int_{\Gamma_2} \bar{g} v d\Gamma$$

in the space V can be easily concluded, hence

$$\exists\, l \in V' : \quad \langle l, v\rangle = \int_\Omega \bar{f} v dx + \epsilon \int_{\Gamma_2} \bar{g} v d\Gamma, \quad \forall\, v \in V. \tag{51}$$

The estimate

$$\left| \int_\Omega (b \cdot \nabla u) v dx \right| \leq d \sup_{x\in\Omega} |b(x)| \|u\| \|v\|$$

(with $d : \|v\|_{L^2(\Omega)} \leq d\|v\| \ \forall\, v \in V$)
is proved by using twice the Cauchy-Schwarz inequality. Now the continuity of the bilinear form in (49) follows in a standard way. Thus, according to the Riesz representation theorem, there exists an operator $\mathcal{A} \subset \mathcal{L}(V, V')$ such that

$$\langle \mathcal{A}u, v\rangle = \epsilon \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega (b \cdot \nabla u) v dx + \int_\Omega cuv dx. \tag{52}$$

Its strong monotonicity can be shown under the additional assumption that

$$c - \frac{1}{2}\text{div } b \geq 0 \ \text{ on } \Omega, \quad \text{and} \ \ \Gamma_2 \subset \Gamma_+$$

which is supposed in the sequel.

Indeed, applying Green's formula to

$$\alpha(u, v) := \int_\Omega (b \cdot \nabla u) v dx$$

we obtain

$$\begin{aligned}
\alpha(u, v) &= -\int_\Omega u \text{div}(vb) dx + \int_{\Gamma_2} uv(b \cdot n) d\Gamma \\
&= -\int_\Omega uv \text{div} b dx - \int_\Omega (b \cdot \nabla v) u dx + \int_{\Gamma_2} uv(b \cdot n) d\Gamma \\
&= -\int_\Omega uv \text{div} b dx - \alpha(v, u) + \int_{\Gamma_2} uv(b \cdot n) d\Gamma.
\end{aligned}$$

Thus

$$\alpha(u, u) = -\frac{1}{2} \int_\Omega u^2 \text{div} b dx + \frac{1}{2} \int_{\Gamma_2} u^2 (b \cdot n) d\Gamma,$$

and $b \cdot n > 0$ in Γ_2 holds because of $\Gamma_2 \subset \Gamma_+$. Now

$$\alpha(u, u) + \int_\Omega cu^2 dx \geq \int_\Omega (c - \frac{1}{2}\text{div} b) u^2 dx \geq 0,$$

and according to (52)

$$\langle \mathcal{A}u, u\rangle \geq \epsilon \|u\|^2, \quad \forall\, u \in V. \tag{53}$$

From the monotonicity and continuity of the operator $\mathcal{A}$ it follows that $\mathcal{A}$ is maximal monotone, and together with (53) this guarantees the existence of a unique $u^* \in V$ such that

$$\langle \mathcal{A}u^* - l, v\rangle = 0, \quad \forall\, v \in V.$$

Because $\alpha(u, v) \neq \alpha(v, u)$, the operator is not symmetric, hence problem (44), (45) cannot be transformed - at least not in a natural way - into an optimization problem.

Conditions on the data of problem (44), (45), which provide $u^* \in H^2(\Omega)$, can be found in [9], [20]. In particular, $u^* \in H^2(\Omega)$ holds in the case that Ω is a convex polygonal domain, $\Gamma_1 = \Gamma$, functions $b,\ c,\ g_1$ are sufficiently smooth, and $f \in L^2(\Omega)$ (see [20], Theorem III.9.1 and Remark III.9.4).

Application of GPP-method to convection-diffusion problems

Applying the GPP-method with $r : v \mapsto \|v\|^2$ and an appropriate parameter sequence $\{\chi_k\}$, we approximate the singularly perturbed elliptic problem (44), (45) by a sequence of problems with unperturbed elliptic operators. Remind that this is not attainable by means of the classical approach of elliptic regularization. On this way, the boundary layers (also inner layers if exist; see [35] for this notion) will be accumulated gradually, because of the term $-\chi_k \nabla r(u^k)$ in the operator of problem (P^k).

