Semi-abelian Categories and Exactness

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Abstract

We show that every semi-abelian category, as defined by Palamodov, possesses a maximal exact structure in the sense of Quillen and that the exact structure of a quasi-abelian category is a special case thereof.

1 Introduction

Palamodov [5] introduced the notion of semi-abelian category as an additive category with kernels and cokernels where the canonical morphism $\tilde{f} : coim(f) \to im(f)$ associated to every morphism $f : X \to Y$ is both an epimorphism and a monomorphism.
There is another, non-additive notion of semi-abelian category in the literature introduced by Janelidze, Màrki and Tholen [3], namely that of a category that is pointed, Barr exact and protomodular with binary products. A semi-abelian category in their sense is additive if and only if it is abelian.
This article is concerned with Palamodov's notion of a semi-abelian category. Quillen [6] introduced the notion of an exact category, which is an additive category C together with a distinguished class $\mathcal{E}$ (called an exact structure on C) of kernel-cokernel pairs, subject to some closure requirements. On every additive category C there exists a smallest exact structure $\mathcal{E}_{min}$, namely the class of all split exact sequences, but in general there is no largest exact structure on C. In this article we show that if C is semi-abelian, such a largest exact structure $\mathcal{E}_{max}$ exists.
Furthermore it follows that a semi-abelian category together with the exact structure $\mathcal{E}_{max}$ is a generalisation of the notion of quasi-abelian category and its largest exact structure (which contains all kernel-cokernel pairs), as used in [7]. Finally we give an example of a semi-abelian category that is not quasi-abelian.

2 Semi-Abelian Categories

In this section we recall the notion of a semi-abelian category and give an overview of their basic properties.
If C is an additive category with kernels and cokernels and $f : X \to Y$ is a morphism in C we will always write $k_f : ker(f) \to X$ for its kernel, $c_f : Y \to coker(f)$ for its cokernel, $ci_f : X \to coim(f)$ for its coimage and $i_f : im(f) \to Y$

for its image. For every such morphism f there is a unique morphism $\tilde{f}$: $coim(f) \to im(f)$ making the following diagram commutative:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "ci_f"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \\ coim(f) \arrow[r, "\tilde{f}"'] & im(f) \arrow[u, "i_f"'] \end{tikzcd}$$

The category C is abelian if the morphism $\tilde{f}$ is always an isomorphism. The following weaker notion was introduced by Palamodov in [5]:

Definition 1 An additive category C with kernels and cokernels is called *semi-abelian* if for every morphism $f : X \to Y$ the canonical morphism $\tilde{f}$: $coim(f) \to im(f)$ is both an epimorphism and a monomorphism.

The dual category C^{op} of a semi-abelian category C is also a semi-abelian category, since the morphism $\tilde{f} : coim(f) \to im(f)$ remains the same.
In analogy to [7] we define a morphism $f : X \to Y$ in a semi-abelian category to be *strict* if the canonical morphism $\tilde{f} : coim(f) \to im(f)$ is an isomorphism.
The following is an important special case of semi-abelian categories (cf. [7, Corollary 1.1.5]):

Definition 2 An additive category C with kernels and cokernels is called *quasi-abelian* if it satisfies the following dual axioms:

(QA) If $\begin{tikzcd} Y \arrow[r, "g"] & Z \\ P \arrow[u, "p_Y"] \arrow[r, "p_T"'] \arrow[ur, phantom, "\circlearrowleft"] & T \arrow[u, "t"'] \end{tikzcd}$ is a pullback square and g a strict epimorphism,

the morphism p_T is also a strict epimorphism.

$(QA)^*$ If $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "s_Y"] \\ T \arrow[r, "s_T"'] & S \end{tikzcd}$ is a pushout square and f a strict monomorphism,

the morphism s_T is also a strict monomorphism.

As shown in [7], every quasi-abelian category is semi-abelian, but the converse is not true in general (a counterexample will be given later in Example 19).
The following holds true in any additive category with kernels and cokernels, as noted in [7]:

Remark 3 i) The kernel $k_f : ker(f) \to X$ of any morphism $f : X \to Y$ in C is a strict monomorphism and its cokernel $c_f : Y \to coker(f)$ is a strict epimorphism.

ii) A morphism is a strict epimorphism if and only if it is the cokernel of its kernel and it is a strict monomorphism if and only if it is the kernel of its cokernel.

iii) A morphism $f : X \to Y$ is strict if and only if it factors as $f = j \circ h$ where j ist a strict monomorphism and h is a strict epimorphism.

In a semi-abelian category we get some useful results for compositions and factorisations of strict epimorphisms and monomorphisms, analogous to those proved in [7]. The following results were shown in [8], we will provide the proofs nonetheless for the sake of completeness.

Proposition 4 Let C be a semi-abelian category.

i) If $f : X \to Y$ is a strict monomorphism and f factors as $f = g \circ h$, then h is also a strict monomorphism.

ii) If $f : X \to Y$ is a strict epimorphism and f factors as $f = g \circ h$, then g is also a strict epimorphism.

iii) If $f : X \to Y$ and $g : Y \to Z$ are strict monomorphisms (resp. epimorphisms), then $g \circ f$ is a strict monomorphism (resp. epimorphism).

iv) If $f : X \to Y$ is a strict morphism, $g : W \to X$ a strict epimorphism and $h : Y \to Z$ a strict monomorphism, then $f \circ g$ and $h \circ f$ are strict morphisms.

