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Using swaps and deletes to make strings match

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Abstract

Given a source and a target string, their swap-delete-edit distance is the minimum number of interchange-consecutive-symbols and delete operations applied to the source string to make it equal the target string. We show that the swap-delete-edit distance of strings over alphabets of bounded size can be computed in polynomial time.

1 Introduction

Wagner and Fischer introduced the edit distance of strings, and considered computational aspects of determining this number [8]. In this early work, Wagner and Fischer considered three string modification – or *edit* – operations: changing a letter, and inserting and deleting a letter. The edit distance of two strings is the smallest number of operation applications to the one – *source* – string to obtain the other – *target* – string. They showed that the edit distance is polynomial-time computable, by applying a dynamic-programming approach [8]. In a subsequent paper, Lowrance and Wagner extended the list of applicable edit operations by interchanging consecutive symbols [5], which is often called a *swap*. Building up on these results, Wagner studied the complexity of computing the edit distance when restricting the allowed edit operations, showing hardness and tractability results [7]. It may be worth to mention that Wagner's considerations included weights on the edit operations, hereby modelling a favour for some edit operations over others.

In this paper, we consider the edit distance problem when restricting edit operation applications to the two operations swap and delete. Wagner showed that computing the edit distance of two arbitrary strings by applying only swap and delete is intractable [7]. A minor but important fact in Wagner's reduction is the use of an infinite set of symbols. When considering string problems in general, and when string problems appear in practical applications in particular, the involved sets of symbols are often fixed. It is therefore an interesting question of practical importance to ask whether the intractability of the edit distance problem remains when restricting the input strings to fixed alphabets. As the main result of this paper, we show that the edit distance problem when applying only swap and delete becomes tractable for fixed alphabets. This particularly shows the necessity of an infinite symbols set for achieving intractability.

The algorithm of this paper computes the unweighted swap-delete-edit distance of two strings. Since the number of delete operations is predetermined by the length difference of the source and target sting, the weighted and unweighted swap-delete-edit distance problems

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are computationally equivalent [7]. The algorithm itself follows a dynamic-programming approach, that has already been applied for the first algorithm by Wagner and Fischer [8]. The algorithm's work can be informally described as follows. The input is two strings, the *source* and *target* string. The source string should not be of smaller length than the target string; otherwise, a solution does not exist. The source and target string are broken into substrings, and the swap-delete-edit distance problem is solved on the substring pairs. For breaking into substrings, a fixed alphabet symbol is used, and substring pairs without that symbol are generated. The problem then is solved on input pairs for an alphabet of smaller size.

The final designed algorithm for computing the swap-delete-edit distance of two strings is an easy-to-implement algorithm, using only standard data structures and elementary techniques. The correctness proof of the algorithm, that is the main part of this paper, is technically involved but based on standard mathematical methods only. We use partial functions on the set of natural numbers to represent solutions for editing the source string into the target string. The partial functions approach strongly resembles the trace diagrams already used by Wagner and Fischer for computing the change-delete-insert-edit distance of two strings [8].

The swap-delete-edit distance problem has a special place in the group of edit distance problems allowing change, insert, delete, swap as edit operations. The swap-delete-edit distance problem, and the swap-insert-edit distance problem equivalently, is the only intractable problem among the possible problem instances [7]. It is a noteworthy side remark that the general edit distance problem, that is the change-insert-delete-swap-edit distance problem, is tractable with unit costs and under more selective cost assumptions [5]. Such a non-monotone behaviour is not often seen in computational complexity theory. A first approach to overcome the intractability of the swap-delete-edit distance problem was by Abu-Khzam et al., who devised first fixed-parameter tractable parametrised algorithms for the swap-delete-edit distance problem on input strings over arbitrary sets of input symbols [1]. They pointed out the problem of considering input strings over fixed-sized alphabets, binary alphabets even, and left the complexity status as an open problem. Spreen considered the binary alphabet case, and solved some special cases [6]. Fernau et al. resolved the complete binary alphabet case [3]. In this paper, we resolve the complexity of the swap-delete-edit distance problem for arbitrary alphabets fully.

A recently studied variant of the edit distance problem discussed here is the interchange-edit distance problem. Given two strings, the one being a permutation of the other, determine the minimum number of *interchange* operations to transform the source string into the target string. An *interchange* of two symbols interchanges the symbols. The interchange operation generalises swap. Amir et al. and Kapah et al. study the computational complexity of this and related edit distance problems under the unit cost and other cost models [2, 4].

2 Definitions and the Splitting lemma

In this section, we define the terminology, notions, and main technical tools to develop our algorithm in Section 5. A Glossary appendix at the end of the paper contains the definitions that are not given here explicitly. Throughout the paper, k denotes a positive integer, that is considered arbitrary but fixed.

A *partial k-colouring* for $\mathbb{N}$ is a total mapping $\psi : \mathbb{N} \to \{0, 1, \ldots, k\}$. The elements of

$\{1, \ldots, k\}$ are considered the k colours, and we use 0 to mean "uncoloured".

Definition 2.1. *Let ψ and χ be partial k-colourings for $\mathbb{N}$.*

1) *The ordered pair (ψ, χ) is called a* k-colour embedding pair.
2) *A* colour-preserving embedding *for (ψ, χ) is a partial injective function φ on $\mathbb{N}$ such that for every $x \in \mathbb{N} \setminus \psi^{-1}(0)$: $\varphi(x)$ is defined and $\psi(x) = \chi(\varphi(x))$.*
3) *By $\mathscr{E}\big(\psi, \chi\big)$, we denote the set of the colour-preserving embeddings for (ψ, χ).*

We can say that a colour-preserving embedding assigns each ψ-coloured integer to an integer of the same colour with χ. We are interested in k-colour embedding pairs (ψ, χ) for which $\mathscr{E}\big(\psi, \chi\big)$ is non-empty. Such pairs are called *satisfiable*.

A *k-colour vector* is a k-tuple over $\mathbb{N} \cup \{\aleph_0\}$. Let ψ be a partial k-colouring for $\mathbb{N}$, and let $B \subseteq \mathbb{N}$. The *colour vector* of (ψ, B) is the k-colour vector $\mathfrak{c}$ satisfying $c_j = \big|\{x \in B : \psi(x) = j\}\big|$ for $1 \leq j \leq k$, that counts the occurrences of the colours in ψ when restricting to the integers in B. The *colour vector* of ψ is the colour vector of $(\psi, \mathbb{N})$. Observe that uncoloured integers are not considered. Let $\mathfrak{a}$ be the colour vector of ψ. We say that a k-colour vector $\mathfrak{e}$ is a *colour vector for ψ* if $\mathfrak{e} \leq \mathfrak{a}$.

Lemma 2.2. *Let (ψ, χ) be a k-colour embedding pair. Let $\mathfrak{a}$ and $\mathfrak{b}$ be the colour vectors of ψ and χ, respectively. Then, (ψ, χ) is satisfiable if and only if $\mathfrak{a} \leq \mathfrak{b}$.*

Proof. If (ψ, χ) is satisfiable then there is $\varphi \in \mathscr{E}\big(\psi, \chi\big)$, and for every $1 \leq j \leq k$:

$$a_j \;=\; \big|\psi^{-1}(j)\big| \;=\; \big|\chi^{-1}(j) \cap \varphi(\mathbb{N} \setminus \psi^{-1}(0))\big| \;\leq\; \big|\chi^{-1}(j)\big| \;=\; b_j\,.$$

For the converse, assume $\mathfrak{a} \leq \mathfrak{b}$. We iteratively define a partial function φ, for $x \in \mathbb{N} \setminus \psi^{-1}(0)$, taken in increasing order:

$$\varphi(x) \;=_{\text{def}}\; \min\; \chi^{-1}(\psi(x)) \setminus \{\varphi(x') : x' < x\}\,.$$

It is straightforward to verify that φ can be defined, i.e., indeed exists, and φ is a colour-preserving embedding for (ψ, χ). Thus, $\varphi \in \mathscr{E}\big(\psi, \chi\big)$, and $\mathscr{E}\big(\psi, \chi\big)$ is non-empty, and (ψ, χ) is satisfiable. ∎

Let (ψ, χ) be a k-colour embedding pair. We group the colour-preserving embeddings for (ψ, χ) by using parameters. Let $p, q \in \mathbb{N}$, and let $\mathfrak{c}$ be a k-colour vector. We call $\big(\psi, \chi; (p, q); \mathfrak{c}\big)$ a *k-input tuple*. Let φ be a partial function on $\mathbb{N}$. We say that φ is *ψ-colour-monotone* if for every pair x, y from $\mathbb{N} \setminus \psi^{-1}(0)$: $\psi(x) = \psi(y)$ and $x < y$ implies $\varphi(x) < \varphi(y)$. We say that φ *satisfies* $(\chi, q, \mathfrak{c})$ if the colour vector of $\big(\chi, \{y \in \varphi(\mathbb{N} \setminus \psi^{-1}(0)) : y < q\}\big)$ is equal to $\mathfrak{c}$. Our restriction on $\mathscr{E}\big(\psi, \chi\big)$ is:

$$\mathscr{E}\big(\psi, \chi; (p, q); \mathfrak{c}\big) \;=_{\text{def}}\; \left\{ \varphi \in \mathscr{E}\big(\psi, \chi\big) : \begin{array}{c} \varphi(p) = q \text{ and } \varphi \text{ satisfies } (\chi, q, \mathfrak{c}) \\ \text{and } \varphi \text{ is } \psi\text{-colour-monotone} \end{array} \right\}.$$

Observe that $\mathscr{E}\big(\psi, \chi; (p, q); \mathfrak{c}\big)$ may be empty, for example when $\psi(p) \neq \chi(q)$. We say that $\big(\psi, \chi; (p, q); \mathfrak{c}\big)$ is *satisfiable* if $\mathscr{E}\big(\psi, \chi; (p, q); \mathfrak{c}\big)$ is non-empty. The main part of this paper deals with $\mathscr{E}\big(\psi, \chi; (p, q); \mathfrak{c}\big)$.

The swap-delete-edit distance problem can be formulated as a function problem minimizing a certain parameter (see the proof of Proposition 5.4 and [7]). That parameter for functions is defined now. Let φ be an injective function on $\mathbb{N}$. The *cross number* of φ is

$$\mathrm{cross}(\varphi) \ =_{\mathrm{def}} \ \left|\{(x,y) : x,y \in \mathbb{N} \text{ and } x < y \text{ and } \varphi(y) < \varphi(x)\}\right|.$$

We can say that the cross number of φ counts the number of *inversions* of φ. Note that the cross number considers only those integer pairs x, y for which $\varphi(x)$ and $\varphi(y)$ are defined.

Based on the cross number of injective functions, we can finally define the parameters to be minimized. Let (ψ, χ) be a satisfiable k-colour embedding pair, and let $\big(\psi, \chi; (p,q); \mathfrak{c}\big)$ be a satisfiable k-input tuple. The *cross numbers* of (ψ, χ) and $\big(\psi, \chi; (p,q); \mathfrak{c}\big)$ are

$$\begin{aligned} \mathrm{cross}(\psi,\chi) \ &=_{\mathrm{def}} \ \min\Big\{\mathrm{cross}(\varphi) : \ \varphi \in \mathscr{E}\big(\psi,\chi\big)\Big\} \\ \mathrm{cross}\big(\psi,\chi;(p,q);\mathfrak{c}\big) \ &=_{\mathrm{def}} \ \min\Big\{\mathrm{cross}(\varphi) : \ \varphi \in \mathscr{E}\big(\psi,\chi;(p,q);\mathfrak{c}\big)\Big\}. \end{aligned}$$

Next, we show that the cross number of embedding pairs can be determined from the cross numbers of input tuples. We also show a characterisation of satisfiable input tuples analogous to Lemma 2.2 for embedding pairs. Let $\big(\psi, \chi; (p,q); \mathfrak{c}\big)$ be a k-input tuple. We call $\big(\psi, \chi; (p,q); \mathfrak{c}\big)$ *good* if the two conditions are satisfied:

- $\psi(p) = \chi(q)$ and $\psi(p) \neq 0$
- $\mathfrak{c}$ is a colour vector for ψ, and $c_{\psi(p)} = \left|\{x \in \mathbb{N} : x < p \text{ and } \psi(x) = \psi(p)\}\right|$.

Note that the second condition about $c_{\psi(p)}$ means that p is the $(c_{\psi(p)}{+}1)$th smallest integer of colour $\psi(p)$ in ψ. We see in the next lemma, among others, that $\big(\psi, \chi; (p,q); \mathfrak{c}\big)$ is satisfiable if and only if $\big(\psi, \chi; (p,q); \mathfrak{c}\big)$ is good. The proof of the second statement relies on a result about the existence of ψ-colour monotone embeddings that we state and prove only in Subsection 3.1.

Lemma 2.3. *Let (ψ, χ) be a satisfiable k-colour embedding pair. Let $p \in \mathbb{N}$ with $\psi(p) \neq 0$. The following two hold:*

1) for every $q \in \mathbb{N}$ and every k-colour vector $\mathfrak{c}$ such that $\big(\psi, \chi; (p,q); \mathfrak{c}\big)$ is a good k-input tuple:
$\mathrm{cross}(\psi,\chi) \leq \mathrm{cross}\big(\psi,\chi;(p,q);\mathfrak{c}\big)$

2) for some $q \in \mathbb{N}$ and some colour vector $\mathfrak{c}$ for ψ:
$\big(\psi, \chi; (p,q); \mathfrak{c}\big)$ *is a good k-input tuple and* $\mathrm{cross}(\psi,\chi) \geq \mathrm{cross}\big(\psi,\chi;(p,q);\mathfrak{c}\big)$.