In particular, for problem (44), (45) with $\Gamma_1 = \Gamma$, $g_1 \equiv 0$, the exact problem (P^k) (with $\delta_k = 0$) consists in the finding of a weak solution of the equation

$$-(\epsilon + 2\chi_k)\Delta u + b \cdot \nabla u + cu = f + 2\chi_k \Delta u^k, \quad \text{in } H_0^1(\Omega).$$

The gradual accumulation of boundary- and inner layers allows us a more successful application of standard finite element methods, and with $\epsilon + 2\chi_k$ in place of ϵ, we obtain a better stability and conditioning of the discretized problems.

Remark 7 In [35], authors analyze situations where boundary- and inner layer functions can be defined a priori - sometimes in explicit form - by using a standard technique from the singular perturbation theory. If such functions are known (exact or approximately), they can be used to choose a starting point in the GPP-method or to correct an approximate solution after certain number of iterations. $\diamond$

Now, we examine the application of the convergence results from Section 3 to the GPP-method for solving the convection-diffusion problem in the form (49). As in the previous case, it is supposed that Ω is a convex polygonal set and that the solution u^* belongs to $H^2(\Omega)$. So, we deal with auxiliary problems (P^k) in the space $V = \{v \in H^1(\Omega) : u|_{\Gamma_1} = 0\}$, in which

$$\mathcal{Q} : v \mapsto \mathcal{A}v - l, \quad \varphi_k \equiv 0, \quad r : v \mapsto \|v\|^2$$

and K^k are given by (28).

Obviously, in this case $K^k \subset K := V$, (2-i) and (16) are satisfied. Assumption (1-ii) is valid with the operator $\mathcal{B} = -\epsilon\Delta$, and in (2-ii) one can take $\tilde{\chi} = 0$. The relation

$$\forall\ w \in K,\ \ \forall\ k,\ \exists w^k \in K^k : \quad \lim_{k\to\infty} \|w - w^k\| = 0$$

follows immediately from the proof of Theorem 3.3 in [10], Section 1.1., and therefore, the continuity of the operator $\mathcal{A}$ implies the validity of (2-iii).

Next, because $u^* \in H^2(\Omega)$, the estimate

$$\|u^* - u^*_{I_k}\| \leq c\|u^*\|_{H^2(\Omega)} h_k, \quad \forall\ k$$

is known, and taking into account that $u^*_{I_k} \in K^k$ and

$$\|\mathcal{Q}(u^*) - \mathcal{Q}(u^*_{I_k})\| \leq \|\mathcal{A}\|_{V'} \|u^* - u^*_{I_k}\|,$$

assumption (2-v) is valid with $\sigma_k := h_k$ if $\sum_{k=1}^{\infty} \frac{h_k}{\chi_k} < \infty$.

Finally, since the operator $\mathcal{B} = -\epsilon\Delta$ is strongly monotone on V, condition (iii) in Theorem 1 holds with $\hat{\mathcal{B}} = 0$.

Therefore, choosing the controlling parameters according to (1), (2) and $\sum_{k=1}^{\infty} \frac{\delta_k}{\chi_k} < \infty$, one can use Theorem 1, which guarantees that the iterates u^k of the GPP-method converge to u^* strongly in V.

4.3 Problems in linear elasticity theory

Now we consider briefly some other applications of the GPP-method.

In [11], [12] the proximal point method was developed for solving variational inequalities in elasticity theory: *two-body contact problems without friction* and *static problems of linear elasticity with given friction* have been investigated.

In distinction to the GPP-method described here, in [11], [12] a so-called multi-step proximal point method has been studied, where at each discretization level proximal iterations are repeated within a special efficiency criterion. This allows one to obtain better approximate solutions at each discretization level. However, a certain unconventional information about the variational inequality $VI(\mathcal{Q}, \varphi, K)$ is needed, in particular, an upper bound d for the norm of some solution and bounds for the image of the operator $\mathcal{Q} + \partial\varphi$ on the set $K \cap \mathbb{B}_d$ (see also [14], [13]).