Proof: ii) is the dual statement of i) and iv) follows from iii) and Remark 3 iii), so it suffices to show i) and iii).
i) It is obvious, that $h : X \to H$ is a monomorphism. Therefore $id_X : X \to X$ is a coimage of f, hence $h = i_h \circ \tilde{h}$. Similarly $f = i_f \circ \tilde{f}$.
Let $c_f : Y \to coker(f)$ be the cokernel of f, then

$$c_f \circ g \circ i_h \circ \tilde{h} = c_f \circ g \circ h = c_f \circ f = 0$$

and since $\tilde{h}$ is an epimorphism it follows that $c_f \circ g \circ i_h = 0$. The universal property of $im(f)$ gives rise to a unique morphism $v : im(h) \to im(f)$ with $i_f \circ v = g \circ i_h$. Since

$$i_f \circ v \circ \tilde{h} = g \circ i_h \circ \tilde{h} = g \circ h = f = i_f \circ \tilde{f}$$

and because i_f is a monomorphism, we have $v \circ \tilde{h} = \tilde{f}$, hence $\tilde{f}^{-1} \circ v \circ \tilde{h} = id_{coim(h)}$. Then

$$\tilde{h} \circ \tilde{f}^{-1} \circ v \circ \tilde{h} = \tilde{h}$$

and since $\tilde{h}$ is both an epimorphism and a monomorphism, it follows that $\tilde{h}$ is an isomorphism, hence h is a strict monomorphism.
iii) Since f and g are strict monomorphisms they are their own images by Remark 3 and since $g \circ f$ is a monomorphism we have $g \circ f = i_{g \circ f} \circ \widetilde{g \circ f}$. Then $c_g \circ i_{g \circ f} \circ \widetilde{g \circ f} = c_g \circ g \circ f = 0$, hence $c_g \circ i_{g \circ f} = 0$ since $\widetilde{g \circ f}$ is an epimorphism. The universal property of $im(g)$ gives rise to a unique morphism $v : im(g \circ f) \to Y = im(g)$ with $i_{g \circ f} = g \circ v$. Then it follows from $g \circ f = i_{g \circ f} \circ g \tilde{\circ} f = g \circ v \circ \widetilde{g \circ f}$ that $f = v \circ \widetilde{g \circ f}$, since g is a monomorphism and from $c_f \circ v \circ \widetilde{g \circ f} = c_f \circ f = 0$ it follows that $c_f \circ v = 0$, since $\widetilde{g \circ f}$ is an

epimorphism. The universal property of $im(f)$ then yields a unique morphism $w : im(g \circ f) \to X$ with $v = f \circ w$.
If $t : T \to Z$ is any morphism with $c_{g\circ f} \circ t = 0$, then there is a unique morphism $\lambda : T \to im(g \circ f)$ with $t = i_{g\circ f} \circ \lambda$. Then

$$t = i_{g\circ f} \circ \lambda = g \circ v \circ \lambda = g \circ f \circ w \circ \lambda$$

and since $g \circ f$ is a monomorphism, the morphism $w \circ \lambda$ is unique with $t = g \circ f \circ w \circ \lambda$. Therefore $g \circ f$ is its own image and hence by Remark 3 a strict monomorphism.

Any additive category C with kernels and cokernels possesses also pullbacks and pushouts. In addition, morphisms under pullbacks have isomorphic kernels and morphisms under pushouts have isomorphic cokernels:

Lemma 5 Let C be an additive category.

i) If $g : Y \to Z$, $t : T \to Z$ are morphisms in C and (P, p_T, p_Y) is their pullback, then there is a morphism $j : X \to P$ making the diagram

$$\begin{tikzcd}
ker(g) \arrow[r, "k_g"] & Y \arrow[r, "g"] & Z \\
ker(g) \arrow[u, "id"] \arrow[r, "j"'] \arrow[ur, phantom, "\circlearrowleft"] & P \arrow[u, "p_Y"'] \arrow[r, "p_T"'] \arrow[ur, phantom, "\circlearrowleft"] & T \arrow[u, "t"']
\end{tikzcd}$$

commutative and j is a kernel of p_T.

ii) If $f : X \to Y$, $t : X \to T$ are morphisms in C and (S, s_T, s_Y) is their pushout, then there is a morphism $C : S \to S$ making the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r, "c_f"] \arrow[d, "s_Y"] \arrow[dr, phantom, "\circlearrowleft"] & coker(f) \arrow[d, "id"] \\
T \arrow[r, "s_T"'] & S \arrow[r, "c"'] & coker(f)
\end{tikzcd}$$

commutative and c is a cokernel of s_T.

Proof: ii) is the dual statement of i), so ist suffices to show i).
Since $g \circ k_g = 0 = t \circ o$, the universal property of (P, p_T, p_Y) gives rise to a unique morphism $j : ker(g) \to P$ with $p_Y \circ j = k_G$ and $p_T \circ j = 0$. We show that j is a kernel of p_T: Let $h : H \to P$ be a morphism with $p_T \circ h = 0$. Then $g \circ p_Y \circ h = t \circ p_T \circ h = 0$, therefore the universal property of $ker(g)$ yields a unique morphism $\lambda : H \to ker(g)$ with $p_Y \circ h = k_G \circ \lambda$, hence $p_Y \circ h = p_Y \circ j \circ \lambda$ and $p_T \circ h = 0 = p_T \circ j \circ \lambda$, so the universal property of (P, p_T, p_Y) shows $h = j \circ \lambda$ and λ is unique with this property. This shows that j is a kernel of p_T.

Pullbacks and pushouts in a semi-abelian category have many useful properties, a thorough discussion of which can be found in [8]. The following proposition, which is also shown [8], will be of importance in our context:

Proposition 6 Let C be a semi-abelian category and

$$\begin{tikzcd} (*)\ X \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "s_Y"] \\ T \arrow[r, "s_T"'] & S \end{tikzcd}$$

be a pushout square.

i) If f is a strict epimorphism, then s_T is a strict epimorphism.

ii) If f or t is a strict monomorphism, then $(*)$ is also a pullback diagram.

