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Clique-width of full bubble model graphs

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Abstract

A bubble model is a 2-dimensional representation of proper interval graphs. We consider proper interval graphs that have bubble models of specific properties. We characterise the maximal such proper interval graphs of bounded clique-width and of bounded linear clique-width and the minimal such proper interval graphs whose clique-width and linear clique-width exceed the bounds. As a consequence, we can efficiently compute the clique-width and linear clique-width of the considered graphs.

1 Introduction

Clique-width is a graph width parameter with applications in efficient graph algorithms [3, 4, 20]. Clique-width generalises treewidth in the sense that graphs of bounded treewidth also have bounded clique-width [5], but graphs of bounded clique-width may have unbounded treewidth. A simple example for the latter relationship are the complete graphs, whose clique-width is at most 2 and whose treewidth is proportional to the number of vertices. Next to their applicational importance, width parameters in general are also and necessarily studied with a focus on their theoretical aspects. The basic questions are about the complexity of recognising graphs of bounded width and about the structure of graphs of bounded width. In this paper, we address these basic questions for clique-width and full bubble model graphs.

Clique-width is a graph parameter that is difficult to deal with. Despite its strong and important applications, only little is known about its properties. It was shown that computing the clique-width for general graphs is hard [6], but no hardness result for restricted graph classes is known. Graphs of clique-width at most 2 can be recognised efficiently and their structural properties are fully known [5]. This is due to the fact that graphs of clique-width at most 2 are exactly the cographs, which are the graphs that do not have a chordless path on four vertices as an induced subgraph. Cographs are well-studied and many of their properties are known [1]. In particular, the maximal graphs of clique-width at most 2 and the minimal graphs of clique-width more than 2 are known, the latter being only the chordless path on four vertices. The situation changes for larger bounds on the clique-width. Graphs of clique-width at most 3 can be recognised efficiently [2], however, little is known about their structure. And no efficient algorithm is known for recognising graphs of bounded clique-width for any bound larger than 3. In fact, not even a moderately exponential-time algorithm is known for computing the clique-width of a graph.

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The situation is not better for restricted versions of the problem. Currently, there are only two positive results known for computing the clique-width for graph classes of unbounded clique-width: for square grids and for path powers. The clique-width of an $n \times n$-grid is exactly $n + 1$ [10], for $n \geq 3$. This also means that there is exactly one such graph for each value of the clique-width, and the $n \times n$-grid is the unique maximal square grid of clique-width at most $n + 1$, and the $n \times n$-grid is the unique minimal square grid of clique-width at least $n + 1$. The clique-width of path powers has a more complex description, and it is fully known [14]. A k-path power is the kth power of a chordless path. The clique-width of a k-path power on sufficiently many vertices is exactly $k + 2$. The clique-width of k-path powers on small numbers of vertices can be related to the independence number. As a result, there are two types of maximal path powers of clique-width at most k, and their is a unique minimal path power of clique-width at least k.

Another restriction for clique-width is its linear variant linear clique-width. Briefly, the relationship between clique-width and linear clique-width can be understood in similarity to the relationship between treewidth and its linear variant pathwidth. Also linear clique-width has been studied [7, 18, 12]. As for the recognition problem, graphs of linear clique-width of at most up to 3 can be recognised efficiently [8, 12], and no efficient algorithm for larger bounds is known. The structure of graphs of linear clique-width at most 2 is fully known [8, 12], and a description of graphs of linear clique-width at most 3 is known [12], that can be used to characterise maximal graphs of linear clique-width at most 3. As for minimal graphs of linear clique-width more than 3, some results are known [12, 13], but no full characterisation of the graphs of linear clique-width at most 3 by forbidden induced subgraphs is known.

In this paper, we study the basic questions for a class of proper interval graphs. We will give efficient algorithms for computing the clique-width and the linear clique-width of the considered graphs, and we fully characterise the minimal and maximal graphs for given clique-width and linear clique-width bounds. Hereby, we extend the previous results for path powers. We briefly sketch our considered graph class. Proper interval graphs admit a characterisation through a 2-dimensional model, the so-called bubble model [11]. The bubble model is a representation for graphs that places the vertices in a grid-like structure, and the edges are implicit. We consider such proper interval graphs that have a bubble model with a special closure property.