The static problem of linear elasticity with given friction

At first we check the application of the GPP-method to the static problem of linear elasticity with given friction. For its mechanical interpretation and theoretical analysis see [3], Chapt. 3.

Let $\Omega \subset \mathbb{R}^2$ be as in previous subsections, Γ_c be a connected part of the boundary Γ, $S \in [L^\infty(\Gamma_c)]^2$ be a given vector-function; the elasticity coefficients a_{klmp} $(k, l, m, p = 1, 2)$ are assumed to be measurable and bounded on Ω. Moreover, the symmetry

$$a_{klmp} = a_{lkmp} = a_{mp,kl}$$

is supposed as well as the existence of a positive constant α_0 such that

$$a_{klmp}(x)\sigma_{kl}\sigma_{mp} \geq \alpha_0 \sigma_{kl}\sigma_{kl} \tag{54}$$

holds for all symmetric matrices $(\sigma_{kl})_{k,l=1,2}$ and almost every $x \in \Omega$. In this description, summation over repeated indices is always assumed.

Applying the notation

$$V = [H^1(\Omega)]^2, \quad S_n = S \cdot n, \quad u_t = u - (u \cdot n)n$$

(n - outward unit normal to Γ_c),

$$\begin{aligned}
\epsilon_{kl}(u) &= \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right), \quad k, l = 1, 2, \\
\langle \mathcal{A}u, v\rangle &= \int_\Omega a_{klmp}\epsilon_{kl}(u)\epsilon_{mp}(v)d\Omega, \quad \forall\, u, v \in V, \\
\mathcal{Q}(v) &= \mathcal{A}v - l, \qquad (l \in V' \text{ is given}), \\
\varphi(u) &= \int_{\Gamma_c} \mu|S_n|\,|u_t|d\Gamma, \qquad (\mu > 0 \text{ a given constant}),
\end{aligned}$$

the problem is formulated as the variational equality

$$find\ u \in V,\ p \in \partial\varphi(u): \quad \langle \mathcal{Q}(u) + p, v\rangle = 0, \quad \forall\, v \in V. \tag{55}$$

The kernel $\ker(\mathcal{A})$ of the operator $\mathcal{A}$ on the space V has the structure

$$\ker(\mathcal{A}) : \{z = (z_1, z_2) :\ z_1 = a_1 - bx_2,\ z_2 = a_2 + bx_1\}$$

with arbitrary $a_1, a_2, b \in \mathbb{R}$.

About the solvability of problem (55) the following result is known.

Proposition 3 (cf. [3])

(i) The condition

$$|\langle l, v\rangle| \le \varphi(v), \quad \forall\, v \in \ker(\mathcal{A})$$

is necessary for the solvability of problem (55).

(ii) A solution exists if

$$|\langle l, v\rangle| < \varphi(v), \quad \forall\, v \in \ker(\mathcal{A}),\ v \neq 0.$$

(iii) If u^* and u^{**} are two solutions of (55), then $u^* - u^{**} \in \ker(\mathcal{A})$.

Applying the GPP-method to problem (55), we approximate the convex functional φ by a sequence of convex continuously differentiable functionals

$$\varphi_k : v \mapsto \int_{\Gamma_c} \mu|S_n|\sqrt{|v_t|^2 + h_k^2}\,d\Gamma.$$

With the use of the trace inequality (50), one can conclude that for all $u, v \in V$

$$\begin{aligned}
|\varphi(u) - \varphi(v)| &\le \int_{\Gamma_c} \mu|S_n|\,|u_t - v_t|d\Gamma \le \int_{\Gamma_c} \mu|S_n|\,|u - v|d\Gamma \\
&\le \left(\int_{\Gamma_c} (\mu|S_n|)^2 d\Gamma\right)^{1/2} \|u - v\|_{[L^2(\Gamma_c)]^2} \le c_0\|u - v\|,
\end{aligned} \tag{56}$$

i.e., the functional φ is Lipschitz continuous on V.
Assumption (2-iv) and the estimate

$$\varphi_k(v) - \varphi(v) \le h_k \int_{\Gamma_c} \mu|S_n|d\Gamma \tag{57}$$

are obviously valid.