Proof: i) By Proposition 4 it suffices to show that s_T is its own coimage. Let $r : T \to R$ be a morphism with $r \circ k_{s_T} = 0$. The diagram $(*)$ induces a unique morphism $\lambda : ker(f) \to ker(s_T)$ with $t \circ k_f = k_{s_T} \circ \lambda$. Then $r \circ t \circ k_f = r \circ k_{s_T} \circ \lambda = 0$. Since f is a strict epimorphism, it is its own coimage, hence there is a unique $\epsilon : Y \to R$ with $r \circ t = \epsilon \circ f$. Then the universal property of the pushout yields a unique morphism with $\eta : S \to R$ with $\epsilon = \eta \circ s_Y$ and $r = \eta \circ s_T$, which shows that s_T is its own coimage.
ii) Let $\omega_Y : Y \to Y \times T$, $\omega_T : T \to Y \times T$, $\pi_Y : Y \times T \to Y$, and $\pi_T : Y \times T \to T$ denote the canonical morphisms. If $p := (f, -t)^t : X \to Y \times T$ then $(coker(p), c_p \circ \omega_Y, c_p \circ \omega_T)$ is a pushout of f and t (this is true in any additive category with kernels and cokernels), so we can assume $S = coker(p)$, $s_Y = c_p \circ \omega_Y$ and $s_T = c_P \circ \omega_T$. Furthermore:

(I) $(im(p), -\pi_Y \circ i_p, \pi_T \circ i_p)$ is a pullback of $c_P \circ \omega_Y$ and $c_p \circ \omega_T$.

Proof of (I): One has

$$-(c_P \circ \omega_T \circ (-\pi_T \circ i_P)) + c_P \circ \omega_Y \circ \pi_Y \circ i_P = c_P \circ (\omega_T \circ \pi_T + \omega_Y \circ \pi_Y) \circ i_P = c_P \circ i_P = 0$$

and if $l_T : L \to T$ and $l_Y : L \to Y$ are morphisms with $c_P \circ \omega_Y \circ l_Y = c_P \circ \omega_T \circ l_T$ we have $c_P \circ (\omega_Y \circ l_Y - \omega_T \circ l_T) = 0$. The image of p gives rise to a unique morphism $h : L \to im(p)$ with $i_P \circ h = \omega_Y \circ l_Y - \omega_T \circ l_T$. Then

$$-\pi_T \circ i_P \circ h = -\pi_T \circ \omega_Y \circ l_Y + \pi_T \circ \omega_T \circ l_T = l_T$$

$$\pi_Y \circ i_P \circ h = \pi_Y \circ \omega_Y \circ l_Y + \pi_Y \circ \omega_T \circ l_T = l_Y$$

and h is unique with this property, which proves (I).
The diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\tilde{p} \circ c i_P"] \arrow[dd, "t"'] & \arrow[d, phantom, "\circlearrowleft"] & Y \arrow[dd, "s_Y"] \\ \circlearrowleft & im(p) \arrow[ur, "\pi_Y \circ i_P"'] \arrow[dl, "-\pi_T \circ i_P"] \arrow[dr, phantom, "\circlearrowleft"] & \\ T \arrow[rr, "s_T"'] & & S \end{tikzcd}$$

is commutative and since C is semi-abelian, the morphism $\tilde{p} \circ ci_P$ is an epimorphism. If f or t is a strict monomorphism, then $\tilde{p} \circ ci_P$ is also a strict monomorphism by Proposition 4, hence an isomorphism. Together with (I) this proves the proposition.

The dual argument shows:

Proposition 7 Let C be a semi-abelian category and

$$\begin{tikzcd} (*) \quad Y \arrow[r, "g"] & Z \\ P \arrow[u, "p_Y"] \arrow[r, "p_T"'] \arrow[ur, phantom, "\circlearrowleft"] & T \arrow[u, "t"'] \end{tikzcd}$$

be a pullback square.

i) If g is a strict monomophism, then p_T is a strict monomorphism.

ii) If g or t is a strict epimorphism, then $(*)$ is also a pushout square.

These propositions show what happens to strict epimorphisms under pullbacks and to strict monomorphisms under pushouts in a semi-abelian category:

Corollary 8 i) If

$$\begin{tikzcd} Y \arrow[r, "g"] & Z \\ P \arrow[u, "p_Y"] \arrow[r, "p_T"'] \arrow[ur, phantom, "\circlearrowleft"] & T \arrow[u, "t"'] \end{tikzcd}$$

is a pullback square and g a strict epimorphism, then p_T is an epimorphism.

ii) If

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "s_Y"] \\ T \arrow[r, "s_T"'] & S \end{tikzcd}$$

is a pushout square and f is a strict monomorphism, then s_T is a monomorphism.

Proof: ii) is the dual statement of i), so it suffices to show i). Since g is a strict epimorphism, the pullback in i) is also a pushout by Proposition 7. Let $c_{p_T} : T \to coker(p_T)$ be the cokernel of p_T, then by Lemma 5 ii) there is a commutative diagram

$$\begin{tikzcd} P \arrow[r, "p_T"] \arrow[d, "p_Y"'] \arrow[dr, phantom, "\circlearrowleft"] & T \arrow[r, "c_{p_T}"] \arrow[d, "t"] \arrow[dr, phantom, "\circlearrowleft"] & coker(p_T) \arrow[d, "id"] \\ Y \arrow[r, "g"'] & Z \arrow[r, "c"'] & coker(p_T) \end{tikzcd}$$

where c is a cokernel of g. Since g is an epimorphism, we have $coker(p_T) = 0$ and thus p_T is also an epimorphism.