Proof. Since $\mathscr{E}\big(\psi,\chi;(p,q);\mathfrak{c}\big) \subseteq \mathscr{E}\big(\psi,\chi\big)$ by definition, the correctness of the first claim follows from the definitions of the cross numbers.

We prove the second claim. Let $\varphi \in \mathscr{E}\big(\psi,\chi\big)$ be ψ-colour-monotone such that $\mathrm{cross}(\psi,\chi) = \mathrm{cross}(\varphi)$; φ exists, since (ψ,χ) is satisfiable according to the assumptions of the lemma and by Lemma 3.2. We choose q and $\mathfrak{c}$, and verify their properties.

Let $q =_{\mathrm{def}} \varphi(p)$, that is well-defined, since $\psi(p) \neq 0$ and therefore $\varphi(p)$ is defined. Since φ is a colour-preserving embedding for (ψ,χ), $\psi(p) = \chi(\varphi(p)) = \chi(q)$ is the case.

Let $\mathfrak{c}$ be the colour vector of $(\chi, \{y \in \varphi(\mathbb{N} \setminus \psi^{-1}(0)) : y < q\})$. It is clear that φ satisfies $(\chi, q, \mathfrak{c})$, which means $c_j \leq \left|\psi^{-1}(j)\right|$ for every $1 \leq j \leq k$ in particular, and thus, $\mathfrak{c}$ is a colour vector for ψ. Let $x \in \psi^{-1}(\psi(p))$. Since φ is ψ-colour-monotone, $x < p$ if and only if $\varphi(x) < \varphi(p)$, i.e., if and only if $\varphi(x) < q$. Thus,

$$\left|\{x \in \psi^{-1}(\psi(p)) :\ x < p\}\right| \ = \ \left|\{y \in \chi^{-1}(\psi(p)) \cap \varphi(\mathbb{N} \setminus \psi^{-1}(0)) :\ y < q\}\right| \ = \ c_{\psi(p)}\,.$$

We conclude: $\big(\psi, \chi; (p, q); \mathfrak{c}\big)$ is a good k-input tuple and $\varphi \in \mathscr{E}\big(\psi, \chi; (p, q); \mathfrak{c}\big)$, and $\text{cross}(\psi, \chi) \geq \text{cross}(\varphi) \geq \text{cross}\big(\psi, \chi; (p, q); \mathfrak{c}\big)$. ■

Let ψ be a partial k-colouring for $\mathbb{N}$, and let $\mathfrak{c}$ be a colour vector for ψ. We define two subcolourings of ψ. For $x \in \mathbb{N}$, let $\psi'(x) =_{\text{def}} \psi(x + 1)$. If $\psi(0) \neq 0$ and $c_{\psi(0)} \geq 1$ then let $\hat{\mathfrak{c}}$ be the k-colour vector satisfying $c_{\psi(0)} = \hat{c}_{\psi(0)} + 1$ and $c_j = \hat{c}_j$ for every $1 \leq j \leq k$ where $j \neq \psi(0)$. The *first-fit subcolouring of* ψ *induced by* $\mathfrak{c}$ is denoted as $\psi[\mathfrak{c}]$ and the *co-first-fit subcolouring of* ψ *induced by* $\mathfrak{c}$ is denoted as $\psi[-\mathfrak{c}]$, and they are inductively defined as follows:

- *first-fit subcolouring*
  if $\psi \equiv 0$ or $\mathfrak{c} = 0$ or $\psi(0) = 0$ or $c_{\psi(0)} = 0$:

$$\psi[\mathfrak{c}] \ =_{\text{def}} \begin{cases} 0 & \text{, if } \psi \equiv 0 \text{ or } \mathfrak{c} = 0 \\ \psi'[\mathfrak{c}] & \text{, if } \mathfrak{c} \neq 0 \text{, and } \psi(0) = 0 \text{ or } c_{\psi(0)} = 0 \end{cases}$$

  and if $\psi \not\equiv 0$ and $\mathfrak{c} \neq 0$ and $\psi(0) \neq 0$ and $c_{\psi(0)} \geq 1$: for $x \in \mathbb{N}$,

$$\psi[\mathfrak{c}](x) \ =_{\text{def}} \begin{cases} \psi(0) & \text{, if } x = 0 \\ \psi'[\hat{\mathfrak{c}}](x - 1) & \text{, if } x \geq 1 \end{cases}$$

- *co-first-fit subcolouring*
  if $\psi \equiv 0$ or $\mathfrak{c} = 0$ or $\psi(0) = 0$ or $c_{\psi(0)} \geq 1$:

$$\psi[-\mathfrak{c}] \ =_{\text{def}} \begin{cases} 0 & \text{, if } \psi \equiv 0 \\ \psi'[-\mathfrak{c}] & \text{, if } \psi \not\equiv 0 \text{ and } \psi(0) = 0 \\ \psi & \text{, if } \mathfrak{c} = 0 \text{ and } \psi(0) \neq 0 \\ \psi'[-\hat{\mathfrak{c}}] & \text{, if } \mathfrak{c} \neq 0 \text{ and } \psi(0) \neq 0 \text{ and } c_{\psi(0)} \geq 1 \end{cases}$$

  and if $\psi \not\equiv 0$ and $\mathfrak{c} \neq 0$ and $\psi(0) \neq 0$ and $c_{\psi(0)} = 0$: for $x \in \mathbb{N}$,

$$\psi[-\mathfrak{c}](x) \ =_{\text{def}} \begin{cases} \psi(0) & \text{, if } x = 0 \\ \psi'[-\mathfrak{c}](x - 1) & \text{, if } x \geq 1\,. \end{cases}$$

Informally, if we view the integers in $\mathbb{N}$ along the number line, we can say that the first-fit subcolouring $\psi[\mathfrak{c}]$ marks the c_j leftmost occurrences of colour j and restricts ψ to the marked integers, and the co-first-fit subcolouring is the restriction to the unmarked integers. The definitions of the two subcolourings appear overly technical at first glance; they are designed to make later results easy to state and prove, such as the three results of Subsection 3.2, and to apply them algorithmically. For instance, leftmost uncoloured integers are removed.

It is clear and obvious that the number of first-fit subcolourings of ψ is equal to the number of colour vectors for ψ. This will be of high importance for our efficient algorithm.

Lemma 2.4 (Splitting). *Let $\big(\psi,\chi;(p,q);\mathfrak{c}\big)$ be a good k-input tuple. For $y \in \mathbb{N}$, let*

$$\chi^L(y) \ =_{\text{def}} \begin{cases} \chi(y) & \text{, if } y < q \\ 0 & \text{, if } y \geq q \end{cases} \qquad \text{and} \qquad \chi^R(y) \ =_{\text{def}} \begin{cases} 0 & \text{, if } y \leq q \\ \chi(y) & \text{, if } y > q\,. \end{cases}$$

Let $\tilde{\mathfrak{c}}$ be the k-colour vector satisfying $c_j = \tilde{c}_j$ for every $1 \leq j \leq k$ where $j \neq \psi(p)$ and $\tilde{c}_{\psi(p)} = c_{\psi(p)} + 1$. Then, $(\psi[\mathfrak{c}],\chi^L)$ and $(\psi[-\tilde{\mathfrak{c}}],\chi^R)$ are satisfiable k-colour embedding pairs, and

$$\text{cross}\big(\psi,\chi;(p,q);\mathfrak{c}\big) \ = \ f(\psi,\mathfrak{c},p) + \text{cross}\big(\psi[\mathfrak{c}],\chi^L\big) + \text{cross}\big(\psi[-\tilde{\mathfrak{c}}],\chi^R\big)\,,$$

and function f is efficiently computable.

We prove the Splitting lemma, Lemma 2.4, mainly in Section 4, by applying the definitions and results of Section 3.

3 Interchange operation on functions

In this section, we prepare the proof of the Splitting lemma, that we give in the next section. We consider an interchange operation on functions, that has similarities with the string interchange operation considered by Amir et al. and Kapah et al. [2, 4]. We prove the existence of ψ-colour-monotone embeddings, and consider an invariance property for first-fit and co-first-fit subcolourings.

We define our interchange operation. Let φ be a function on $\mathbb{N}$, and let $a,b \in \mathbb{N}$ where $\varphi(a)$ and $\varphi(b)$ are defined. The *a,b-interchange* of φ, denoted as $\varphi^{(a\leftrightarrow b)}$, is defined as follows, for $x \in \mathbb{N}$ where $\varphi(x)$ is defined:

$$\varphi^{(a\leftrightarrow b)}(x) \ =_{\text{def}} \begin{cases} \varphi(x) & \text{, if } x \neq a \text{ and } x \neq b \\ \varphi(b) & \text{, if } x = a \\ \varphi(a) & \text{, if } x = b\,. \end{cases}$$

Note that φ and $\varphi^{(a\leftrightarrow b)}$ are defined on the same integers. The interchange operation will be the basis for all results in this paper.

3.1 Easy properties of the interchange operation

We consider the cross number of functions after applying an interchange operation, and we conclude the useful colour-monotonicity result as a direct consequence.

Let φ be a function on $\mathbb{N}$, and let $A \subseteq \mathbb{N}$ and $p \in \mathbb{N}$. The *cross set* of φ with respect to A and with respect to (A,p) are:

$$\begin{aligned} \text{CROSS}(\varphi;A) \ &=_{\text{def}} \ \Big\{(x,y):\ x,y \in A \text{ and } x < y \text{ and } \varphi(y) < \varphi(x)\Big\} \\ \text{CROSS}(\varphi;A,p) \ &=_{\text{def}} \ \Big\{x \in A:\ \big\{(x,p),(p,x)\big\} \cap \text{CROSS}(\varphi;\mathbb{N}) \neq \emptyset\Big\}\,. \end{aligned}$$

Recall from the definition of cross(φ) that only pairs (x,y) are considered where $\varphi(x)$ and $\varphi(y)$ are defined.

Lemma 3.1. *Let φ be a function on $\mathbb{N}$. Let $(a,b) \in \mathrm{CROSS}(\varphi;\mathbb{N})$. Let $A =_{\mathrm{def}} \mathbb{N} \setminus \{a,b\}$ and $B =_{\mathrm{def}} \{x \in \mathbb{N} : x < a \text{ or } b < x\}$ and $C =_{\mathrm{def}} \{x \in \mathbb{N} : a < x < b\}$. Then,*

1) $(a,b) \notin \mathrm{CROSS}(\varphi^{(a\leftrightarrow b)};\mathbb{N})$

2) $\mathrm{CROSS}(\varphi;A) = \mathrm{CROSS}(\varphi^{(a\leftrightarrow b)};A)$

3) $\mathrm{CROSS}(\varphi;B,a) \;=\; \mathrm{CROSS}(\varphi^{(a\leftrightarrow b)};B,b)$ *and*
$\mathrm{CROSS}(\varphi;B,b) \;=\; \mathrm{CROSS}(\varphi^{(a\leftrightarrow b)};B,a)$

4) $\mathrm{CROSS}(\varphi^{(a\leftrightarrow b)};C,a) \;\subseteq\; \mathrm{CROSS}(\varphi;C,a) \setminus \mathrm{CROSS}(\varphi;C,b)$ *and*
$\mathrm{CROSS}(\varphi^{(a\leftrightarrow b)};C,b) \;\subseteq\; \mathrm{CROSS}(\varphi;C,b) \setminus \mathrm{CROSS}(\varphi;C,a)$.

Proof. Observe that $(a,b) \in \mathrm{CROSS}(\varphi;\mathbb{N})$ means $a < b$ and $\varphi(b) < \varphi(a)$.

Claim 1. Since $\varphi^{(a\leftrightarrow b)}(a) = \varphi(b) < \varphi(a) = \varphi^{(a\leftrightarrow b)}(b)$, we observe $(a,b) \notin \mathrm{CROSS}(\varphi^{(a\leftrightarrow b)};\mathbb{N})$ all right.

Claim 2. Let $x,y \in A$, and assume $\varphi(x)$ and $\varphi(y)$ defined. Since $\varphi(x) = \varphi^{(a\leftrightarrow b)}(x)$ and $\varphi(y) = \varphi^{(a\leftrightarrow b)}(y)$, we observe $(x,y) \in \mathrm{CROSS}(\varphi;A)$ if and only if $(x,y) \in \mathrm{CROSS}(\varphi^{(a\leftrightarrow b)};A)$.

Claim 3. Let $x \in B$, and assume $\varphi(x)$ defined. Note $\varphi^{(a\leftrightarrow b)}(x) = \varphi(x)$, and $x < a$ if and only if $x < b$. So:

$$\begin{aligned}\big(x \in \mathrm{CROSS}(\varphi;B,a)\big) \quad &\text{if and only if} \quad \big(x < a \;\Leftrightarrow\; \varphi(a) < \varphi(x)\big)\\ &\text{if and only if} \quad \big(x < b \;\Leftrightarrow\; \varphi^{(a\leftrightarrow b)}(b) < \varphi^{(a\leftrightarrow b)}(x)\big)\\ &\text{if and only if} \quad \big(x \in \mathrm{CROSS}(\varphi^{(a\leftrightarrow b)};B,b)\big)\,.\end{aligned}$$

Analogously the proof of $\mathrm{CROSS}(\varphi;B,b) = \mathrm{CROSS}(\varphi^{(a\leftrightarrow b)};B,a)$.