A successive approximation of V by a sequence of subspaces $\{V^k\}$ can be performed as above, by using the finite element method on a sequence of triangulations $\mathcal{T}_k$ with parameter h_k; namely,

$$V^k := \{v \in [C(\bar{\Omega})]^2 : \; v|_T \in [\mathfrak{P}_1(T)]^2, \quad \forall \; T \in \mathcal{T}_k\}$$

and $K^k := V^k$.

For the following estimates we keep the assumption that Ω is a polygonal set and refer to [10], Chapt. 2 for approximations on non-polygonal domains. We also suppose that some solution u^* of problem (55) belongs to $[H^2(\Omega)]^2$. Then, according to Proposition 3, $u \in [H^2(\Omega)]^2$ holds for any solution of (55).

The inclusion $K^k \subset K := V$ is evident. Moreover, the relations

$$w \in K \quad \Rightarrow \quad w_{I_k} \in K^k \; \forall \; k, \qquad \lim_{k\to\infty} \|w - w_{I_k}\| = 0, \tag{58}$$

and

$$v \in [H^2(\Omega)]^2 \subset K \quad \Rightarrow \quad \|v - v_{I_k}\| \le \bar{c}\|v\|_{[H^2(\Omega)]^2} h_k, \; \forall \; k \tag{59}$$

hold true. Now using (56)-(59) together with

$$\|\mathcal{Q}(v) - \mathcal{Q}(u)\| \le \|\mathcal{A}\|_{V'} \|u - v\|, \quad \forall \; u, v \in V,$$

we immediately conclude the fulfillment of the assumptions (2-v) (for any solution of (55) and with $\sigma_k = h_k$ in (3)) and (2-iii).

Taking into account relation (54), the operator

$$\mathcal{B} : \quad \langle \mathcal{B}u, v\rangle = \alpha_0 \int_\Omega \epsilon_{kl}(u)\epsilon_{kl}(v)dx, \quad \forall \; u, v \in V$$

satisfies (1-ii), and the second Korn inequality (see, for instance, [31], Chapt. 1) allows one to guarantee the validity of assumption (2-ii) with the regularizing functional

$$r : u \mapsto \|u\|^2_{[L_2(\Omega)]^2}, \tag{60}$$

which meets also assumption (2-i). Of course, $r : u \mapsto \|u\|^2$ is also a possible choice, but (60) is more preferable from the numerical point of view.

Remark 8 Using the regularizing functional (60) we recover here the proximal method with weak regularization (cf. [11], [17]).Some applications and numerical results showing a significant acceleration of the convergence compared with the standard proximal regularization can be found in [36], [37]. ◇

Thus, we have established the fulfillment of Assumptions 1 and 2. In order to apply Theorem 1, let us mention that (16) is evident and the condition on the operator $\hat{\mathcal{B}}$ in Theorem 1 is valid for

$$\hat{\mathcal{B}} : \quad \langle \hat{\mathcal{B}}u, v\rangle = \int_\Omega uv d\Omega, \quad \forall \; u, v \in V.$$

On this way we obtain the following result.

Theorem 4

(i) The problems (P^k), corresponding to the variational equation (55), are solvable.

(ii) If the controlling sequences $\{\chi_k\}$, $\{h_k\}$, $\{\delta_k\}$ of the GPP-method satisfy (1), (2) and $\sum_{k=1}^{\infty} \frac{\delta_k}{\chi_k} < \infty$ and a solution of (55) belongs to $[H^2(\Omega)]^2$, then the iterates of this method converge strongly in V to some solution of (55).