The morphisms p_T and s_T of corollary 8 are not necessarily strict morphisms (see example 19). We now turn our attention to those strict epimorphisms and monomorphisms where this will be the case.

3 l-strict Epimorphisms and Monomorphisms

A sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in a semi-abelian category C is called exact, if $g \circ f = 0$, f is a strict monomorphism, g is a strict epimorphism and the canonical morphism $\lambda : im(f) \to ker(g)$ is an isomorphism.

Lemma 9 For a short sequence $(*)$ $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in a semi-abelian category the following are equivalent:

i) $(*)$ is exact.

ii) f is a kernel of g and g is a cokernel of f.

Proof: *i)* $\Rightarrow$ *ii)* If $(*)$ is exact it is enough to show that f is a kernel of g, since g is a strict epimorphism, hence by Remark 3 a cokernel of its kernel. Since f is a strict monomorphism $\tilde{f} \circ ci_f$ is an isomorphism and since $(*)$ is exact, the canonical morphism $\lambda : im(f) \to ker(g)$ with $i_f = k_g \circ \lambda$ is also an isomorphism. Then

$$f = i_f \circ \tilde{f} \circ ci_f = k_g \circ \mu \circ \tilde{f} \circ ci_f,$$

which shows that f is a kernel of g.
ii) $\Rightarrow$ *i)* If f is a kernel of g and g is a cokernel of f, then f is a strict monomorphism and g is a strict epimorphism by Remark 3 and $g \circ f = 0$. Since g is a cokernel of f, the kernel of g is an image of f, hence the canonical morphism $\lambda : im(f) \to ker(g)$ with $i_f = k_g \circ \lambda$ is an isomorphism. Therefore $(*)$ is exact.

Given a short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ and a morphism $t : T \to Z$, Lemma 5 i) yields a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
X \arrow[u, "id"] \arrow[r, "k"'] \arrow[ur, phantom, "\circlearrowleft"] & P \arrow[u, "p_Y"'] \arrow[r, "p_T"'] \arrow[ur, phantom, "\circlearrowleft"] & T \arrow[u, "t"']
\end{tikzcd}$$

where (P, p_Y, p_T) is the pullback of g and t and k is a kernel of p_T. By Proposition 8 the morphism p_T is an epimorphism, but it need not be a strict epimorphism, so the bottom row is in general not exact. Dually, given a morphism $t : X \to T$ we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r, "g"] \arrow[d, "s_Y"] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[d, "id"] \\ T \arrow[r, "s_T"'] & P \arrow[r, "c"'] & Z \end{tikzcd}$$

where (s, s_Y, s_T) is the pushout of f and t and c is a cokernel of s_T, the morphism s_T is a monomorphism, but not necessarily a strict monomorphism. Those strict epimorphisms and monomorphisms that are preserved under pullback, resp. pushout, deserve a special name:

Definition 10 Let C be an additive category with kernels and cokernels.

i) A strict epimorphism $g : Y \to Z$ is called an l-strict epimorphism (lifting-strict), if for every pullback square

$$\begin{tikzcd} Y \arrow[r, "g"] & Z \\ P \arrow[u, "p_Y"] \arrow[r, "p_T"'] \arrow[ur, phantom, "\circlearrowleft"] & T \arrow[u, "t"'] \end{tikzcd}$$

the morphism p_T is also a strict epimorphism.

ii) A strict monomorphism $f : X \to Y$ is called an l-strict monomorphism, if for every pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "s_Y"] \\ T \arrow[r, "s_T"'] & S \end{tikzcd}$$

the morphism s_T is also a strict monomorphism.

Using these notions a quasi-abelian category is a semi-abelian category in which every strict epimorphism and every strict monomorphism is l-strict.

Remark 11

Because of the transitivity of both pullbacks and pushouts the morphisms p_T and s_T in definition 10 are again l-strict.

Since retractions are stable under pullbacks, coretractions are stable under pushouts and isomorphisms are stable under both, we obtain:

Proposition 12 i) Retractions are l-strict epimorphisms.

ii) Coretractions are l-strict monomorphisms.

iii) Isomorphisms are l-strict epimorphisms and l-strict monomorphisms.

In a semi-abelian category we obtain additionally:

Proposition 13 Let C be a semi-abelian category.

i) If $g : Y \to Z$ and $g' : Z \to Z'$ are l-strict epimorphisms, then $g' \circ g$ is an l-strict epimorphism.

ii) If $g : Y \to Z$ is an l-strict epimorphism that factors as $g = p \circ q$, then p is an l-strict epimorphism, too.

iii) If $f : X \to Y$ and $f' : Y \to Y'$ are l-strict monomorphisms, then $f' \circ f$ is an l-strict monomorphism.

iv) If $f : X \to Y$ is an l-strict monomorphism that factors as $f = i \circ j$, then j is an l-strict monomorphism, too.

Proof: iii) and iv) are the dual statements of i) and ii), so it suffices to show these.
i) Because of the transitivity of the pullback, the diagram

$$\begin{tikzcd} Y \arrow[r, "g"] & Z \arrow[r, "g'"] & Z' \\ Q \arrow[u, "q_Y"] \arrow[r, "q_P"'] \arrow[ur, phantom, "\circlearrowleft"] & P \arrow[u, "p_Z"'] \arrow[r, "p_T"'] \arrow[ur, phantom, "\circlearrowleft"] & T \arrow[u, "t"'] \end{tikzcd}$$

is a pullback of $g' \circ g$ and t, if (P, p_T, p_Z) is a pullback of g' and t and (Q, q_P, q_Y) is a pullback of g and p_Z. By Proposition 4 iii) the morphism $p_T \circ q_P$ is a strict epimorphism, hence $g' \circ g$ is an l-strict epimorphism.
ii) This follows as above from the transitivity of the pullback by using Proposition 4 ii) instead of iii).