Claim 4. If $x \in \mathrm{CROSS}(\varphi^{(a\leftrightarrow b)};C,a)$ then $\varphi^{(a\leftrightarrow b)}(x) < \varphi^{(a\leftrightarrow b)}(a)$, and thus, $\varphi(x) < \varphi(b) < \varphi(a)$, and $x \in \mathrm{CROSS}(\varphi;C,a) \setminus \mathrm{CROSS}(\varphi;C,b)$. And if $x \in \mathrm{CROSS}(\varphi^{(a\leftrightarrow b)};C,b)$ then $\varphi(b) < \varphi(a) < \varphi(x)$, and $x \in \mathrm{CROSS}(\varphi;C,b) \setminus \mathrm{CROSS}(\varphi;C,a)$. ∎

Lemma 3.1 implies the colour-monotonicity result for colour-preserving embeddings of smallest cross number.

Lemma 3.2. *Let (ψ,χ) be a satisfiable k-colour embedding pair. There is $\varphi \in \mathscr{E}(\psi,\chi)$ that is ψ-colour-monotone and* $\mathrm{cross}(\psi,\chi) = \mathrm{cross}(\varphi)$.

Proof. Let $\varphi \in \mathscr{E}(\psi,\chi)$. If φ is not ψ-colour-monotone then there is $(a,b) \in \mathrm{CROSS}(\varphi;\mathbb{N})$ such that $\psi(a) \neq 0$ and $\psi(b) \neq 0$ and $\psi(a) = \psi(b)$, and $\varphi^{(a\leftrightarrow b)} \in \mathscr{E}(\psi,\chi)$, and $\mathrm{cross}(\varphi^{(a\leftrightarrow b)}) \leq \mathrm{cross}(\varphi)$ due to Lemma 3.1. Thus, for every $\varphi \in \mathscr{E}(\psi,\chi)$, there is $\varphi^* \in \mathscr{E}(\psi,\chi)$ that is ψ-colour-monotone and that satisfies $\mathrm{cross}(\varphi^*) \leq \mathrm{cross}(\varphi)$, where we can use the function defined in the second part of the proof of Lemma 2.2 as φ^* in case of $\mathrm{cross}(\psi,\chi) = \aleph_0$. This proves the claim. ∎

3.2 Subcolourings and the interchange operation

Let ψ_1 and ψ_2 be partial k-colourings for $\mathbb{N}$. We write $\psi_1 \sim \psi_2$ if $\mathrm{cross}(\psi_1, \psi_2) = \mathrm{cross}(\psi_2, \psi_1) = 0$. We can say that $\psi_1 \sim \psi_2$ means that there is a monotone bijective colour-preserving embedding for (ψ_1, ψ_2). Informally, we can also say that $\psi_1 \sim \psi_2$ means that ψ_1 and ψ_2 are equal when restricting to the coloured integers.

As an auxiliary result, we show that above's similarity property transfers to subcolourings. Throughout this subsection, let $\mathfrak{c}$ be a suitable k-colour vector.

Lemma 3.3. *Let ψ_1 and ψ_2 be partial k-colourings for $\mathbb{N}$, and assume $\psi_1 \sim \psi_2$. Then, $\psi_1[\mathfrak{c}] \sim \psi_2[\mathfrak{c}]$ and $\psi_1[-\mathfrak{c}] \sim \psi_2[-\mathfrak{c}]$.*

Proof. If $\psi_1 \equiv 0$ then $\psi_2 \equiv 0$, and $\psi_1[\mathfrak{c}] \equiv \psi_1[-\mathfrak{c}] \equiv 0$ and $\psi_2[\mathfrak{c}] \equiv \psi_2[-\mathfrak{c}] \equiv 0$. Otherwise, $\psi_1 \not\equiv 0$ and $\psi_2 \not\equiv 0$. Let $p_1, p_2 \in \mathbb{N}$ be smallest such that $\psi_1(p_1) \neq 0$ and $\psi_2(p_2) \neq 0$, that exist. If $\mathfrak{c} = 0$ then $\psi_1[\mathfrak{c}] \equiv \psi_2[\mathfrak{c}] \equiv 0$ and $\psi_1[-\mathfrak{c}](x) = \psi_1(x + p_1)$ and $\psi_2[-\mathfrak{c}](x) = \psi_2(x + p_2)$ for every $x \in \mathbb{N}$, and $\psi_1[-\mathfrak{c}] \sim \psi_2[-\mathfrak{c}]$ follows. Henceforth, assume $\mathfrak{c} \neq 0$. Without loss of generality, we may assume $p_1 \geq p_2$.

If $p_1 \geq 1$ then $\psi_1[\mathfrak{c}] = (\psi_1)'[\mathfrak{c}]$ and $\psi_1[-\mathfrak{c}] = (\psi_1)'[-\mathfrak{c}]$, and since $\psi_1 \sim (\psi_1)'$, we conclude by induction. Otherwise, $p_1 = p_2 = 0$. Recall in the following that $\psi_1(0) = \psi_2(0)$ is the case. We distinguish between the two cases about $c_{\psi_1(0)}$.

Assume $c_{\psi_1(0)} = 0$. By an inductive argument, we obtain $(\psi_1)'[\mathfrak{c}] \sim (\psi_2)'[\mathfrak{c}]$ and $(\psi_1)'[-\mathfrak{c}] \sim (\psi_2)'[-\mathfrak{c}]$, so that $\psi_1[\mathfrak{c}] \sim \psi_2[\mathfrak{c}]$ and $\psi_1[-\mathfrak{c}] \sim \psi_2[-\mathfrak{c}]$ directly follow.

Assume $c_{\psi_1(0)} \geq 1$. Let $\hat{\mathfrak{c}}$ be the k-colour vector such that $c_j = \hat{c}_j$ for every $1 \leq j \leq k$ with $j \neq \psi_1(0)$ and $c_{\psi_1(0)} = \hat{c}_{\psi_1(0)} + 1$. By induction, $(\psi_1)'[\hat{\mathfrak{c}}] \sim (\psi_2)'[\hat{\mathfrak{c}}]$ and $(\psi_1)'[-\hat{\mathfrak{c}}] \sim (\psi_2)'[-\hat{\mathfrak{c}}]$, so that $\psi_1[\mathfrak{c}] \sim \psi_2[\mathfrak{c}]$ and $\psi_1[-\mathfrak{c}] \sim \psi_2[-\mathfrak{c}]$ directly follow. ■

We apply Lemma 3.3 to prove two results about the interchange operation and subcolourings. We make these assumptions for the remaining part of this subsection. Let ψ be a partial k-colouring for $\mathbb{N}$, and we assume $\psi \not\equiv 0$. Let $t \in \mathbb{N}$, and we assume $\psi(t) \neq 0$ and $\psi(t+1) \neq 0$ and $\psi(t) \neq \psi(t+1)$. As a remark, if one of these assumptions is not satisfied, we can conclude below's results by a direct application of Lemma 3.3, as we will do so in later applications. We assume that $\mathfrak{c}$ is a colour vector for ψ. And let $\mathfrak{f}$ be the colour vector of $(\psi, \{x \in \mathbb{N} : x \leq t+1\})$.

Lemma 3.4. *Assume $c_{\psi(t)} < f_{\psi(t)}$. Then, $\psi[\mathfrak{c}] = \psi^{(t \leftrightarrow t+1)}[\mathfrak{c}]$.*

Proof. If $\mathfrak{c} = 0$ then $\psi[\mathfrak{c}] \equiv 0$ and $\psi^{(t \leftrightarrow t+1)}[\mathfrak{c}] \equiv 0$, and the claim follows. We henceforth assume $\mathfrak{c} \neq 0$. We prove the claim by induction on t.

Assume $t = 0$. Clearly: $\psi^{(t \leftrightarrow t+1)} = \psi^{(0 \leftrightarrow 1)}$ and $0 = c_{\psi(0)} < f_{\psi(0)} = 1$. Thus, $\psi[\mathfrak{c}] = \psi'[\mathfrak{c}]$. If $c_{\psi(1)} = 0$ then $\psi'[\mathfrak{c}] = \psi''[\mathfrak{c}]$ and $\psi^{(0 \leftrightarrow 1)}[\mathfrak{c}] = (\psi^{(0 \leftrightarrow 1)})'[\mathfrak{c}] = (\psi^{(0 \leftrightarrow 1)})''[\mathfrak{c}] = \psi''[\mathfrak{c}]$. Otherwise, $c_{\psi(1)} \geq 1$. Let $\hat{\mathfrak{c}}$ be the k-colour vector satisfying $c_j = \hat{c}_j$ for every $1 \leq j \leq k$ where $j \neq \psi(1)$ and $c_{\psi(1)} = \hat{c}_{\psi(1)} + 1$. Then, for $x \in \mathbb{N}$,

$$\psi[\mathfrak{c}](x) \;=\; \psi'[\mathfrak{c}](x) \;=\; \begin{cases} \psi'(0) & , \; x = 0 \\ \psi''[\hat{\mathfrak{c}}](x-1) & , \; x \geq 1 \end{cases}$$

and

$$\psi^{(0 \leftrightarrow 1)}[\mathfrak{c}](x) \;=\; \begin{cases} \psi^{(0 \leftrightarrow 1)}(0) & , \; x = 0 \\ (\psi^{(0 \leftrightarrow 1)})'[\hat{\mathfrak{c}}](x-1) & , \; x \geq 1\,. \end{cases}$$

Since $\psi'(0) = \psi(1) = \psi^{(0\leftrightarrow 1)}(0)$, it remains to verify $\psi''[\hat{\mathfrak{c}}] = (\psi^{(0\leftrightarrow 1)})'[\hat{\mathfrak{c}}]$. If $\hat{\mathfrak{c}} = 0$ then $\psi''[\hat{\mathfrak{c}}] \equiv (\psi^{(0\leftrightarrow 1)})'[\hat{\mathfrak{c}}] \equiv 0$, and if $\hat{\mathfrak{c}} \neq 0$ then $(\psi^{(0\leftrightarrow 1)})'(0) = \psi^{(0\leftrightarrow 1)}(1) = \psi(0)$ and $\hat{c}_{\psi(0)} = c_{\psi(0)} = 0$ implies $(\psi^{(0\leftrightarrow 1)})'[\hat{\mathfrak{c}}] = ((\psi^{(0\leftrightarrow 1)})')'[\hat{\mathfrak{c}}] = (\psi^{(0\leftrightarrow 1)})''[\hat{\mathfrak{c}}] = \psi''[\hat{\mathfrak{c}}]$.

Assume $t \geq 1$. Observe $\psi(0) = \psi^{(t\leftrightarrow t+1)}(0)$. If $\psi(0) = 0$ or $c_{\psi(0)} = 0$ then, by the induction hypothesis, $\psi[\mathfrak{c}] = \psi'[\mathfrak{c}] = (\psi')^{(t-1\leftrightarrow t)}[\mathfrak{c}] = \big(\psi^{(t\leftrightarrow t+1)}\big)'[\mathfrak{c}] = \psi^{(t\leftrightarrow t+1)}[\mathfrak{c}]$. Otherwise, where $\hat{\mathfrak{c}}$ is the k-colour vector satisfying $c_j = \hat{c}_j$ for every $1 \leq j \leq k$ where $j \neq \psi(0)$ and $c_{\psi(0)} = \hat{c}_{\psi(0)} + 1$, for every $x \in \mathbb{N}$

$$\psi[\mathfrak{c}](x) = \left\{ \begin{array}{ll} \psi(0) & , \ x = 0 \\ \psi'[\hat{\mathfrak{c}}](x-1) & , \ x \geq 1 \end{array} \right\} = \left\{ \begin{array}{ll} \psi^{(t\leftrightarrow t+1)}(0) & , \ x = 0 \\ (\psi')^{(t-1\leftrightarrow t)}[\hat{\mathfrak{c}}](x-1) & , \ x \geq 1 \end{array} \right\} = \psi^{(t\leftrightarrow t+1)}[\mathfrak{c}](x)\,.$$

For all applications of the induction hypothesis, it is important to note that the assumption about $c_{\psi(t)} < f_{\psi(t)}$ is analogously satisfied. ∎

Lemma 3.5. *Assume $c_{\psi(t)} \leq f_{\psi(t)}$ and $c_{\psi(t+1)} \geq f_{\psi(t+1)}$. Then, $\psi[-\mathfrak{c}] = \psi^{(t\leftrightarrow t+1)}[-\mathfrak{c}]$.*

Proof. Note $\mathfrak{c} \neq 0$ because of $c_{\psi(t+1)} \geq f_{\psi(t+1)} \geq 1$. We adopt the proof of Lemma 3.4.