Our results present a first real picture of the structure of graphs of bounded clique-width. As already mentioned, previous comparable results were only known for graphs of clique-width at most 2. We also and implicitly establish the proper interval graphs as a fundamental graph class for studying clique-width. A result to support this is by Lozin, who studied questions about minimal graph classes of unbounded clique-width [17].

The challenges in this paper are manifold. The main technical results of this paper are lower and upper bounds on the clique-width and linear clique-width of special proper interval graphs. Usually, upper bounds are easier to obtain than lower bounds, and this is also the case here. The challenge about upper bounds is to present these bounds in a “readable” and “understandable” form. There is no established language and technique to construct upper-bound results. Similarly, and much more demanding, is the development of lower-bound results. Our lower bounds are obtained by studying situations and classifying them by applying combinatorial arguments. There is no catalogue of situations that can be applied here. A major goal of our study, that goes beyond the mere results, is to develop techniques for proving such lower bounds and to provide a more detailed picture of the two width parameters.

Organisation of the paper. Section 2 presents the graph-theoretic background, definitions and notations, and clique-width and linear clique-width and related notions and results. Especially, a characterisation of clique-width and linear clique-width is presented, that will be the basis for the lower-bound proofs. In Section 3, we consider proper interval graphs, characterisations, and the bubble model. We define classes of proper interval graphs, for which we will give upper bounds on their clique-width or linear clique-width in Section 4. In Sections 5 and 6, we show our lower bounds for two classes of proper interval graphs, that are two out of three classes of minimal graphs to exceed clique-width bounds. The third such class are the minimal path powers. Finally, in Section 7, we combine the obtained lower- and upper-bound results to a complete characterisation of the clique-width and linear clique-width of our proper interval graphs. This characterisation directly implies a simple and efficient algorithm for computing the clique-width and linear clique-width of our proper interval graphs.

2 Graph preliminaries and clique-width

The graphs in this paper are simple, finite, undirected. A graph G is an ordered pair (V, E) where $V = V(G)$ is the *vertex set* of G and $E = E(G)$ is the *edge set* of G. Edges are denoted as uv. Let u, v be a vertex pair of G with $u \neq v$. If uv is an edge of G then u and v are *adjacent* in G, and u is a *neighbour* of v in G, and vice versa. If uv is not an edge of G then u and v are *non-adjacent* in G. The *neighbourhood* of a vertex u of G, denoted as $N_G(u)$, is the set of the neighbours of u in G, and $N_G[u] =_{\text{def}} N_G(u) \cup \{u\}$ is the *closed neighbourhood* of u in G. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a set X of vertices of G, the subgraph of G *induced* by X, denoted as $G[X]$, has vertex set X, and for each vertex pair u, v from X with $u \neq v$, uv is an edge of $G[X]$ if and only if uv is an edge of G. By $G \setminus X$, we denote the induced subgraph $G[V(G) \setminus X]$ of G, and by $G-x$ for x a vertex of G, we denote the induced subgraph $G \setminus \{x\}$ of G. A *clique* of G is a set of vertices that are pairwise adjacent in G, and a *maximal clique* of G is a clique of G that is not contained in any other clique of G.

Let G be a graph. Let u, v be a vertex pair of G. A *u, v-path of G* is a sequence $(x_0, \ldots, x_r)$ of pairwise different vertices of G where $x_0 = u$ and $x_r = v$ and $x_i x_{i+1} \in E(G)$ for every $0 \leq i < r$. If for every vertex pair u, v of G, there is a u, v-path of G, we call G *connected*; otherwise, if there is a vertex pair u, v of G such that there is no u, v-path of G, G is called *disconnected*. The *connected components* of G are the maximal connected induced subgraphs of G.

An important graph operation throughout the paper is the disjoint union of two graphs. Let G and H be two vertex-disjoint graphs, which means that $V(G) \cap V(H) = \emptyset$. The *disjoint union* of G and H, denoted as $G \oplus H$, is the graph $(V(G) \cup V(H), E(G) \cup E(H))$.

2.1 Clique-width, linear clique-width and expressions

These definitions are relevant mainly for the upper-bound constructions of Section 4.