4 Semi-Abelian Categories as Exact Categories

The notion of exact category was introduced by Quillen [6]. An excellent elementary exposition of the theory of exact categories can be found in [2]. We use the following definition of exact category which is due to Keller [4] and has a minimal set of axioms:

Definition 14 Let C be an additive category and $\mathcal{E}$ be a class of pairs (f, g), called *conflations*, of composable morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

such that f is a kernel of g and g is a cokernel of f. A morphism f is called an *inflation*, if $(f, g) \in \mathcal{E}$ for some morphism g and a morphism g is called a *deflation* if $(f, g) \in \mathcal{E}$ for some morphism f. An exact structure on C is a class $\mathcal{E}$ of conflations that is closed under isomorphisms and satisfies the following axioms:

(E0) $0 \to 0$ is a deflation

(E1) If $Y \xrightarrow{g} Z$ and $Z \xrightarrow{g'} V$ are deflations, then $g' \circ g$ is a deflation.

(E2) If $g : Y \to Z$ is a deflation and $t : T \to Z$ is a morphism, then the pullback

$$\begin{tikzcd} Y \arrow[r, "g"] & Z \\ P \arrow[u, "p_Y"] \arrow[r, "p_T"'] \arrow[ur, phantom, "\circlearrowleft"] & T \arrow[u, "t"'] \end{tikzcd}$$

of g and t exists and p_T is a deflation.

(E3) If $f : X \to Y$ is an inflation and $t : X \to T$ is a morphism, then the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "s_Y"] \\ T \arrow[r, "s_T"'] & S \end{tikzcd}$$

of f and t exists and s_T is an inflation.

An *exact category* is a pair $(C, \mathcal{E})$ consisting of an additive category C and an exact structure $\mathcal{E}$ on C.

Remark 15 If C is a semi-abelian category and $\mathcal{E}$ is an exact structure on C, then $\mathcal{E}$ is a subclass of all short exact sequences of C by Lemma 9.

Let now C be a semi-abelian category and let E_C be the class of pairs (f, g) of composable morphisms $f : X \to Y$ and $g : Y \to Z$ with

(1) $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact,

(2) g is an l-strict epimorphism,

(3) f is an l-strict monomorphism.

We call E_C the class of admissible short exact sequences of C.

T

The following theorem is the main result of this article:

Theorem 16 If C is a semi-abelian category, then the class E_C of admissible short exact sequences is an exact structure on C.

Proof: (E0) is satisfied by Proposition 12. We show that E_C is closed under isomorphisms: Let $(f, g) \in E_C$ and let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r, "g"] \arrow[d, "i_Y"] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[d, "i_Z"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \end{tikzcd}$$

be a commutative square in C with isomorphisms i_X, i_Y and i_Z.
Then the sequence $(*)$ $0 \to X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \to 0$ is exact. In fact, since $i_Z \circ g = g' \circ i_Y$ is a strict epimorphism, g' is also a strict epimorphism and f' is obviously a kernel of g'. Since g' is a strict epimorphism it is the cokernel of its kernel by Proposition 4. Therefore (f', g') is a kernel-cokernel pair, hence $(*)$ is exact.
Every commutative square

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & F \arrow[d, "\psi"] \\ E' \arrow[r, "h'"'] & F' \end{tikzcd}$$

in C with isomorphisms ϕ and ψ is a pullback square as well as a pushout square, hence f' is an l-strict monomorphism and g' an l-strict epimorphism by Remark 11 which shows $(f', g') \in E_C$.
(E2): The pullback of any two morphisms does exist in C, since C has products and kernels. Let $(f, g) \in E_C$, let

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, phantom, "\circlearrowleft"] & Z \\ P \arrow[u, "p_Y"] \arrow[r, "p_T"'] & T \arrow[u, "t"'] \end{tikzcd}$$

be a pullback square and $k : K \to P$ be a kernel of p_T. Then p_T is an l-strict epimorphism and the sequence $0 \to K \xrightarrow{k} Y \xrightarrow{p_T} T \to 0$ is exact, so it only remains to be shown that k is an l-strict monomorphism.
Lemma 5 i) shows that there is a unique isomorphism $j : K \to X$ such that $p_y \circ k = f \circ j$. Let then

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "r"'] \arrow[dr, phantom, "\circlearrowleft"] & P \arrow[d, "s_P"] \\ R \arrow[r, "s_R"'] & S \end{tikzcd} \tag{1}$$

be a pushout square. First construct pushouts

$$\begin{tikzcd} K \arrow[r, "j"] \arrow[d, "r"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\phi_1"] \\ T \arrow[r, "\psi_1"'] & Q_1 \end{tikzcd} \tag{2}$$

and

$$\begin{tikzcd} P \arrow[r, "p_Y"] \arrow[d, "s_p"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "\phi_2"] \\ T \arrow[r, "\psi_2"'] & Q_2 \end{tikzcd} . \tag{3}$$

Then ψ_1 is an isomorphism. We have

$$\psi_2 \circ s_R \circ r = \psi_2 \circ s_P \circ k = \phi_2 \circ p_Y \circ k = \phi_2 \circ f \circ j,$$

hence by the universal property of (1) there is a unique morphism $\epsilon : Q_1 \to Q_2$ with $\psi_2 \circ s_R = \epsilon \circ \psi_1$ and $\phi_2 \circ f = \epsilon \circ \phi_1$, so that the following cube is commutative:

$$\begin{tikzcd}
& Y \arrow[rr, "\phi_2"] & & Q_2 \\
X \arrow[ur, "f"] \arrow[rr, "\phi_1"] & & Q_1 \arrow[ur, "\epsilon"] & \\
& P \arrow[uu, "p_Y"'] \arrow[rr, "s_P"'] & & S \arrow[uu, "\psi_2"'] \\
K \arrow[uu, "j"] \arrow[ur, "k"'] \arrow[rr, "r"'] & & R \arrow[uu, "\psi_1"'] \arrow[ur, "s_R"'] &
\end{tikzcd}$$

We have:

(I) (Q_2, ϵ, ϕ_2) is a pushout of f and ϕ_1.