Assume $t = 0$. This means: $f_{\psi(0)} = f_{\psi(1)} = 1$ and $c_{\psi(0)} \leq 1$. If $c_{\psi(0)} = 1$ then, for $\hat{\hat{\mathfrak{c}}}$ the k-colour vector satisfying $c_j = \hat{\hat{c}}_j$ for every $1 \leq j \leq k$ where $j \neq \psi(0)$ and $j \neq \psi(1)$ and $c_{\psi(0)} = \hat{\hat{c}}_{\psi(0)} + 1$ and $c_{\psi(1)} = \hat{\hat{c}}_{\psi(1)} + 1$,

$$\psi[-\mathfrak{c}] \;=\; \psi''[-\hat{\hat{\mathfrak{c}}}] \;=\; (\psi^{(0\leftrightarrow 1)})''[-\hat{\hat{\mathfrak{c}}}] \;=\; \psi^{(0\leftrightarrow 1)}[-\hat{\hat{\mathfrak{c}}}]\,.$$

If $c_{\psi(0)} = 0$ then, for $\hat{\mathfrak{c}}$ the k-colour vector satisfying $c_j = \hat{c}_j$ for every $1 \leq j \leq k$ where $j \neq \psi(1)$ and $c_{\psi(1)} = \hat{c}_{\psi(1)} + 1$,

$$\begin{array}{rcl} \psi[-\mathfrak{c}](x) & = & \left\{ \begin{array}{ll} \psi(0) & , \ x = 0 \\ \psi'[-\mathfrak{c}](x-1) & , \ x \geq 1 \end{array} \right\} = \left\{ \begin{array}{ll} \psi(0) & , \ x = 0 \\ \psi''[-\hat{\mathfrak{c}}](x-1) & , \ x \geq 1 \end{array} \right\} \\ \| & & \\ \psi^{(0\leftrightarrow 1)}[-\mathfrak{c}](x) & = & (\psi^{(0\leftrightarrow 1)})'[-\hat{\mathfrak{c}}](x) = \left\{ \begin{array}{ll} (\psi^{(0\leftrightarrow 1)})'(0) & , \ x = 0 \\ (\psi^{(0\leftrightarrow 1)})''[-\hat{\mathfrak{c}}](x-1) & , \ x \geq 1 \end{array} \right\}. \end{array}$$

Assume $t \geq 1$. If $\psi(0) = 0$ then $\psi[-\mathfrak{c}] = \psi'[-\mathfrak{c}] = (\psi')^{(t-1\leftrightarrow t)}[-\mathfrak{c}] = \psi^{(t\leftrightarrow t+1)}[-\mathfrak{c}]$, and if $\psi(0) \neq 0$ and $c_{\psi(0)} \geq 1$ then, for $\hat{\mathfrak{c}}$ the appropriately defined colour vector, $\psi[-\mathfrak{c}] = \psi'[-\hat{\mathfrak{c}}] = (\psi')^{(t-1\leftrightarrow t)}[-\hat{\mathfrak{c}}] = \psi^{(t\leftrightarrow t+1)}[-\mathfrak{c}]$. Otherwise,

$$\psi[-\mathfrak{c}](x) = \left\{ \begin{array}{ll} \psi(0) & , \ x = 0 \\ \psi'[-\mathfrak{c}](x-1) & , \ x \geq 1 \end{array} \right\} = \left\{ \begin{array}{ll} \psi^{(t\leftrightarrow t+1)}(0) & , \ x = 0 \\ (\psi')^{(t-1\leftrightarrow t)}[-\mathfrak{c}](x-1) & , \ x \geq 1 \end{array} \right\} = \psi^{(t\leftrightarrow t+1)}[-\mathfrak{c}](x)\,.$$

The claim of the lemma follows. ∎

4 Proof of the Splitting lemma

We prove the Splitting lemma. We execute a proof by induction on the distance between $\mathfrak{c}$ of the given input tuple and the colour vector of $(\psi, \{x \in \mathbb{N} : x < p\})$. This distance represents the number of iteration steps. We partition the proof into smaller results, that are presented in the three subsections of this section.

Throughout this section, let $\big(\psi, \chi; (p, q); \mathfrak{c}\big)$ be a good k-input tuple. Particularly note $\psi \not\equiv 0$ because of $\psi(p) \neq 0$. Let $\mathfrak{e}$ be the colour vector of $(\psi, \{x \in \mathbb{N} : x < p\})$. Recall $e_{\psi(p)} = c_{\psi(p)}$ according to the definition of good input tuples. Let $\tilde{\mathfrak{c}}$ be the k-colour vector satisfying $c_j = \tilde{c}_j$ for every $1 \leq j \leq k$ where $j \neq \psi(p)$ and $\tilde{c}_{\psi(p)} = c_{\psi(p)} + 1$. Observe that $\tilde{\mathfrak{c}}$ is the colour vector of $(\psi, \{x \in \mathbb{N} : x \leq p\})$.

4.1 Part I of the proof

We prove the induction base case, when the distance of $\mathfrak{c}$ and $\mathfrak{e}$ is 0, i.e., when $\mathfrak{c}$ and $\mathfrak{e}$ are equal. We assume this to be the case for the two results of this subsection.

We split ψ and χ into two each. For $x \in \mathbb{N}$, let

$$\psi^L(x) \ =_{\text{def}} \begin{cases} \psi(x) & \text{, if } x < p \\ 0 & \text{, if } x \geq p \end{cases} \qquad \text{and} \qquad \psi^R(x) \ =_{\text{def}} \begin{cases} 0 & \text{, if } x \leq p \\ \psi(x) & \text{, if } x > p\,, \end{cases}$$

and for $y \in \mathbb{N}$, let

$$\chi^L(y) \ =_{\text{def}} \begin{cases} \chi(y) & \text{, if } y < q \\ 0 & \text{, if } y \geq q \end{cases} \qquad \text{and} \qquad \chi^R(y) \ =_{\text{def}} \begin{cases} 0 & \text{, if } y \leq q \\ \chi(y) & \text{, if } y > q\,. \end{cases}$$

It is to note that $x = p$ and $y = q$ play a special role, since $\psi^L(p) = \psi^R(p) = 0$ and $\chi^L(q) = \chi^R(q) = 0$.

Lemma 4.1. $\psi[\mathfrak{c}] \sim \psi^L$ *and* $\psi[-\tilde{\mathfrak{c}}] \sim \psi^R$

Proof. Assume $\mathfrak{c} = \mathfrak{e} = 0$. This means $\psi(0) = \cdots = \psi(p-1) = 0$ and $\psi[\mathfrak{c}] \equiv 0$, and thus, $\psi[\mathfrak{c}] \sim \psi^L$. For $\psi[-\tilde{\mathfrak{c}}]$, note $\tilde{c}_{\psi(p)} = 1$, and $\psi[-\tilde{\mathfrak{c}}] \sim \psi^R$ follows from an iterative application of the definition of co-first-fit subcolourings.

Assume $\mathfrak{c} \neq 0$. This particularly means $p \geq 1$. If $\psi(0) = 0$ then $\psi[\mathfrak{c}] = \psi'[\mathfrak{c}]$ and $\psi[-\tilde{\mathfrak{c}}] = \psi'[-\tilde{\mathfrak{c}}]$, and $\psi[\mathfrak{c}] \sim \psi'[\mathfrak{c}] \sim \psi^L$ and $\psi[-\tilde{\mathfrak{c}}] \sim \psi'[-\tilde{\mathfrak{c}}] \sim \psi^R$ follow by induction. Otherwise, $\psi(0) \neq 0$, which particularly means $\tilde{c}_{\psi(0)} \geq c_{\psi(0)} = e_{\psi(0)} \geq 1$. Let $\hat{\mathfrak{c}}$ and $\hat{\tilde{\mathfrak{c}}}$ be the k-colour vectors satisfying $\hat{c}_j = c_j$ and $\hat{\tilde{c}}_j = \tilde{c}_j$ for every $1 \leq j \leq k$ where $j \neq \psi(0)$ and $c_{\psi(0)} = \hat{c}_{\psi(0)} + 1$ and $\tilde{c}_{\psi(0)} = \hat{\tilde{c}}_{\psi(0)} + 1$. So, $\psi[-\tilde{\mathfrak{c}}] = \psi'[-\hat{\tilde{\mathfrak{c}}}]$ by definition, and $\psi'[-\hat{\tilde{\mathfrak{c}}}] \sim (\psi^R)' \sim \psi^R$ by induction, and thus, $\psi[-\tilde{\mathfrak{c}}] \sim \psi^R$. Similarly, $\psi'[\hat{\mathfrak{c}}] \sim (\psi^L)'$ by induction, and $\psi[\mathfrak{c}] \sim \psi^L$ follows. ■

Lemma 4.2. $(\psi[\mathfrak{c}], \chi^L)$ *and* $(\psi[-\tilde{\mathfrak{c}}], \chi^R)$ *are satisfiable k-colour embedding pairs, and*

$$\text{cross}\big(\psi, \chi; (p, q); \mathfrak{c}\big) \ = \ \text{cross}\big(\psi[\mathfrak{c}], \chi^L\big) + \text{cross}\big(\psi[-\tilde{\mathfrak{c}}], \chi^R\big)\,.$$

Proof. As an intermediate result, we prove $\operatorname{cross}\big(\psi,\chi;(p,q);\mathfrak{c}\big) = \operatorname{cross}\big(\psi^L,\chi^L\big)+\operatorname{cross}\big(\psi^R,\chi^R\big)$ first. We prove the two inequalities.

For the first inequality, let $\varphi^L \in \mathscr{E}\big(\psi^L,\chi^L\big)$ and $\varphi^R \in \mathscr{E}\big(\psi^R,\chi^R\big)$. Due to Lemma 3.2, we can choose φ^L to be ψ^L-colour-monotone and φ^R to be ψ^R-colour-monotone. We define a function φ on $\mathbb{N}$. For $x \in \mathbb{N}$ with $\psi(x) \neq 0$:

$$\varphi(x) \ =_{\text{def}} \begin{cases} \varphi^L(x) & \text{, if } x < p \\ q & \text{, if } x = p \\ \varphi^R(x) & \text{, if } x > p\,. \end{cases}$$

Due to the definitions of χ^L and χ^R, we observe about the image of φ:

$$\begin{aligned} \varphi\big(\{x \in \mathbb{N} : x < p\}\big) &\subseteq (\chi^L)^{-1}(\{1,\ldots,k\}) \subseteq \{y \in \mathbb{N} : y < q\} \quad \text{and} \\ \varphi\big(\{x \in \mathbb{N} : x > p\}\big) &\subseteq (\chi^R)^{-1}(\{1,\ldots,k\}) \subseteq \{y \in \mathbb{N} : y > q\}\,. \end{aligned}$$

Thus, the injectiveness of φ^L and φ^R implies that φ is injective, too, and

$$\begin{aligned} \operatorname{cross}(\varphi) &= \big|\mathrm{CROSS}(\varphi;\{x \in \mathbb{N} : x < p\})\big| + \big|\mathrm{CROSS}(\varphi;\{x \in \mathbb{N} : x > p\})\big| \\ &\leq \big|\mathrm{CROSS}(\varphi^L;\{x \in \mathbb{N} : x < p\})\big| + \big|\mathrm{CROSS}(\varphi^R;\{x \in \mathbb{N} : x > p\})\big| \\ &\leq \operatorname{cross}(\varphi^L) + \operatorname{cross}(\varphi^R)\,. \end{aligned}$$

We verify that φ is a colour-preserving embedding for (ψ,χ). For $x \in \mathbb{N}$ with $\psi(x) \neq 0$, observe, also by applying above's observation about the image of φ:

$$\psi(x) \ = \begin{cases} \psi^L(x) = \chi^L(\varphi^L(x)) = \chi(\varphi(x)) & \text{, if } x < p \\ \chi(q) = \chi(\varphi(x)) & \text{, if } x = p \\ \psi^R(x) = \chi^R(\varphi^R(x)) = \chi(\varphi(x)) & \text{, if } x > p\,. \end{cases}$$

Thus, $\varphi \in \mathscr{E}\big(\psi,\chi\big)$. It remains to show $\varphi \in \mathscr{E}\big(\psi,\chi;(p,q);\mathfrak{c}\big)$.

Note that the colour vector of ψ^L is equal to $\mathfrak{e}$, and thus equal to $\mathfrak{c}$, and the colour vectors of $(\psi,\{x \in \mathbb{N} : x < p\})$ and $(\chi,\{y \in \varphi^L(\mathbb{N}\setminus\psi^{-1}(0)) : y < q\})$ are equal. Thus, φ satisfies $(\chi,q,\mathfrak{c})$. Also note that φ is ψ-colour-monotone, as an easy consequence of the definition of φ. Thus, $\varphi \in \mathscr{E}\big(\psi,\chi;(p,q);\mathfrak{c}\big)$, and $\operatorname{cross}\big(\psi,\chi;(p,q);\mathfrak{c}\big) \leq \operatorname{cross}(\psi^L,\chi^L) + \operatorname{cross}(\psi^R,\chi^R)$ follows.