Let k be an integer with $k \geq 1$. Consider the following inductive definition of k-expressions and linear k-expressions:

(1) for $o \in \{1, \ldots, k\}$ and u a vertex name, $o(u)$ is a k-expression and a linear k-expression

(2) for α a k-expression and $s, o \in \{1, \dots, k\}$ with $s \neq o$, $\eta_{s,o}(\alpha)$ and $\rho_{s\to o}(\alpha)$ are k-expressions; if α is a linear k-expression then $\eta_{s,o}(\alpha)$ and $\rho_{s\to o}(\alpha)$ are linear k-expressions

(3) for α and δ vertex-disjoint k-expressions, $(\alpha \oplus \delta)$ is a k-expression

(4) for δ a linear k-expression and $o \in \{1, \dots, k\}$ and u a vertex name that does not occur in δ, $(o(u) \oplus \delta)$ is a linear k-expression.

Observe that linear k-expressions are k-expressions of restricted form. A *k-labelled graph* is an ordered pair (G, ℓ) where G is a graph and $\ell : V(G) \to \{1, \dots, k\}$ is a mapping assigning a label from $\{1, \dots, k\}$ to each vertex of G; the *vertices* of (G, ℓ) are the vertices of G. By $[(G, \ell)]$, we denote the graph G. A k-expression defines a k-labelled graph. Let α be a k-expression. The k-labelled graph defined by α is denoted as $\text{val}(\alpha)$ and is inductively defined as follows:

(1) if $\alpha = o(u)$ then $\text{val}(\alpha)$ is the k-labelled graph with the single vertex u and u has label o

(2) if $\alpha = \eta_{s,o}(\delta)$ where δ is a k-expression then $\text{val}(\alpha)$ is obtained from $\text{val}(\delta)$ by adding all missing edges between the vertices with label s and the vertices with label o

(3) if $\alpha = \rho_{s\to o}(\delta)$ where δ is a k-expression then $\text{val}(\alpha)$ is obtained from $\text{val}(\delta)$ by assigning label o to all vertices that have label s

(4) if $\alpha = (\gamma \oplus \delta)$ where γ and δ are k-expressions then $\text{val}(\alpha)$ is the disjoint union of $\text{val}(\gamma)$ and $\text{val}(\delta)$; this means, for $\text{val}(\gamma) = (G, \ell)$ and $\text{val}(\delta) = (G', \ell')$, $\text{val}(\alpha) = (G \oplus G', \ell \cup \ell')$.

The *graph* that is defined by α is $[\text{val}(\alpha)]$, and we say that α is a k-expression for a graph G if $G = [\text{val}(\alpha)]$, and we say that α is a linear k-expression for G if α is a linear k-expression and $G = [\text{val}(\alpha)]$.

Let G be a graph. The *clique-width* of G, denoted as $\text{cwd}(G)$, is the smallest integer k such that G has a k-expression. And the *linear clique-width* of G, denoted as $\text{lcwd}(G)$, is the smallest integer k such that G has a linear k-expression. Since linear k-expressions are k-expressions, it clearly holds that $\text{cwd}(G) \leq \text{lcwd}(G)$. In this algebraic context, it is important to remark that we only consider non-empty graphs, i.e., our considered graphs have vertices, so that the smallest integer k indeed exists. It is known that clique-width and linear clique-width are monotone for induced subgraphs. So, for H an induced subgraph of G, $\text{cwd}(H) \leq \text{cwd}(G)$ and $\text{lcwd}(H) \leq \text{lcwd}(G)$.

We will prove lower and upper bounds on the clique-width and linear clique-width of graphs. For proving upper bounds, the following notion will be important. Let k be an integer with $k \geq 1$, and let α be a k-expression. A label i with $i \in \{1, \dots, k\}$ is called *inactive* in α if for all $\eta_{s,o}$ in α, $s \neq i$ and $o \neq i$, and for all $\rho_{s\to o}$ in α, $s \neq i$. Informally, we can say that i is inactive in α if i is not used to create edges. Note that $i(u)$ can appear in α and i is an inactive label in α, namely, if u has no neighbours in $[\text{val}(\alpha)]$.

We will make use of the following simplifications for denoting expressions. Let k be an integer with $k \geq 1$. Let $s \in \{1, \dots, k\}$ and let $A \subseteq \{1, \dots, k\}$ with $s \notin A$ and where $A = \{a_1, \dots, a_r\}$. Instead of writing $\eta_{s,a_1}(\cdots \eta_{s,a_r}(\delta) \cdots)$, we will simply write $\eta_{s,A}(\delta)$. Analogously, we will shorten sequences of ρ-operations by writing, as an example, $\rho_{2\to 3\to 4}(\delta)$ instead of $\rho_{2\to 3}(\rho_{3\to 4}(\delta))$, which can be seen as “label shift” operations. Here, it is important to note that the corresponding ρ-operations are applied from right to left.