Proof of (I): Let $l_1 : Q_1 \to L$ and $l_Y : Y \to L$ be morphisms such that $l_Y \circ f = l_1 \circ \phi_1$. Then $l_Y \circ p_Y \circ k \circ j^{-1} = l_1 \circ \psi_1 \circ r \circ j^{-1}$, hence $l_Y \circ p_Y \circ k = l_1 \circ \psi_1 \circ r$. The universal property of (1) then yields a unique morphism $\mu_1 : S \to L$ with $l_Y \circ p_Y = \mu_1 \circ s_P$ and $l_1 \circ \psi_1 = \mu_1 \circ s_R$ and the universal property of (3) yields a unique morphism $\mu_2 : Q_2 \to L$ with $l_Y = \mu_2 \circ \phi_2$ and $\mu_1 = \mu_2 \circ \psi_2$. Then

$$l_1 \circ \psi_1 = \mu_1 \circ s_R = \mu_2 \circ \psi_2 \circ s_R = \mu_2 \circ \epsilon \circ \psi_1,$$

hence $l_1 = \mu_2 \circ \epsilon$, since ψ_1 is an isomorphism, making the following diagram commutative:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi_1"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "\phi_2"] \arrow[ddr, bend left, "l_Y"] & \\
Q_1 \arrow[r, "\epsilon"'] \arrow[drr, bend right, "l_1"'] & Q_2 \arrow[dr, "\mu_2"] \arrow[r, phantom, "\circlearrowleft"] \arrow[d, phantom, "\circlearrowleft"] & \\
& {} & L
\end{tikzcd}.$$

The morphism μ_2 is unique with this property because of the universal properties of (1) and (3). This shows (I).
Since f is an l-strict monomorphism it follows from (I) that ϵ is a strict monomorphism. Then $\epsilon \circ \psi_1 = \psi_2 \circ s_R$ is also a strict monomorphism and by Proposition 4 it follows that s_R is a strict monomorphism, hence k is an l-strict monomorphism which shows $(k, p_T) \in E_C$.
(E3): Since C has products and cokernels, the pushout of any two morphisms does exist. The pair (f, g) is in E_C if and only if (g^{op}, f^{op}) is in $E_{C^{op}}$. Then (E3) follows from (E2) by duality.
(E1): Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ and $0 \to X' \xrightarrow{f'} Z \xrightarrow{g'} Z' \to 0$ be short exact

sequences with $(f,g),(f',g') \in E_C$ and let $k : K \to Y$ be a kernel of $g' \circ g$. Since $g' \circ g$ is a strict epimorphism, the sequence $0 \to K \xrightarrow{k} Y \xrightarrow{g' \circ g} Z' \to 0$ is exact and by Proposition 13 the epimorphism $g' \circ g$ is l-strict, so it remains to show:

(II) k is an l-strict monomorphism.

Proof of (II): Since $g' \circ g \circ k = 0$, there exists a unique $\alpha : K \to X'$ with $f' \circ \alpha = g \circ k$. We have:

(II.1) The diagram

$$\begin{tikzcd} Y \arrow[r, "g"] & Z \\ K \arrow[u, "k"] \arrow[r, "\alpha"'] \arrow[ur, phantom, "\circlearrowleft"] & X \arrow[u, "f'"'] \end{tikzcd}$$

is a pullback square.

Proof of (II.1): Let $l_Y : L \to Y$ and $l_{X'} : L \to X'$ be morphisms with $f' \circ l_{X'} = g \circ l_Y$. Then $g' \circ g \circ l_Y = g' \circ f' \circ l_{X'} = 0$, hence there exists a unique $\eta : L \to K$ with $l_Y = k \circ \eta$. This yields $f' \circ l_{X'} = g \circ l_Y = g \circ k \circ \eta = f' \circ \alpha \circ \eta$ and from this follows $l_{X'} = \alpha \circ \eta$, since f' is a monomorphism, hence the following diagram commutes:

$$\begin{tikzcd} & Y \arrow[r, "g"] & Z \\ & K \arrow[u, "k"] \arrow[r, "\alpha"'] \arrow[ur, phantom, "\circlearrowleft"] & X' \arrow[u, "f'"'] \\ L \arrow[uur, bend left, "l_Y"] \arrow[ur, "\eta"] \arrow[urr, bend right, "l_{X'}"'] & & \end{tikzcd} .$$

Since k is a monomorphism, η is unique with this property, hence the diagram in (II.1) is a pullback square.
We need the following Lemma:

(II.2) If $(f,g) \in E_C$ and

$$\begin{tikzcd} Y \arrow[r, "g"] & Z \\ P \arrow[u, "p_Y"] \arrow[r, "p_R"'] \arrow[ur, phantom, "\circlearrowleft"] & R \arrow[u, "r"'] \end{tikzcd}$$

is a pullback square diagram then the sequence

$$0 \to P \overset{(-p_R,p_Y)^t}{\to} R \times Y \overset{(r,g)}{\to} Z \to 0 \tag{4}$$

is an admissible short exact sequence, that is $((-p_R,p_Y)^t,(r,g)) \in E_C$.