For the second inequality, let $\varphi \in \mathscr{E}\big(\psi,\chi;(p,q);\mathfrak{c}\big)$. We may assume $\varphi^{-1}(0) = \emptyset$. Since φ is ψ-colour-monotone, the following is the case for every $1 \leq j \leq k$:

$$e_j \ = \ \big|(\psi^L)^{-1}(j)\big| \ = \ \big|\psi^{-1}(j) \cap \{x \in \mathbb{N} : x < p\}\big| \ = \ \big|\chi^{-1}(j) \cap \varphi(\mathbb{N}) \cap \{y \in \mathbb{N} : y < q\}\big| \ = \ c_j\,.$$

We prove $\varphi\big(\{x \in \mathbb{N} : x < p\}\big) = \varphi(\mathbb{N}) \cap \{y \in \mathbb{N} : y < q\}$. Let $x \in \mathbb{N}$ satisfying $x < p$ and $\psi(x) \neq 0$, and suppose $\varphi(x) > q$ for a contradiction. Since φ is monotone for colour $\psi(x)$ in particular,

$$\begin{aligned} c_{\psi(x)} &= \big|\chi^{-1}(\psi(x)) \cap \varphi(\mathbb{N}) \cap \{y \in \mathbb{N} : y < q\}\big| \\ &\leq \big|\chi^{-1}(\psi(x)) \cap \varphi(\mathbb{N}) \cap \{y \in \mathbb{N} : y < \varphi(x)\}\big| \\ &= \big|\psi^{-1}(\psi(x)) \cap \{x' \in \mathbb{N} : x' < x\}\big| \\ &< \big|\psi^{-1}(\psi(x)) \cap \{x' \in \mathbb{N} : x' < p\}\big| \ = \ e_{\psi(x)}\,, \end{aligned}$$

a contradiction. Similarly, let $x \in \mathbb{N}$ satisfying $x > p$ and $\psi(x) \neq 0$, and suppose $\varphi(x) < q$:

$$\begin{aligned} c_{\psi(x)} &= \left|\chi^{-1}(\psi(x)) \cap \varphi(\mathbb{N}) \cap \{y \in \mathbb{N} : y < q\}\right| \\ &\geq \left|\chi^{-1}(\psi(x)) \cap \varphi(\mathbb{N}) \cap \{y \in \mathbb{N} : y \leq \varphi(x)\}\right| \\ &= \left|\psi^{-1}(\psi(x)) \cap \{x' \in \mathbb{N} : x' \leq x\}\right| \\ &> \left|\psi^{-1}(\psi(x)) \cap \{x' \in \mathbb{N} : x' < p\}\right| \;=\; e_{\psi(x)} \,. \end{aligned}$$

As a consequence, also because of $\varphi(p) = q$,

$$\begin{aligned} \varphi(\mathbb{N}) \cap \{y \in \mathbb{N} : y < q\} &= \varphi\big(\{x \in \mathbb{N} : x < p\}\big) \\ \varphi(\mathbb{N}) \cap \{y \in \mathbb{N} : y > q\} &= \varphi\big(\{x \in \mathbb{N} : x > p\}\big) \,. \end{aligned}$$

We define functions φ^L and φ^R on $\mathbb{N}$, and verify. For $x \in \mathbb{N}$ with $\psi(x) \neq 0$:

- if $x < p$, let $\varphi^L(x) =_{\text{def}} \varphi(x)$; then: $\psi^L(x) = \psi(x) = \chi(\varphi(x)) = \chi^L(\varphi^L(x))$
- if $x > p$, let $\varphi^R(x) =_{\text{def}} \varphi(x)$; then: $\psi^R(x) = \psi(x) = \chi(\varphi(x)) = \chi^R(\varphi^R(x))$.

So, φ^L is a colour-preserving embedding for (ψ^L, χ^L) and φ^R is a colour-preserving embedding for (ψ^R, χ^R), meaning $\varphi^L \in \mathscr{E}\big(\psi^L, \chi^L\big)$ and $\varphi^R \in \mathscr{E}\big(\psi^R, \chi^R\big)$. This particularly shows that (ψ^L, χ^L) and (ψ^R, χ^R) are satisfiable.

It remains to observe $\text{cross}(\varphi) \geq \text{cross}(\varphi^L) + \text{cross}(\varphi^R) \geq \text{cross}(\psi^L, \chi^L) + \text{cross}(\psi^R, \chi^R)$.

We have proved $\text{cross}\big(\psi, \chi; (p, q); \mathfrak{c}\big) = \text{cross}\big(\psi^L, \chi^L\big) + \text{cross}\big(\psi^R, \chi^R\big)$. Since $\psi[\mathfrak{c}] \sim \psi^L$ and $\psi[-\tilde{\mathfrak{c}}] \sim \psi^R$ due to Lemma 4.1, it follows $\text{cross}\big(\psi^L, \chi^L\big) = \text{cross}\big(\psi[\mathfrak{c}], \chi^L\big)$ and $\text{cross}\big(\psi^R, \chi^R\big) = \text{cross}\big(\psi[-\tilde{\mathfrak{c}}], \chi^R\big)$, particularly showing that $(\psi[\mathfrak{c}], \chi^L)$ and $(\psi[-\tilde{\mathfrak{c}}], \chi^R)$ are satisfiable, and we can conclude the claim of the lemma. ∎

4.2 Part II of the proof

We prove a subroutine for our induction step case, that will be applied when $\mathfrak{c}$ and $\mathfrak{e}$ are not equal. We can implicitly assume this to be the case for the results of this subsection, however it is not required. We provide a subroutine for the inductive improvement step.

We define two sets, that we call the *left* and *right reaction set* of $\big(\psi, \chi; (p, q); \mathfrak{c}\big)$:

$$\begin{aligned} \mathcal{L} &=_{\text{def}} \big\{x \in \mathbb{N} : \; x < p \text{ and } \psi(x) \neq 0 \text{ and } e_{\psi(x)} > c_{\psi(x)}\big\} \\ \mathcal{R} &=_{\text{def}} \big\{x \in \mathbb{N} : \; x > p \text{ and } \psi(x) \neq 0 \text{ and } e_{\psi(x)} < c_{\psi(x)}\big\} \,. \end{aligned}$$

Informally, we can say that the left reaction set is for colours that occur too often in $\{x \in \mathbb{N} : x < p\}$ and the right reaction set is for colours that are rare.

We first show a technical result about the elements of $\mathcal{L}$ and $\mathcal{R}$ closest to p.

Lemma 4.3. *Let* $\varphi \in \mathscr{E}\big(\psi, \chi; (p, q); \mathfrak{c}\big)$.

1) Assume $\mathcal{L} \neq \emptyset$*, and let* $t =_{\text{def}} \max \mathcal{L}$*. Then,* $\varphi(t + 1) < \varphi(t)$*.*

2) Assume $\mathcal{R} \neq \emptyset$*, and let* $t =_{\text{def}} \min \mathcal{R}$*. Then,* $\varphi(t) < \varphi(t - 1)$*.*

Proof. We prove the first claim. First, suppose $\varphi(t) < q$. Since $\psi(x) \neq \psi(t)$ for all $t < x \leq p$ by the choice of t, we observe

$$e_{\psi(t)} \;=\; \Big|\big\{x' \in \mathbb{N} :\; x' < p \text{ and } \psi(x') = \psi(t)\big\}\Big| \;=\; \Big|\big\{x' \in \mathbb{N} :\; x' \leq t \;\text{ and } \psi(x') = \psi(t)\big\}\Big|.$$

Since φ is injective and ψ-colour-monotone, we obtain

$$\begin{aligned}
e_{\psi(t)} &= \Big|\big\{x' \in \mathbb{N} :\; x' \leq t \;\text{ and } \psi(x') = \psi(t)\big\}\Big| \\
&= \Big|\varphi\Big(\{x' \in \mathbb{N} :\; x' \leq t\} \cap \psi^{-1}\big(\psi(t)\big)\Big)\Big| \\
&= \Big|\{y' \in \varphi(\mathbb{N} \setminus \psi^{-1}(0)) :\; y' \leq \varphi(t)\} \cap \chi^{-1}(\psi(t))\Big| \\
&\leq \Big|\{y' \in \varphi(\mathbb{N} \setminus \psi^{-1}(0)) :\; y' < q\} \cap \chi^{-1}(\psi(t))\Big| \;=\; c_{\psi(t)}\,,
\end{aligned}$$

contradicting $t \in \mathcal{L}$. So, $q \leq \varphi(t)$. Next, suppose $q \leq \varphi(t) < \varphi(t+1)$. Observe $t+1 \neq p$, so that $t < t+1 < p$. This, together with the ψ-colour-monotonicity of φ, implies $e_{\psi(t+1)} > c_{\psi(t+1)}$, and thus $t+1 \in \mathcal{L}$, a contradiction to the maximum choice of t.

The second claim is proved analogous to the first claim. If $q \leq \varphi(t)$ then

$$\begin{aligned}
e_{\psi(t)} &= \Big|\{x' \in \mathbb{N} :\; x' < p\} \;\cap\; \psi^{-1}(\psi(t))\Big| \\
&= \Big|\{x' \in \mathbb{N} :\; x' \leq t\} \;\cap\; \psi^{-1}(\psi(t))\Big| - 1 \\
&= \Big|\varphi(\{x' \in \mathbb{N} \setminus \psi^{-1}(0) :\; x' \leq t\}) \;\cap\; \chi^{-1}(\psi(t))\Big| - 1 \\
&= \Big|\{y' \in \varphi(\mathbb{N} \setminus \psi^{-1}(0)) :\; y' \leq \varphi(t)\} \cap \chi^{-1}(\psi(t))\Big| - 1 \\
&\geq \Big|\{y' \in \varphi(\mathbb{N} \setminus \psi^{-1}(0)) :\; y' < q\} \cap \chi^{-1}(\psi(t))\Big| \;=\; c_{\psi(t)}\,,
\end{aligned}$$

a contradiction, and if $\varphi(t-1) < \varphi(t) < q$ then $p < t-1 < t$ and $e_{\psi(t-1)} < c_{\psi(t-1)}$, a contradiction. ∎

The inductive improvement step subroutine is the result of the next lemma.

Lemma 4.4. *1) Assume $\mathcal{L} \neq \emptyset$. Let $t =_{\text{def}} \max \mathcal{L}$, and let $t^* =_{\text{def}} t+1$. Then,*

$$\operatorname{cross}\big(\psi, \chi; (p,q); \mathfrak{c}\big) \;=\; \begin{cases} \operatorname{cross}\big(\psi^{(t \leftrightarrow t^*)}, \chi; (p,q); \mathfrak{c}\big) & ,\ \text{if } t^* < p \text{ and } \psi(t^*) = 0 \\ 1 + \operatorname{cross}\big(\psi^{(t \leftrightarrow t^*)}, \chi; (p,q); \mathfrak{c}\big) & ,\ \text{if } t^* < p \text{ and } \psi(t^*) \neq 0 \\ 1 + \operatorname{cross}\big(\psi^{(t \leftrightarrow t^*)}, \chi; (p-1,q); \mathfrak{c}\big) & ,\ \text{if } t^* = p\,. \end{cases}$$

2) Assume $\mathcal{R} \neq \emptyset$. Let $t =_{\text{def}} \min \mathcal{R}$, and let $t^* =_{\text{def}} t-1$. Then,

$$\operatorname{cross}\big(\psi, \chi; (p,q); \mathfrak{c}\big) \;=\; \begin{cases} \operatorname{cross}\big(\psi^{(t \leftrightarrow t^*)}, \chi; (p,q); \mathfrak{c}\big) & ,\ \text{if } t^* > p \text{ and } \psi(t^*) = 0 \\ 1 + \operatorname{cross}\big(\psi^{(t \leftrightarrow t^*)}, \chi; (p,q); \mathfrak{c}\big) & ,\ \text{if } t^* > p \text{ and } \psi(t^*) \neq 0 \\ 1 + \operatorname{cross}\big(\psi^{(t \leftrightarrow t^*)}, \chi; (p+1,q); \mathfrak{c}\big) & ,\ \text{if } t^* = p\,. \end{cases}$$

Proof. We prove the two claims of the lemma simultaneously.

Let t and t^* be chosen according to the assumptions of the lemma, where $t < p$ implies $t^* = t+1$ and $t > p$ implies $t^* = t-1$. Recall $\psi(t) \neq \psi(t^*)$, as argued in the proof of Lemma 4.3. Let φ be a function on $\mathbb{N}$. Observe

$$\begin{aligned} \varphi(\mathbb{N} \setminus \psi^{-1}(0)) &= \varphi^{(t\leftrightarrow t^*)}(\mathbb{N} \setminus \psi^{-1}(0)) \\ \varphi(\mathbb{N} \setminus \psi^{-1}(0)) \cap \{y \in \mathbb{N} : y < q\} &= \varphi^{(t\leftrightarrow t^*)}(\mathbb{N} \setminus \psi^{-1}(0)) \cap \{y \in \mathbb{N} : y < q\}\,. \end{aligned}$$

So,

$$\begin{array}{rcl} \varphi \in \mathscr{E}\big(\psi, \chi\big) & \text{if and only if} & \varphi^{(t\leftrightarrow t^*)} \in \mathscr{E}\big(\psi^{(t\leftrightarrow t^*)}, \chi\big) \\ \varphi \text{ is } \psi\text{-colour-monotone} & \text{if and only if} & \varphi^{(t\leftrightarrow t^*)} \text{ is } \psi^{(t\leftrightarrow t^*)}\text{-colour-monotone} \\ \varphi \text{ satisfies } (\chi, q, \mathfrak{c}) & \text{if and only if} & \varphi^{(t\leftrightarrow t^*)} \text{ satisfies } (\chi, q, \mathfrak{c}), \end{array}$$

and we conclude

$$\varphi \in \mathscr{E}\big(\psi, \chi; (p,q); \mathfrak{c}\big) \quad \text{if and only if} \quad \varphi^{(t\leftrightarrow t^*)} \in \begin{cases} \mathscr{E}\big(\psi^{(t\leftrightarrow t^*)}, \chi; (p,q); \mathfrak{c}\big) & \text{, if } t^* \neq p \\ \mathscr{E}\big(\psi^{(t\leftrightarrow t^*)}, \chi; (t,q); \mathfrak{c}\big) & \text{, if } t^* = p\,. \end{cases}$$

As a consequence: for considering $\mathrm{cross}\big(\psi, \chi; (p,q); \mathfrak{c}\big)$ and $\mathrm{cross}\big(\psi^{(t\leftrightarrow t^*)}, \chi; (p,q); \mathfrak{c}\big)$, it suffices to consider φ and $\varphi^{(t\leftrightarrow t^*)}$.