We partition expressions into parts, that we define informally. Let k be an integer with $k \geq 1$, and let α, γ and δ be k-expressions. Assume that α can be written as $\beta(\gamma \oplus \delta)$ or $\beta(\delta)$. We call δ a *beginning* of α and β an *end* of α. Note that an end is not a k-expression, but it can extend k-expressions.

Clique-width has some invariance properties, that are useful for the characterisation of graphs of bounded clique-width and for computing the clique-width of a graph. For a graph G and a vertex pair u, v of G with $u \neq v$, u and v are *true twins* of G if $N_G(u) \cup \{u\} = N_G(v) \cup \{v\}$. The true-twin relation is an equivalence relation on the vertex set, and the maximal sets of pairwise true twins of a graph can be computed in linear time.

Lemma 2.1 ([5]). *Let G be a graph.*

1) *The clique-width of G is equal to the maximum clique-width of the connected components of G.*

2) *If G has clique-width at least 2 then adding true twins to G does not increase the clique-width.*

A consequence of Lemma 2.1 is that it suffices to consider connected graphs without true twins. We only mention here that linear clique-width does not have such nice general invariance properties, which can already be seen on graphs built from induced paths [13]: adding true twins may increase the linear clique-width, and the linear clique-width of a disconnected graph may be strictly larger than the linear clique-width of each of its connected components.

2.2 Clique-width characterisation, and useful results

These definitions and results are relevant mainly for the lower-bound proofs of Sections 5 and 6.

2.2.1 Groups and supergroups

Let G be a graph and let H be a subgraph of G. Let u, v be a vertex pair of G where u is a vertex of H and uv is an edge of G. If $uv \notin E(H)$, we say that uv is *non-visible* in H and v is a *non-visible neighbour* of u in H with respect to G. The set of the non-visible neighbours of u in H with respect to G is denoted as $\text{nonVis}_{H\subseteq G}(u)$; if the context G is clear, we may shortly write $\text{nonVis}_H(u)$. Observe the following easy properties: $\text{nonVis}_H(x) \subseteq N_G(x)$, and if x and y are vertices of H then $y \in \text{nonVis}_H(x)$ if and only if $x \in \text{nonVis}_H(y)$. The following two notions are important in the context of clique-width.

Definition 2.2 ([14]). *Let G be a graph and let H be a subgraph of G. Let X be a set of vertices of H.*

1) *X is called a* group *of H with respect to G if every vertex pair u, v from X satisfies the* group condition*: $\text{nonVis}_{H\subseteq G}(u) = \text{nonVis}_{H\subseteq G}(v)$.*

2) *X is called a* supergroup *of H with respect to G if every vertex pair u, v from X satisfies the* supergroup condition*: $\text{nonVis}_{H\subseteq G}(u) \subseteq N_G(v)$.*

It is important to remember throughout the paper that subsets of groups and supergroups are also groups and supergroups; this is a direct consequence of the definitions.

Let G be a graph. We call (B, C) a *partial partition of* $V(G)$ if $B \subseteq V(G)$ and $C \subseteq V(G)$ and $B \cap C = \emptyset$. In particular, B and C may be empty sets. The following results are easy but important facts about groups and supergroups.

Lemma 2.3 ([14])**.** *Let G be a graph, let (B, C) be a partial partition of $V(G)$, and let $H =_{\text{def}} G[B] \oplus G[C]$. The following is the case with respect to G.*

1) Every group of H is a supergroup of H.

2) Every supergroup of H that is a subset of B is a group of $G[B]$.

3) Every group of $G[B]$ is a group of H.

A direct consequence of Lemma 2.3 is the fact that the notions of groups and supergroups coincide on induced subgraphs.

Another easy property of supergroups is shown in the next lemma.