Proof of (II.2): By Lemma 5 i) we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\ X \arrow[u, "id"] \arrow[r, "k"'] \arrow[ur, phantom, "\circlearrowleft"] & P \arrow[u, "p_Y"'] \arrow[r, "p_T"'] \arrow[ur, phantom, "\circlearrowleft"] & T \arrow[u, "t"'] \end{tikzcd}$$

such that k is a kernel of p_R and by (E2) the pair (k, p_R) is in E_C.
If (S, s_Y, s_P) is the pushout of f and k, Lemma 5 ii) yields a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "k"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r, "g"] \arrow[d, "s_Y"] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[d, "id_Z"] \\ P \arrow[r, "s_p"'] \arrow[d, "p_R"'] \arrow[dr, phantom, "\circlearrowleft"] & S \arrow[r, "c"'] \arrow[d, "c'"] & Z \\ R \arrow[r, "id_R"'] & R & \end{tikzcd}$$

such that c is a cokernel of s_P and c' is a cokernel of s_Y. By (E3) we have $(s_p, c), (s_y, c') \in E_C$.
We have $(-p_R, p_Y)^t \circ k = \omega_Y \circ f$, hence the universal property of the pushout yields a unique morphism $\lambda : S \to R \times Y$ with $(-p_R, p_Y) = \lambda \circ s_P$ and $\omega_Y = \lambda \circ s_Y$.
Then λ is an isomorphism. In fact, because of $(s_Y \circ p_Y - s_P) \circ k = 0$ there exists a unique $\gamma : R \to S$ with $s_Y \circ p_Y - s_P = \gamma \circ p_R$. This in turn gives rise to a unique morphism $\mu : R \times Y \to S$ with $\gamma = \mu \circ \omega_R$ and $s_Y = \mu \circ \omega_Y$.
With the help of the universal properties of coproduct and pushout it is easy to check that $\lambda \circ \mu = id_{R\times Y}$ and $\mu \circ \lambda = id_S$.
The universal property of the coproduct shows that $c \circ \mu = (r, g)$, so the diagram

$$\begin{tikzcd} P \arrow[r, "s_P"] \arrow[d, "id_p"'] \arrow[dr, phantom, "\circlearrowleft"] & S \arrow[r, "c"] \arrow[d, "\lambda"] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[d, "id"] \\ P \arrow[r, "(-p_R, p_Y)^t"'] & R \times Y \arrow[r, "(r,g)"'] & Z \end{tikzcd}$$

commutes. This proofs (II.2), since E_C is closed under isomorphisms.
Returning to the proof of (II), we know by (II.2) that the pair (p, q) of morphisms $p := (-\alpha, k)^t : K \to X' \times Y$ and $q := (f', g) : X' \times Y \to Z$ lies in E_C.
We have a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\omega_{X'}"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[d, "\omega_Z"] \\ X' \times Y \arrow[r, "r"'] & Z \times Y \end{tikzcd} \tag{5}$$

where $r := \begin{pmatrix} f' & 0 \\ 0 & id_Y \end{pmatrix}$. Furthermore:

(II.3) (5) is a pushout square.

Proof of (II.3): Let $l_{X'\times Y} : X' \times Y \to L$ and $l_Z : Z \to L$ be morphisms with $l_Z \circ f' = l_{X'\times Y} \circ \omega_{X'}$. This yields a unique morphism $\delta : Z \times Y \to L$ with $l_Z = \delta \circ \omega_Y$ and we have:

$$l_{X'\times Y} \circ \omega_{X'} = l_Z \circ f' = \delta \circ \omega_Z \circ f' = \delta \circ r \circ \omega_{X'}$$

$$\begin{aligned} l_{X' \times Y} \circ \omega_Y &= \delta \omega_Y = \delta \circ \omega_Y \pi_Y \circ r \circ \omega_Y \\ &= \delta \circ (id_{Z \times Y} - \omega_Z \circ \pi_Z) \circ r \circ \omega_Y \\ &= \delta \circ r \circ \omega_Y \end{aligned}$$

Hence the universal property of the coproduct yields $l_{X' \times Y} = \delta \circ r$, making the following diagram commutative:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\omega_{X'}"'] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[d, "\omega_Z"] \arrow[ddr, bend left, "l_Z"] & \\ X' \times Y \arrow[r, "r"'] \arrow[drr, bend right, "l_{X' \times Y}"'] & Z \times Y \arrow[dr, "\delta"] \arrow[r, phantom, "\circlearrowleft"] \arrow[d, phantom, "\circlearrowleft"] & \\ & & L \end{tikzcd}$$

The uniqueness of δ follows from the universal property of the coproduct, which proves (II.3).
Then r is an l-strict monomorphism and by Proposition 13 the composition $r \circ p$ is also an l-strict monomorphism. Define $\sigma := (-g, id_Y)^t$, then we have:

$$r \circ p = \begin{pmatrix} f' & 0 \\ 0 & id_Y \end{pmatrix} \begin{pmatrix} -\alpha \\ k \end{pmatrix} = \begin{pmatrix} -f' \circ \alpha \\ k \end{pmatrix} = \begin{pmatrix} -g \circ k \\ k \end{pmatrix} = \sigma \circ k$$

Since $r \circ p$ is an l-strict monomorphism, it follows from Proposition 13, that k is an l-strict monomorphism. That proves (II) and thus the theorem. $\square$

Remark 17 The structure of the proof of (E1) in theorem 16 is basically the same as that of the proof of $Ex1^{op}$ in [4], we just substitute Proposition 13 for some properties of exact categories. A careful analysis of the proof of Theorem 16 shows, that the theorem also holds true for any additive category C which has kernels and cokernels and in which Proposition 13 holds.