Let $A =_{\mathrm{def}} \mathbb{N} \setminus \{t, t^*\}$. Let $\varphi \in \mathscr{E}\big(\psi, \chi; (p,q); \mathfrak{c}\big)$, and we assume $\psi^{-1}(0) \cap \varphi^{-1}(\mathbb{N}) = \emptyset$. We claim the following three cross set equalities, whose correctness we will verify at the end of the proof:

$$\begin{aligned} \mathrm{CROSS}(\varphi; A) &= \mathrm{CROSS}(\varphi^{(t\leftrightarrow t^*)}; A) \\ \mathrm{CROSS}(\varphi; A, t) &= \mathrm{CROSS}(\varphi^{(t\leftrightarrow t^*)}; A, t^*) \\ \mathrm{CROSS}(\varphi; A, t^*) &= \mathrm{CROSS}(\varphi^{(t\leftrightarrow t^*)}; A, t)\,. \end{aligned}$$

If $\psi(t^*) = 0$ then $t^* \neq p$ and $\mathrm{CROSS}(\varphi; \mathbb{N}, t^*) = \mathrm{CROSS}(\varphi^{(t\leftrightarrow t^*)}; \mathbb{N}, t) = \emptyset$, and

$$\begin{aligned} \mathrm{CROSS}(\varphi; \mathbb{N}) &= \mathrm{CROSS}(\varphi; A) \cup \mathrm{CROSS}(\varphi; A, t) \\ \mathrm{CROSS}(\varphi^{(t\leftrightarrow t^*)}; \mathbb{N}) &= \mathrm{CROSS}(\varphi^{(t\leftrightarrow t^*)}; A) \cup \mathrm{CROSS}(\varphi^{(t\leftrightarrow t^*)}; A, t^*)\,, \end{aligned}$$

so that $\mathrm{cross}(\varphi) = \mathrm{cross}(\varphi^{(t\leftrightarrow t^*)})$.

Otherwise, assume $\psi(t^*) \neq 0$. We apply Lemma 4.3: if $t < t^*$ then $(t, t^*) \in \mathrm{CROSS}(\varphi)$, and if $t^* < t$ then $(t^*, t) \in \mathrm{CROSS}(\varphi)$. Thus,

$$\begin{aligned} \mathrm{CROSS}(\varphi; \mathbb{N}) &= \mathrm{CROSS}(\varphi; A) \\ &\quad \cup \mathrm{CROSS}(\varphi; A, t) \ \cup \ \mathrm{CROSS}(\varphi; A, t^*) \ \cup \ \begin{cases} \big\{(t, t^*)\big\} & \text{, if } t < t^* \\ \big\{(t^*, t)\big\} & \text{, if } t > t^* \end{cases} \\ \mathrm{CROSS}(\varphi^{(t\leftrightarrow t^*)}) &= \mathrm{CROSS}(\varphi^{(t\leftrightarrow t^*)}; A) \\ &\quad \cup \mathrm{CROSS}(\varphi^{(t\leftrightarrow t^*)}; A, t^*) \ \cup \ \mathrm{CROSS}(\varphi^{(t\leftrightarrow t^*)}; A, t)\,, \end{aligned}$$

so that $\mathrm{cross}(\varphi) = 1 + \mathrm{cross}(\varphi^{(t \leftrightarrow t^*)})$.

To complete the proof, we verify the validity of above's cross set equalities. If $\psi(t^*) \neq 0$ then Lemma 3.1 is applicable, and the equalities directly follow. If $\psi(t^*) = 0$ then the third equality is already validated, and the first and second equality follow analogous to the proof of the corresponding claims of Lemma 3.1. ∎

Observe – implicitly from the result and explicitly from the proof of Lemma 4.4 – that the input tuples considered in Lemma 4.4 are good input tuples always.

4.3 Part III of the proof

We are ready to prove the Splitting lemma. We first show the completed inductive improvement step, and then, we provide the final proof of the Splitting lemma. The inductive improvement step is partitioned into two lemmas, for better understandability. Recall the initial assumptions from the beginning of the section.

Lemma 4.5. *Let $\mathcal{L}$ be the left reaction set of $\big(\psi, \chi; (p,q); \mathfrak{c}\big)$, and assume $\mathcal{L} \neq \emptyset$. Let $t =_{\mathrm{def}} \max \mathcal{L}$. Let $\psi^* =_{\mathrm{def}} (\cdots \psi^{(t \leftrightarrow t+1)} \cdots)^{(p-1 \leftrightarrow p)}$. The four claims hold true:*

1) $\mathrm{cross}\big(\psi, \chi; (p,q); \mathfrak{c}\big) = \big|\{x \in \mathbb{N} : t < x \leq p \text{ and } \psi(x) \neq 0\}\big| + \mathrm{cross}\big(\psi^*, \chi; (p{-}1, q); \mathfrak{c}\big)$
2) $\big(\psi^*, \chi; (p{-}1, q); \mathfrak{c}\big)$ *is a good k-input tuple*
3) *for $\mathfrak{e}^*$ the colour vector of $(\psi^*, \{x \in \mathbb{N} : x < p{-}1\})$, $e_j = e^*_j$ for every $1 \leq j \leq k$ where $j \neq \psi(t)$ and $e_{\psi(t)} = e^*_{\psi(t)} + 1$*
4) $\psi^*[\mathfrak{c}] \sim \psi[\mathfrak{c}]$ *and* $\psi^*[-\tilde{\mathfrak{c}}] \sim \psi[-\tilde{\mathfrak{c}}]$.

Proof. Let $\psi_t =_{\mathrm{def}} \psi$, and for $t \leq i < p$, let $\psi_{i+1} =_{\mathrm{def}} (\psi_i)^{(i \leftrightarrow i+1)}$. Note $\psi_p = \psi^*$. If Lemma 4.4 is applicable, we obtain the following equalities for every $t < i \leq p$

$$\mathrm{cross}\big(\psi_{i-1}, \chi; (p,q); \mathfrak{c}\big) \;=\; \big|\{i : \psi(i) \neq 0\}\big| + \begin{cases} \mathrm{cross}\big(\psi_i, \chi; (p,q); \mathfrak{c}\big) & \text{, if } i < p \\ \mathrm{cross}\big(\psi^*, \chi; (p{-}1,q); \mathfrak{c}\big) & \text{, if } i = p\,, \end{cases}$$

which together yields

$$\mathrm{cross}\big(\psi, \chi; (p,q); \mathfrak{c}\big) \;=\; \big|\{x \in \mathbb{N} : t < x \leq p \text{ and } \psi(x) \neq 0\}\big| + \mathrm{cross}\big(\psi^*, \chi; (p{-}1,q); \mathfrak{c}\big)\,.$$

We verify the applicability of Lemma 4.4:

- for $t \leq i \leq p$, $\big(\psi_i, \chi; (p,q); \mathfrak{c}\big)$ is a good k-input tuple,
  in case of $i = t$ by assumption, and in case of $i > t$ by inductively applying Lemma 4.4
- for $t < i \leq p$:
  $\mathfrak{c}$ is equal to the colour vector of $(\psi_{i-1}, \{x \in \mathbb{N} : x < p\})$, and $(\mathcal{L} \setminus \{t\}) \cup \{i-1\}$ is the left reaction set of $\big(\psi_{i-1}, \chi; (p,q); \mathfrak{c}\big)$, and $\max\big(\mathcal{L} \setminus \{t\}) \cup \{i-1\}\big) = i-1$.

Thus, the first and the second claim hold. The third claim is verified straightforward.

We prove the fourth claim of the lemma. By Lemma 3.3 and if the assumptions of Lemmas 3.4 and 3.5 are satisfied then, for every $t \le i < p$, $\psi[\mathfrak{c}] \sim \psi_t[\mathfrak{c}] \sim \psi_i[\mathfrak{c}] \sim \psi_{i+1}[\mathfrak{c}] \sim \psi_p[\mathfrak{c}] \sim \psi^*[\mathfrak{c}]$ and $\psi[-\tilde{\mathfrak{c}}] \sim \psi_t[-\tilde{\mathfrak{c}}] \sim \psi_i[-\tilde{\mathfrak{c}}] \sim \psi_{i+1}[-\tilde{\mathfrak{c}}] \sim \psi_p[-\tilde{\mathfrak{c}}] \sim \psi^*[-\tilde{\mathfrak{c}}]$ follow.

Let $t \le i < p$, and recall $e_{\psi_i(i)} > c_{\psi_i(i)}$ according to the definition of the left reaction set. We assume $\psi(i+1) \neq 0$. Let $\mathfrak{f}$ be the colour vector of $(\psi_i, \{x \in \mathbb{N} : x \le i+1\})$. It clearly holds $f_{\psi_i(i)} = e_{\psi_i(i)}$, and thus, $f_{\psi_i(i)} > c_{\psi_i(i)}$, and Lemma 3.4 is indeed applicable.

We consider the co-first-fit subcolourings and the applicability of Lemma 3.5. We have to verify $\tilde{c}_{\psi_i(i+1)} \ge f_{\psi_i(i+1)}$ and $\tilde{c}_{\psi_i(i)} \le f_{\psi_i(i)}$. Recall that $i+1$ is not contained in the left reaction set of $\big(\psi_i, \chi; (p,q); \mathfrak{c}\big)$, which means $e_{\psi_i(i+1)} \le c_{\psi_i(i+1)}$, that is also the case for $i+1 = p$, which cannot be concluded from the definition of the left reaction set. Also recall $\psi(t) \neq \psi(p)$, and thus, $\psi_i(i) \neq \psi(p)$, so that $\tilde{c}_{\psi_i(i)} = c_{\psi_i(i)}$, and $\psi_i(i) \neq \psi_i(i+1)$. Since $e_{\psi_i(i)} = f_{\psi_i(i)}$, we conclude $f_{\psi_i(i)} = e_{\psi_i(i)} > c_{\psi_i(i)} = \tilde{c}_{\psi_i(i)}$ all right. For the other inequality, we consider two cases: if $\psi(i+1) = \psi(p)$ then $\tilde{c}_{\psi(i+1)} \ge f_{\psi(i+1)}$, and if $\psi(i+1) \neq \psi(p)$, which particularly means $i+1 < p$, then $\tilde{c}_{\psi(i+1)} = c_{\psi(i+1)} \ge e_{\psi(i+1)} \ge f_{\psi(i+1)}$.

We conclude that the forth claim holds. ■

Lemma 4.6. *Let $\mathcal{R}$ be the right reaction set of $\big(\psi, \chi; (p,q); \mathfrak{c}\big)$, and assume $\mathcal{R} \neq \emptyset$. Let $t =_{\text{def}} \min \mathcal{R}$. Let $\psi^* =_{\text{def}} (\cdots \psi^{(t \leftrightarrow t-1)} \cdots)^{(p+1 \leftrightarrow p)}$. The four claims hold true:*

1) $\mathrm{cross}\big(\psi, \chi; (p,q); \mathfrak{c}\big) = \big|\{x \in \mathbb{N} : p \le x < t \text{ and } \psi(x) \neq 0\}\big| + \mathrm{cross}\big(\psi^*, \chi; (p{+}1, q); \mathfrak{c}\big)$
2) $\big(\psi^*, \chi; (p{+}1, q); \mathfrak{c}\big)$ *is a good k-input tuple*
3) *for $\mathfrak{e}^*$ the colour vector of $(\psi^*, \{x \in \mathbb{N} : x < p{+}1\})$, $e_j = e^*_j$ for every $1 \le j \le k$ where $j \neq \psi(t)$ and $e^*_{\psi(t)} = e_{\psi(t)} + 1$*
4) $\psi^*[\mathfrak{c}] \sim \psi[\mathfrak{c}]$ *and* $\psi^*[-\tilde{\mathfrak{c}}] \sim \psi[-\tilde{\mathfrak{c}}]$.

Proof. Let $\psi_t =_{\text{def}} \psi$, and for $p < i \le t$, let $\psi_{i-1} =_{\text{def}} (\psi_i)^{(i \leftrightarrow i-1)}$. The proofs of the first, second, and third claim are purely analogous to the corresponding proofs for Lemma 4.5.

The proof of the fourth claim follows as in the proof of Lemma 4.5. Let $p < i \le t$. If $\psi(i-1) = 0$ then $\psi_i \sim \psi_{i-1}$, and $\psi_i[\mathfrak{c}] \sim \psi_{i-1}[\mathfrak{c}]$ and $\psi_i[-\tilde{\mathfrak{c}}] \sim \psi_{i-1}[-\tilde{\mathfrak{c}}]$ due to Lemma 3.3. Otherwise, $\psi(i-1) \neq 0$. Let $\mathfrak{f}$ be the colour vector of $(\psi_i, \{x \in \mathbb{N} : x \le i\})$. Observe: $e_{\psi(p)} = c_{\psi(p)} = \tilde{c}_{\psi(p)} - 1 < f_{\psi(p)}$, and $\tilde{c}_{\psi_i(i)} = c_{\psi_i(i)}$ and $e_{\psi_i(i)} = f_{\psi_i(i)}$, and due to the definition of the right reaction set, $f_{\psi(i-1)} > e_{\psi(i-1)} \ge c_{\psi(i-1)}$ and $f_{\psi_i(i)} = e_{\psi_i(i)} < c_{\psi_i(i)} = \tilde{c}_{\psi_i(i)}$.