Lemma 2.4. *Let G be a graph and let H be a subgraph of G. Let $A \subseteq V(H)$. If A is a supergroup of H with respect to G then $H[A] = G[A]$.*

Proof. Assume that there is a vertex pair u, v from A such that $uv \in E(G)$ and $uv \notin E(H)$. Note that uv is a non-visible edge of H with respect to G, so that $v \in \text{nonVis}_{H \subseteq G}(u)$. Since $v \notin N_G(v)$, the vertex pair u, v does not satisfy the supergroup condition, and thus, $\{u, v\}$ is not a supergroup of H with respect to G. It follows that A is not a supergroup of H with respect to G. ∎

We will often argue that two vertices cannot be contained in the same supergroup, by providing a witness. Let G be a graph and let H be a subgraph of G. For a vertex triple u, v, y of G where u and v are vertices of H, we say that y *s-distinguishes* u and v in H if one of the two cases applies: (1) $y \in \text{nonVis}_{H \subseteq G}(u)$ and $y \notin N_G(v)$ or (2) $y \in \text{nonVis}_{H \subseteq G}(v)$ and $y \notin N_G(u)$. Note this important example: if $v \in \text{nonVis}_H(u)$ then v s-distinguishes u and v, since $v \notin N_G(v)$.

Lemma 2.5 ([14])**.** *Let G be a graph and let H be a subgraph of G. For every vertex pair u, v of H, $\{u, v\}$ is a supergroup of H if and only if there is no vertex of G that s-distinguishes u and v in H.*

Let G be a graph and let H be a subgraph of G. A *maximal group* of H is a group of H that is not contained in any other group of H, all with respect to G. It is important to observe that the group condition of Definition 2.2 is an equivalence relation, and the maximal groups of H therefore define a partition of $V(H)$.

2.2.2 Clique-width characterisation

Clique-width and linear clique-width can be characterised through labelled partition trees, that are based on rooted binary trees. We repeat the necessary definitions: (1) a tree on a single node $\underline{u}$ is a *rooted binary tree* with root $\underline{u}$, and (2) for T' and T'' vertex-disjoint rooted binary trees with roots $\underline{b}$ and $\underline{c}$, respectively, and for $\underline{a}$ a new node, $T =_{\text{def}} (V(T' \oplus T'') \cup \{\underline{a}\}, E(T' \oplus T'') \cup \{\underline{a}\underline{b}, \underline{a}\underline{c}\})$ is a *rooted binary tree* with root $\underline{a}$, and the nodes $\underline{b}$ and $\underline{c}$ are the *children* of $\underline{a}$ in T. Every node of a rooted binary tree with a child is an *inner node*, and every node without a child is a *leaf*. Observe that every inner node has exactly two children, and the root is an inner node, unless the tree consists of one node only. Node names of rooted binary trees are highlighted by underlines, to distinguish them from vertex names of the studied graphs.

We label the nodes of rooted binary trees by special partitions of subgraphs of graphs. Let G be a graph and let H be a subgraph of G. Let A and B be sets of vertices of H. We call A and B *compatible* in H with respect to G if at least one of the two cases applies: (1) there is no vertex pair u, v of H with $u \in A$ and $v \in B$ and $v \in \text{nonVis}_{H \subseteq G}(u)$ or (2) $y \in N_G(x)$ for all vertex pairs x, y of H with $x \in A$ and $y \in B$. A *compatible supergroup partition* for H with respect to G is a partition of $V(H)$ into supergroups of H with respect to G, and the supergroups are pairwise compatible in H with respect to G.

Definition 2.6 ([14])**.** *Let k be an integer with $k \geq 1$. Let G be a graph. A* k-supergroup tree *for G is a rooted binary tree T whose nodes are labelled with partitions of subsets of $V(G)$ such that the following conditions are satisfied for every node $\underline{a}$ of T:*

1) if $\underline{a}$ is a leaf of T:
the label of $\underline{a}$ in T is $\{\{x\}\}$ for x some vertex of G

2) if $\underline{a}$ is the root node of T:
the label of $\underline{a}$ in T is a partition of $V(G)$

3) if $\underline{a}$ is an inner node of T:
let $\underline{b}$ and $\underline{c}$ be the children of $\underline{a}$ in T and let $\{A_1, \ldots, A_p\}$, $\{B_1, \ldots, B_q\}$ and $\{C_1, \ldots, C_r\}$ be the labels of respectively $\underline{a}$, $\underline{b}$ and $\underline{c}$ in T; then, the following conditions are satisied:

a) $p \leq k$
b) $(B_1 \cup \cdots \cup B_q) \cap (C_1 \cup \cdots \cup C_r) = \emptyset$
c) $\{B_1, \ldots, B_q, C_1, \ldots, C_r\}$ is a partition-refinement of $\{A_1, \ldots, A_p\}$
d) $\{A_1, \ldots, A_p\}$ is a compatible supergroup partition for $G[B_1 \cup \cdots \cup B_q] \oplus G[C_1 \cup \cdots \cup C_r]$.