The exact structure E_C is the largest exact structure on C:

Proposition 18 Let $\mathcal{E}$ be an exact structure on the semi-abelian category C. Then $\mathcal{E} \subset E_C$.

Proof: If $(f, g) \in \mathcal{E}$, the sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact by Lemma 9. If

$$\begin{tikzcd} Y \arrow[r, "g"] & Z \\ P \arrow[u, "p_Y"] \arrow[r, "p_T"'] \arrow[ur, phantom, "\circlearrowleft"] & T \arrow[u, "t"'] \end{tikzcd}$$

is a pullback square, the morphism p_T is a deflation by (E2), hence it is a cokernel of its kernel and thus a strict epimorphism, hence g is an l-strict epimorphism. Analogously, by (E3) the morphism f is an l-strict monomorphism, which shows $(f, g) \in E_C$.

We hence define $\mathcal{E}_{max} := E$.
The above shows, that the exact structure of a quasi-abelian category C is a special case of the exact structure $\mathcal{E}_{max}$, namely the case of $\mathcal{E}_{max}$ containing all short exact sequences of C.

The following is an example of a semi-abelian category that is not quasi-abelian:

Example 19 A subset B of a locally convex space X is called *bornivorous* if it is absolutely convex and absorbs every bounded set. The locally convex space X is called *bornological* if every bornivorous subset of X is a zero-neighbourhood. Equivalently, X is bornological if and only if every locally bounded linear map $f : X \to Z$ into a locally convex space Z is continuous.
Let $(LC)^{bor}$ be the category of bornological locally convex spaces and continuous linear maps. It is a preadditive full subcategory of the category (LC) of locally convex spaces.
The universal property of the final topology shows:

(1) If $(X_i)_{i \in I}$ is a family of bornological locally convex spaces, X a locally convex space, $f_i : X_i \to X$ a family of linear maps and $\mathcal{T}$ the final topology on X with regard to the f_i, then $(X, \mathcal{T})$ is bornological.

Therefore direct sums and quotients of bornological locally convex spaces are again bornological, so $(LC)^{bor}$ has coproducts and cokernels.
If $X = (X, \mathcal{T})$ is a locally convex space, we define X^{bor} to be the vector space X with the coarsest bornological topology on X that is finer than $\mathcal{T}$. If $f : X \to Y$ is a morphism in $(LC)^{bor}$, the inclusion $j : L^{bor} \hookrightarrow X$, where $L := f^{-1}(\{0\})$, is a kernel of f:
If $t : T \to X$ is another morphism in $(LC)^{bor}$ with $f \circ t = 0$, there is a unique continuous linear mapping $\lambda : T \to L$ with $t = j \circ \lambda$. If V is a zero-neighbourhood in L^{bor}, then $\lambda^{-1}(V)$ is a bornivorous subset in T, since λ is continuous, so it is a neighbourhood of zero in T. Since $\lambda(\lambda^{-1}(V)) \subseteq V$, this shows that $\lambda : T \to L^{bor}$ remains continuous, therefore j is a kernel of f in $(LC)^{bor}$.
It follows that $(LC)^{bor}$ is an additive category with kernels and cokernels and the canonical factorization of a morphism $f : X \to Y$ is given by:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, "ci_f"'] \arrow[drr, phantom, "\circlearrowleft"] & & Y \\
X/L^{bor} \arrow[rr, "\tilde{f}"'] & & f(X)^{bor} \arrow[u, "i_f"']
\end{tikzcd}$$

Since the algebraic structure of $coim(f)$ and $im(f)$ doesn't change, $\tilde{f}$ is bijective and therefore it is both an epimorphism and a monomorphism in $(LC)^{bor}$. This shows that $(LC)^{bor}$ is a semi-abelian category.
However, it is not a quasi-abelian category, since in [1] the authors construct

morphisms $g : Y \to Z$ and $t : T \to Z$ in $(LC)^{bor}$ so that in the pullback square

$$\begin{tikzcd} Y \arrow[r, "g"] & Z \\ P \arrow[u, "p_Y"] \arrow[r, "p_T"'] \arrow[ur, phantom, "\circlearrowleft"] & T \arrow[u, "t"'] \end{tikzcd}$$

the morphism p_T is not a strict morphism.

References

[1] J. Bonet, S. Dierolf. *The Pullback for bornological and ultrabornological spaces*. Note di Matematica 25, n.1, 2005/2006.

[2] T. Bühler. *Exact Categories*. arXiv:0811.1480, preprint.

[3] G. Janelidze, G. M. Kelly. *Galois theory and a general notion of central extension*. J. Pure Appl. Algebra 97 (1994), 135-161.

[4] B. Keller. *Chain complexes and stable categories*. Manuscripta Math. 67 (1990), no. 4.

[5] V. P. Palamodov. *Homological methods in the theory of locally convex spaces*. Uspeki Mat. Nauk, 26(1(157)):3-65, 1971. English transl.: Russian Math Surveys, 26:1-64, 1971.

[6] D. Quillen. *Higher algebraic K-theory*. Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, pp. 171–176. Canad. Math. Congress, Montreal, Que., 1975.

[7] J. P. Schneiders. *Quasi-abelian categories and sheaves*. Mem. Soc. Math. France (N.S.), 76 (1999).

[8] X. Karidopoulou. *Ein Splittingsatz für gradierte L-zahme Räume*. Dissertation, Wuppertal, 2006.