To conclude $\psi_i[\mathfrak{c}] \sim \psi_{i-1}[\mathfrak{c}]$ by an application of Lemma 3.4, we need to verify $c_{\psi_i(i-1)} < f_{\psi_i(i-1)}$, that is indeed the case because of $\psi_i(i-1) = \psi(i-1)$. To conclude $\psi_i[-\tilde{\mathfrak{c}}] \sim \psi_{i-1}[-\tilde{\mathfrak{c}}]$ by an application of Lemma 3.5, we need to verify $\tilde{c}_{\psi_i(i-1)} \le f_{\psi_i(i-1)}$ and $\tilde{c}_{\psi_i(i)} \ge f_{\psi_i(i)}$. This is indeed the case. ■

We are ready to prove the Splitting lemma. The Splitting lemma also claims the existence of a polynomial-time computable function f. Instead of giving an explicit and separate definition of f, we apply and define f at the same time in the following proof.

Proof of Lemma 2.4. Let $\mathfrak{e}$ be the colour vector of $(\psi, \{x \in \mathbb{N} : x < p\})$. We consider the distance of $\mathfrak{c}$ and $\mathfrak{e}$, that is $|\mathfrak{c} - \mathfrak{e}|$, as a measure. The distance can be seen as a “deficiency” measure. We prove the lemma by induction on $|\mathfrak{c} - \mathfrak{e}|$. If $\mathfrak{c} = \mathfrak{e}$ then the situation considered in Subsection 4.1 occurs and Lemma 4.2 is applicable, and the claim of the lemma directly follows. In this case, $f(\psi, \mathfrak{c}, p) = 0$. Otherwise, $\mathfrak{c} \neq \mathfrak{e}$.

Let $\mathcal{L}$ and $\mathcal{R}$ be the respectively left and right reaction sets of $\big(\psi, \chi; (p,q); \mathfrak{c}\big)$. Observe: if $\mathfrak{c} \not\geq \mathfrak{e}$ then $\mathcal{L}$ is non-empty, and if $\mathfrak{c} \not\leq \mathfrak{e}$ then $\mathcal{R}$ is non-empty. Assume $\mathcal{L} \neq \emptyset$, and let $t =_{\text{def}} \max \mathcal{L}$. Let ψ^* and $\mathfrak{e}^*$ be as in Lemma 4.5. Then,

- $\text{cross}\big(\psi, \chi; (p,q); \mathfrak{c}\big) = \big|\{x \in \mathbb{N} : t < x \leq p \text{ and } \psi(x) \neq 0\}\big| + \text{cross}\big(\psi^*, \chi; (p-1,q); \mathfrak{c}\big)$
- $\psi[\mathfrak{c}] \sim \psi^*[\mathfrak{c}]$ and $\psi[-\tilde{\mathfrak{c}}] \sim \psi^*[-\tilde{\mathfrak{c}}]$
- $|\mathfrak{c} - \mathfrak{e}^*| < |\mathfrak{c} - \mathfrak{e}|$.

By induction,

$$\text{cross}\big(\psi^*, \chi; (p-1,q); \mathfrak{c}\big) \;=\; f(\psi^*, \mathfrak{c}, p-1) + \text{cross}\big(\psi^*[\mathfrak{c}], \chi^L\big) + \text{cross}\big(\psi^*[-\tilde{\mathfrak{c}}], \chi^R\big)$$

holds, and we conclude

$$\begin{aligned}
\text{cross}\big(\psi, \chi; (p,q); \mathfrak{c}\big) \;&=\; \big|\{x \in \mathbb{N} : t < x \leq p \text{ and } \psi(x) \neq 0\}\big| \\
&\qquad + \text{cross}\big(\psi^*, \chi; (p-1,q); \mathfrak{c}\big) \\
&=\; \big|\{x \in \mathbb{N} : t < x \leq p \text{ and } \psi(x) \neq 0\}\big| \\
&\qquad + f(\psi^*, \mathfrak{c}, p-1) + \text{cross}\big(\psi^*[\mathfrak{c}], \chi^L\big) + \text{cross}\big(\psi^*[-\tilde{\mathfrak{c}}], \chi^R\big) \\
&=\; f(\psi, \mathfrak{c}, p) + \text{cross}\big(\psi^*[\mathfrak{c}], \chi^L\big) + \text{cross}\big(\psi^*[-\tilde{\mathfrak{c}}], \chi^R\big)\,,
\end{aligned}$$

where

$$f(\psi, \mathfrak{c}, p) \;=\; \big|\{x \in \mathbb{N} : t < x \leq p \text{ and } \psi(x) \neq 0\}\big| + f(\psi^*, \mathfrak{c}, p-1)\,,$$

the claimed result. If $\mathcal{R} \neq \emptyset$, we apply Lemma 4.6 and analogously conclude by induction

$$\begin{aligned}
\text{cross}\big(\psi, \chi; (p,q); \mathfrak{c}\big) \;&=\; \big|\{x \in \mathbb{N} : p \leq x < t \text{ and } \psi(x) \neq 0\}\big| \\
&\qquad + \text{cross}\big(\psi^*, \chi; (p+1,q); \mathfrak{c}\big) \\
&=\; \big|\{x \in \mathbb{N} : p \leq x < t \text{ and } \psi(x) \neq 0\}\big| \\
&\qquad + f(\psi^*, \mathfrak{c}, p+1) + \text{cross}\big(\psi^*[\mathfrak{c}], \chi^L\big) + \text{cross}\big(\psi^*[-\tilde{\mathfrak{c}}], \chi^R\big) \\
&=\; f(\psi, \mathfrak{c}, p) + \text{cross}\big(\psi^*[\mathfrak{c}], \chi^L\big) + \text{cross}\big(\psi^*[-\tilde{\mathfrak{c}}], \chi^R\big)\,.
\end{aligned}$$

As a remark, note that $\mathcal{L}$ and $\mathcal{R}$ can be considered in arbitrary order, since $\mathfrak{c} - \mathfrak{e}$ induces a partition on the colours. ∎

5 Algorithmic consequences

The Splitting lemma is the main theoretical result of the paper. We employ the Splitting lemma to compute the cross number of k-colour embedding pairs. We then apply this algorithm to compute the swap-delete-edit distance of strings over alphabets of bounded size.

We begin with an auxiliary result about first-fit subcolourings.

Lemma 5.1. *Let ψ be a partial k-colouring for $\mathbb{N}$. Let $\mathfrak{c}$ and $\mathfrak{e}$ be colour vectors for ψ, and assume $\mathfrak{c} \geq \mathfrak{e}$. Then, $(\psi[\mathfrak{c}])[\mathfrak{e}] = \psi[\mathfrak{e}]$.*

Proof. If $\psi \equiv 0$ or $\mathfrak{e} = 0$ then $\psi[\mathfrak{e}] \equiv (\psi[\mathfrak{c}])[\mathfrak{e}] \equiv 0$, and if $\psi(0) = 0$ or $c_{\psi(0)} = 0$ then $(\psi[\mathfrak{c}])[\mathfrak{e}] = (\psi'[\mathfrak{c}])[\mathfrak{e}]$ and $\psi[\mathfrak{e}] = \psi'[\mathfrak{e}]$, because of $e_{\psi(0)} = 0$ clearly, and $(\psi'[\mathfrak{c}])[\mathfrak{e}] = \psi'[\mathfrak{e}]$ by induction. Otherwise, $\psi(0) \neq 0$ and $c_{\psi(0)} \geq 1$, which implies $(\psi[\mathfrak{c}])(0) = \psi(0)$. Let $\hat{\mathfrak{c}}$ be the k-colour vector satisfying $c_j = \hat{c}_j$ for every $1 \leq j \leq k$ where $j \neq \psi(0)$ and $c_{\psi(0)} = \hat{c}_{\psi(0)} + 1$. Observe $(\psi[\mathfrak{c}])' = \psi'[\hat{\mathfrak{c}}]$.

Assume $e_{\psi(0)} = 0$. Then, $(\psi[\mathfrak{c}])[\mathfrak{e}] = (\psi[\mathfrak{c}])'[\mathfrak{e}] = (\psi'[\hat{\mathfrak{c}}])[\mathfrak{e}]$ and $\psi[\mathfrak{e}] = \psi'[\mathfrak{e}]$, and since $\mathfrak{e} \leq \hat{\mathfrak{c}}$, $(\psi'[\hat{\mathfrak{c}}])[\mathfrak{e}] = \psi'[\mathfrak{e}]$ by induction.

Assume $e_{\psi(0)} \geq 1$. Then, $\psi[\mathfrak{c}](0) = (\psi[\mathfrak{c}])[\mathfrak{e}](0) = \psi[\mathfrak{e}](0) = \psi(0)$. Let $\hat{\mathfrak{e}}$ be the k-colour vector satisfying $e_j = \hat{e}_j$ for every $1 \leq j \leq k$ where $j \neq \psi(0)$ and $e_{\psi(0)} = \hat{e}_{\psi(0)} + 1$. Then, $(\psi[\mathfrak{e}])' = \psi'[\hat{\mathfrak{e}}]$, and $(\psi[\mathfrak{c}])'[\hat{\mathfrak{e}}] = (\psi'[\hat{\mathfrak{c}}])[\hat{\mathfrak{e}}] = \psi'[\hat{\mathfrak{e}}]$ by induction, so that $(\psi[\mathfrak{c}])[\mathfrak{e}] = \psi[\mathfrak{e}]$. ∎

We prepare our algorithm for computing the cross number of embedding pairs. Let (ψ, χ) be a $(k{+}1)$-colour embedding pair. Let $q', q \in \mathbb{N}$ where $q' < q$. The *restriction of* χ *to* $[q', q)$ is denoted as $\chi^q_{q'}$, and it is, for $y \in \mathbb{N}$:

$$\chi^q_{q'}(y) \;=_{\text{def}}\; \begin{cases} \chi(y + q') & \text{, if } y < q - q' \\ 0 & \text{, otherwise.} \end{cases}$$

For simplicity, we assume that (ψ, χ) is "finite". Assume there are $s, t \in \mathbb{N}$ such that $\psi(x) = \chi(y) = 0$ for every $x, y \in \mathbb{N}$ where $x > s$ and $y > t$; we choose s and t smallest possible, and we call (s, t) the *limit* of (ψ, χ). An embedding pair that has a limit is called *finite*. Finite embedding pairs have canonical representations, by listing the colours simply.

Let (s, t) be the limit of (ψ, χ). Let $Q =_{\text{def}} \{y \in \mathbb{N} : \chi(y) = k + 1\} \cup \{t + 1\}$, whose elements we will use as "split points". Our algorithm will fill the following 3-dimensional table T. For every colour vector $\mathfrak{c}$ for ψ and every $q \in Q$:

$$T(\mathfrak{c}, c_{k+1}, q) \;=_{\text{def}}\; \begin{cases} \operatorname{cross}\big(\psi[\mathfrak{c}], \chi^q_0\big) & \text{, if } (\psi[\mathfrak{c}], \chi^q_0) \text{ is satisfiable} \\ \infty & \text{, otherwise;} \end{cases}$$

recall that $\operatorname{cross}\big(\psi[\mathfrak{c}], \chi^q_0\big)$ is not defined for embedding pairs $(\psi[\mathfrak{c}], \chi^q_0)$ that are not satisfiable. If $\mathfrak{c}$ is equal to the colour vector of ψ then $\psi \sim \psi[\mathfrak{c}]$, and $T(\mathfrak{c}, c_{k+1}, t+1) = \operatorname{cross}\big(\psi, \chi\big)$, which is the desired value. Our algorithm fills the table by a dynamic-programming approach, iteratively filling $T(\,\cdot\,, \ell, \,\cdot\,)$ for increasing values of ℓ.

To simplify the description in the following, for every $(k{+}1)$-colour vector $\mathfrak{a}$, $\tilde{\mathfrak{a}}$ denotes the $(k{+}1)$-colour vector satisfying $a_j = \tilde{a}_j$ for every $1 \leq j \leq k$ and $\tilde{a}_{k+1} = a_{k+1} + 1$. In our applications, $\mathfrak{a}$ is always chosen such that $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ are colour vectors for ψ. Also, for every $\mathfrak{a}$, let $\mathfrak{E}(\mathfrak{a})$ be the set of the colour vectors $\mathfrak{e}$ for ψ satisfying $\mathfrak{e} \leq \mathfrak{a}$ and $a_{k+1} = e_{k+1} + 1$.

Our algorithm is the result of the next Iteration lemma.