The two-body contact problem

The two-body contact problem in the form of a variational inequality $VI(\mathcal{Q}, K)$ was studied in [12] under the following assumptions:

- $\Omega', \Omega'' \subset \mathbb{R}^2$ are bounded polyhedral domains, $\Omega' \cap \Omega'' = \emptyset$, with a common boundary part $\Gamma_c = \partial\Omega' \cap \partial\Omega''$, $mes\ \Gamma_c > 0$;
- $V := \{(v', v'') \in [H^1(\Omega')]^2 \times [H^1(\Omega'')]^2 :\ v' = 0 \text{ on } \Gamma_u\}$, where $\Gamma_u \in \partial\Omega'$, $\Gamma_u \cap \Gamma_c = \emptyset$ and $mes\ \Gamma_u > 0$;
- $K := \{v \in V :\ v' \cdot n - v'' \cdot n \leq 0 \text{ on } \Gamma_c\}$, n denotes the unit normal to Γ_c pointed outward Ω';
- $\mathcal{Q} : v \mapsto \mathcal{A}v - l$, where $l \in V'$ and $\mathcal{A} : V \to V'$ is a linear continuous and monotone operator with a finite dimensional kernel on $[H^1(\Omega')]^2 \times [H^1(\Omega'')]^2$ such that

  $$\exists\, m_0 > 0 : \quad \langle \mathcal{A}v, v\rangle + \|v''\|_{[L^2(\Omega'')]^2} \geq m_0 \|v\|^2, \quad \forall\, v \in V$$

  (in [12], $VI(\mathcal{Q}, K)$ is considered with a certain linear elasticity operator $\mathcal{Q}$ possessing these properties).

The use of the finite element method, like described in Subsection 4.2 of the mentioned paper, provides the inclusion $K^k \subset K\ \forall\, k$. If some solution of $VI(\mathcal{Q}, K)$ belongs to $[H^1(\Omega')]^2 \times [H^1(\Omega'')]^2$, then choosing the regularization functional

$$r : \quad (v', v'') \mapsto \|v''\|^2_{[L^2(\Omega)]^2}$$

and following the analysis of the previous problem, one can satisfy all conditions of Theorem 1. This guaranties the strong convergence in V for the iterates of the GPP-method to some solution of the two-body contact problem.

5 Conclusion

Typically, standard discretization methods in mathematical physics are not efficient when applied to degenerate and singularly perturbed elliptic variational inequalities, and there are numerous investigations addressed to the creation of special discretization procedures, for instance finite element methods with upwinding, streamline diffusion finite element methods, etc. [5], [35]. Also special algorithms for solving the arising discretized problems are needed.

In this paper, we develop a quite different idea, which may be presented as follows: Using the proximal regularization, the original variational inequality is approximated by a sequence of uniformly elliptic problems, which can be treated with standard finite element techniques and standard solvers. Moreover, only a single discretization is used for each regularized problem, with a mild rule for decreasing of the triangulation parameter in the outer process.

In various schemes of proximal point methods the conditions on data approximation are of Mosco's type with order $\alpha > 0$ (see [28] for the definition as well as [21], [1]), or outer approximations of the set K are used [16]. These conditions are certainly not suitable if we deal with problems in mathematical physics and use finite element or finite-difference methods. Indeed, in this case $K^k \not\supset K$, and for an arbitrary element $u \in K$ at best the relation

$$\lim_{k\to\infty} \min_{v\in K^k} \|u - v\| = 0$$

can be concluded (without any estimate for the rate of convergence), i.e., the Mosco convergence with order $\alpha > 0$ cannot be guaranteed.

Abstract assumptions admitting the described peculiarity were introduced first in [14]. Here we use a weaker form of these assumptions.

Growth conditions used so far to obtain a more qualitative convergence of proximal point methods (see for instance [34], [25], [15]) are not fulfilled in the case of the operator $\mathcal{Q}$ in the minimal surface problem. Therefore, the known convergence results provide only weak convergence in $H^1(\Omega)$ of the proximal point methods when applied to this problem or related variational inequalities studied in Subsection 4.1. Theorem 2 of the present paper, based on a new growth condition (Assumption 3), allows to establish a convergence of the iterates of the GPP-method for these problems in the norm of the space $W^{1,1}(\Omega)$, see also Theorem 3 for the exact result.

We conclude with the remark that an application of the elliptic proximal regularization method (without discretization) for solving parabolic variational inequalities was studied in [15].

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