Lemma 5.2 (Iteration)**.** *Let $\mathfrak{c}$ be a colour vector for ψ, and let $q \in Q$. Assume that $(\psi[\mathfrak{c}], \chi^q_0)$ is satisfiable, and assume $c_{k+1} \geq 1$. Let $p' \in \mathbb{N}$ be with $\psi(p') = k + 1$ such that $\big|\{x \in \mathbb{N} : x \leq p'$ and $\psi(x) = k + 1\}\big| = c_{k+1}$. Then,*

$$T(\mathfrak{c}, c_{k+1}, q) \;=\; \min_{\substack{\mathfrak{e} \in \mathfrak{E}(\mathfrak{c}) \\ q' \in Q \text{ where } q' < q}} \; f(\psi[\mathfrak{c}], \mathfrak{e}, p') + \operatorname{cross}\big((\psi[\mathfrak{c}])[-\tilde{\mathfrak{e}}], \chi^q_{q'+1}\big) + T(\mathfrak{e}, c_{k+1} - 1, q')\,.$$

Proof. By the assumptions of the lemma, the definition of T, and by an application of Lemma 2.3:

$$T(\mathfrak{c}, c_{k+1}, q) \;=\; \text{cross}\big(\psi[\mathfrak{c}], \chi_0^q\big) \;=\; \min_{\substack{\mathfrak{e}\in\mathfrak{E}(\mathfrak{c})\\ q'\in Q \text{ where } q'<q}} \text{cross}\big(\psi[\mathfrak{c}], \chi_0^q; (p', q'); \mathfrak{e}\big)\,.$$

For the restriction of $q' < q$ on q', recall $\chi_0^q(y) = 0$ for every $y \in \mathbb{N}$ where $y \geq q$. Also note here that $\big(\psi[\mathfrak{c}], \chi_0^q; (p', q'); \mathfrak{e}\big)$ may not be satisfiable, but this is only due to an unfortunate choice of q', and such q' are easily detectable.

We next apply Lemma 2.4 to the satisfiable $(k{+}1)$-input tuples $\big(\psi[\mathfrak{c}], \chi_0^q; (p', q'); \mathfrak{e}\big)$, and simplify already:

$$\begin{aligned}
\text{cross}\big(\psi[\mathfrak{c}], \chi_0^q; (p', q'); \mathfrak{e}\big) \;&=\; f(\psi[\mathfrak{c}], \mathfrak{e}, p') + \text{cross}\big((\psi[\mathfrak{c}])[\mathfrak{e}], \chi_0^{q'}\big) + \text{cross}\big((\psi[\mathfrak{c}])[-\tilde{\mathfrak{e}}], \chi_{q'+1}^q\big)\\
&=\; f(\psi[\mathfrak{c}], \mathfrak{e}, p') + \text{cross}\big((\psi[\mathfrak{c}])[-\tilde{\mathfrak{e}}], \chi_{q'+1}^q\big) + \text{cross}\big(\psi[\mathfrak{e}], \chi_0^{q'}\big)\\
&=\; f(\psi[\mathfrak{c}], \mathfrak{e}, p') + \text{cross}\big((\psi[\mathfrak{c}])[-\tilde{\mathfrak{e}}], \chi_{q'+1}^q\big) + T(\mathfrak{e}, e_{k+1}, q')\,.
\end{aligned}$$

Recall $\mathfrak{c} \geq \mathfrak{e}$ and therefore $(\psi[\mathfrak{c}])[\mathfrak{e}] = \psi[\mathfrak{e}]$ due to Lemma 5.1. Also recall $e_{k+1} = c_{k+1} - 1$. ∎

We can conclude the tractability result for computing the cross number of finite embedding pairs.

Proposition 5.3. *For $k \geq 1$, the cross number of finite k-colour embedding pairs can be computed in polynomial time.*

Proof. We prove the result by induction on k. Let $\omega_k(s,t)$ denote the running time for computing the cross number of finite k-colour embedding pairs, where (s,t) is the limit of the input embedding pair. If $k = 1$ then ω_1 is of polynomial order, since computing the cross number of 1-colour embedding pairs reduces to a satisfiability check, that is equivalent to determining and comparing the colour vectors of the two input partial 1-colourings.

Next, we consider ω_{k+1}. Let (ψ, χ) be a finite $(k{+}1)$-colour embedding pair with limit (s,t). There are at most $(s+1)^{k+1}$ colour vectors for ψ. Thus, the size of table T is bounded from above by $(s{+}1)^{k+1}\cdot(t{+}1)$. Recall here that the second component of $T(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ directly depends on the first component.

We consider the time for computing the entries of the table. We proceed in an orderly fashion, by computing $T(\,\cdot\,,\ell,\,\cdot\,)$ for increasing values of ℓ. Let $\mathfrak{c}$ and q be given, and we assume $c_{k+1} = \ell$. Assume $\ell = 0$. Then, $\psi[\mathfrak{c}]$ contains no occurrence of colour $k{+}1$, and $\psi[\mathfrak{c}]$ is a partial k-colouring for $\mathbb{N}$, and $T(\mathfrak{c}, 0, q)$ can be computed in $\omega_k(s,t)$ time. Note that all occurrences of colour $k{+}1$ in χ can be replaced by colour 0.

Assume $\ell \geq 1$. Note that $\big|\{x \in \mathbb{N} : \psi[\mathfrak{c}](x) = k+1\}\big| = \ell$. Let $p' =_{\text{def}} \max\{x \in \mathbb{N} : \psi[\mathfrak{c}](x) = k+1\}$. For the computation of $T(\mathfrak{c}, \ell, q)$, we apply Lemma 5.2. Recall that we can assume $\big|\{y \in \mathbb{N} : y < q \text{ and } \chi(y) = k+1\}\big| \geq \ell$, since otherwise, $T(\mathfrak{c}, \ell, q) = \infty$.

We determine the necessary numbers. The cardinality of $\mathfrak{E}(\mathfrak{c})$ is bounded from above by $(s{+}1)^k$, since $e_{k+1} = \ell{-}1$ for every $\mathfrak{e} \in \mathfrak{E}(\mathfrak{c})$. The cardinality of Q is at most $t{+}1$. Together, there are at most $(s{+}1)^k$ values of $f(\psi[\mathfrak{c}], \mathfrak{e}, p')$ to be computed, and there are at most $(s+1)^k \cdot (t+1)$ values of $\text{cross}\big((\psi[\mathfrak{c}])[-\tilde{\mathfrak{e}}], \chi_{q'+1}^q\big)$ to be computed. What needs to be noted is: $(\psi[\mathfrak{c}])[-\tilde{\mathfrak{e}}]$ is also a partial k-colouring for $\mathbb{N}$, since $(\psi[\mathfrak{c}])[-\tilde{\mathfrak{c}}]$ contains no occurrence of colour $k{+}1$, and thus, $\mathrm{cross}\big((\psi[\mathfrak{c}])[-\tilde{\mathfrak{c}}], \chi^q_{q'+1}\big)$ can be computed in time $\omega_k(s,t)$.

We sum up. Let $\omega'(s)$ be a running-time polynomial for function f, where s is the limit of ψ. We then obtain as an upper bound on the running time of our algorithm:

$$\begin{aligned} & (s+1)^{k+1} \cdot (t+1) \cdot \Big((s+1)^k \cdot (t+1) \cdot \big(\omega'(s) + \omega_k(s,t)\big) \Big) \\ = \; & (s+1)^{2k+1} \cdot (t+1)^2 \cdot \big(\omega'(s) + \omega_k(s,t)\big)\,, \end{aligned}$$

which is a polynomial running time, if ω_k is a polynomial running time. ∎

As a remark, note from the running-time upper bound shown in the proof of Proposition 5.3 that the exponent of the running-time polynomial increases with k. The increase is only linear in k.

We complete the paper with an algorithm for our initial problem. Our initial problem was to compute the swap-delete-edit distance of strings over alphabets of bounded size. We apply the algorithm of Proposition 5.3.

Proposition 5.4. *Let Σ be an alphabet. The swap-delete-edit distance problem on input strings over Σ can be computed in polynomial time.*

Proof. We reduce the swap-delete-edit distance problem to the embedding pair cross number problem. For convenience, we may assume $\Sigma = \{1, \dots, k\}$. Let (u,v) be an input string pair over Σ, where v is the target string and u is the source string. Let $u = u_0 \cdots u_s$ and $v = v_0 \cdots v_t$. If $t < s$ then we can already reject, since the problem has no solution. Otherwise, let ψ and χ be the following partial k-colourings for $\mathbb{N}$, for $x, y \in \mathbb{N}$:

$$\psi(x) \;=_{\mathrm{def}}\; \begin{cases} u_x & \text{, if } x \le s \\ 0 & \text{, if } x > s \end{cases} \qquad \text{and} \qquad \chi(y) \;=_{\mathrm{def}}\; \begin{cases} v_y & \text{, if } y \le t \\ 0 & \text{, if } y > t\,. \end{cases}$$

Observe that (ψ,χ) is a finite k-colour embedding pair with limit (s,t). If (ψ,χ) is not satisfiable, again, we can already reject.

We claim that the swap-delete-edit distance of (u,v) is equal to $|t-s| + \mathrm{cross}(\psi,\chi)$. The arguments follow those of Wagner's NP-completeness proof [7], that is why an informal description may suffice. Let $\varphi \in \mathscr{E}\big(\psi,\chi\big)$. To edit v into u, delete the letters of v that correspond to the integers that are not in $\varphi(\{0,\dots,s\})$, and then, iteratively swap consecutive symbol pairs in $\mathrm{CROSS}(\varphi^{-1})$. The number of executed swap operations is exactly $\mathrm{cross}(\varphi)$, since $\mathrm{cross}(\varphi) = \mathrm{cross}(\varphi^{-1})$.

For the converse, it suffices to observe that each sequence of edit operations will contain exactly $|t-s|$ delete operations, and tracing the swap operations defines a colour-preserving embedding φ for (ψ,χ) such that the swap-delete-edit distance of (u,v) is at least $|t{-}s|{+}\mathrm{cross}(\varphi)$.

The polynomial running time follows from Proposition 5.3. ∎

6 Final remark

Wagner showed that the swap-delete-edit distance problem is NP-complete for strings over alphabets of unbounded size [7], since the alphabet is part of the input. Proposition 5.4 shows that bounding the size of the alphabet makes the problem tractable. These two results together establish the alphabet size as the relevant complexity parameter. The alphabet size determined the exponents of our running times. Is it possible to reduce the exponent to a constant? The intractability result implies that it is unlikely for the running time to be independent of the alphabet size. However, a running-time function of the form $h(|\Sigma|) \cdot (s+t)^c$ is possible, but requires h to be "superpolynomial" in $s + t$. Abu-Khzam et al. worked in this direction and showed the fixed-parameter tractability of the swap-delete-edit distance problem when parametrising with an upper bound on the swap-delete-edit distance [1].

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Glossary

Throughout the paper, k denotes an arbitrary but fixed positive integer, i.e., $k \geq 1$.

The set of the non-negative integers is denoted by $\mathbb{N}$, which is the set of the natural numbers. A mapping $\varphi : \mathbb{N} \to \mathbb{N}$ is called a *function on* $\mathbb{N}$. We often additionally say *partial function* to emphasise that φ may not be defined for all integers in $\mathbb{N}$. For $y \in \mathbb{N}$ and $X, Y \subseteq \mathbb{N}$:

$$\begin{aligned} \varphi(X) &= \big\{ \varphi(x) : x \in X \big\}, \text{ and} \\ \varphi^{-1}(y) &= \big\{ x \in \mathbb{N} : \varphi(x) \text{ is defined and } \varphi(x) = y \big\} \\ \varphi^{-1}(Y) &= \big\{ x \in \mathbb{N} : \varphi(x) \text{ is defined and } \varphi(x) \in Y \big\}. \end{aligned}$$

A mapping φ on $\mathbb{N}$ is *total* if $\varphi(x)$ is defined for every $x \in \mathbb{N}$. A mapping φ on $\mathbb{N}$ is *injective* if $\varphi(x) = \varphi(x')$ implies $x = x'$ for every $x, x' \in \mathbb{N}$.

Let φ be a function on $\mathbb{N}$. By φ', we denote the *left-shift* on φ, yielding, for $x \in \mathbb{N}$:

$$\varphi'(x) = \begin{cases} \varphi(x+1) & \text{, if } \varphi(x+1) \text{ is defined} \\ \text{undefined} & \text{, if } \varphi(x+1) \text{ is undefined.} \end{cases}$$

By φ'', we denote $(\varphi')'$.

Let X be a (countable) set. By $|X|$, we denote the *cardinality* of X, that is the number of elements of X. If X is infinite then $|X| = \aleph_0$.

We extend the usual arithmetic operations and relations over $\mathbb{N}$ to $\mathbb{N} \cup \{\aleph_0\}$. For $x \in \mathbb{N} \cup \{\aleph_0\}$, $x + \aleph_0 = \aleph_0 + x = \aleph_0$, and for $x \in \mathbb{N}$, $x - \aleph_0 = -\aleph_0$ and $\aleph_0 - x = \aleph_0$, and $\aleph_0 - \aleph_0 = 0$, and $|\aleph_0| = |-\aleph_0| = \aleph_0$, and $x < \aleph_0$ for every $x \in \mathbb{N}$.

A *k-colour vector* is a k-tuple over $\mathbb{N} \cup \{\aleph_0\}$. Let $\mathfrak{b}$ be a k-colour vector. The entries of $\mathfrak{b}$ are denoted as $b_1, \ldots, b_k$, i.e., $\mathfrak{b} = (b_1, \ldots, b_k)$. Let $\mathfrak{a}$ and $\mathfrak{c}$ be k-colour vectors. We write $\mathfrak{a} \leq \mathfrak{c}$ if $a_j \leq c_j$ for every $1 \leq j \leq k$. The *distance* between $\mathfrak{a}$ and $\mathfrak{c}$ is measured in the 1-norm, it is shortly denoted as $|\mathfrak{a} - \mathfrak{c}|$, and it is the sum $|a_1 - c_1| + \cdots + |a_k - c_k|$.