It follows from the definition of supergroup trees that there is a bijection between the leaves of T and the vertices of G. A *partition-refinement* is a partition consisting of subsets of partition classes.

By $\Sigma_T(\underline{a})$, we denote the union of the supergroups in the partition that is assigned to node $\underline{a}$ in T. In other words, if $\{A_1, \ldots, A_p\}$ is the partition that is assigned to $\underline{a}$ in T then $\Sigma_T(\underline{a}) = A_1 \cup \cdots \cup A_p$. Note here that we will always and implicitly associate the assigned partition labels with T.

Let k be an integer with $k \geq 1$. A *k-supergroup caterpillar tree* is a k-supergroup tree where each inner node has a child that is a leaf.

Theorem 2.7 ([14])**.** *Let k be an integer with $k \geq 1$. Let G be a graph.*

1) $\text{cwd}(G) \leq k$ if and only if G has a k-supergroup tree.

2) $\text{lcwd}(G) \leq k$ if and only if G has a k-supergroup caterpillar tree.

2.2.3 Restrictions to induced subgraphs

We will prove our lower-bound results also be arguing about groups and supergroups in induced subgraphs. The following definitions and results provide the technical means.

Lemma 2.8. *Let G be a graph, let H be a subgraph of G, and let $F \subseteq V(G)$. Let A be a supergroup of H with respect to G. Then, $A \setminus F$ is a supergroup of $H \setminus F$ with respect to $G \setminus F$.*

Proof. Let $A' =_{\text{def}} A \setminus F$ and $G' =_{\text{def}} G \setminus F$ and $H' =_{\text{def}} H \setminus F$. Let u, v be a vertex pair from A', and we verify the supergroup condition for u, v. Observe that $N_{G'}(v) = N_G(v) \setminus F$ and $\text{nonVis}_{H' \subseteq G'}(u) = \text{nonVis}_{H \subseteq G}(u) \setminus F$, and since $\text{nonVis}_{H \subseteq G}(u) \subseteq N_G(v)$ by assumption, we directly conclude that u, v indeed satisfies the supergroup condition. ▪

Lemma 2.8 is often applied to show bounds on the number of maximal groups. This often-applied application is exemplified in the proof of the following direct consequence of Lemma 2.8. We give a full proof of this consequence, also in order to provide more intuition about the technical notions and arguments.

Corollary 2.9. *Let G be a graph, and let H be an induced subgraph of G. Let $B \subseteq V(H)$. The number of maximal groups of $H[B]$ with respect to H is bounded from above by the number of maximal groups of $G[B]$ with respect to G.*

Proof. Let $F =_{\text{def}} V(G) \setminus V(H)$. Note that $H = G \setminus F$ and $H[B] = G[B] \setminus F$. Let A be a group of $G[B]$ with respect to G. Due to the first statement of Lemma 2.3, A is a supergroup of $G[B]$ with respect to G. Since $A = A \setminus F$, A is a supergroup of $H[B]$ with respect to H due to Lemma 2.8, and thus, A is a group of $H[B]$ with respect to H due to the second statement of Lemma 2.3. Thus, every group of $G[B]$ with respect to G is a group of $H[B]$ with respect to H. In particular, every maximal group of $G[B]$ with respect to G is a group of $H[B]$ with respect to H, so that each maximal group of $H[B]$ with respect to H is the union of maximal groups of $G[B]$ with respect to G, and the claim follows. ▪

Let t be an integer with $t \geq 1$. Let G be a graph and let $X \subseteq V(G)$. Let T be a t-supergroup tree for G. We define the reduced supergroup tree, that shall be the restriction of T to the induced subgraph $G[X]$ of G. We define the reduced supergroup tree inductively by associating reduced supergroup trees with the nodes of T. So, let $\underline{a}$ be a node of T and let $\{A_1, \ldots, A_p\}$ be the assigned partition:

- if $\underline{a}$ is a leaf of T:

  if $A_1 \subseteq X$ then the *X-reduced supergroup tree of T at $\underline{a}$* has a single node and the node label is $\{A_1\}$, and if $A_1 \not\subseteq X$ then the *X-reduced supergroup tree of T at $\underline{a}$* is an empty tree (recall that $p = 1$ and $|A_1| = 1$)

- if $\underline{a}$ is an inner node of T:
  let $\underline{b}$ and $\underline{c}$ be the children of $\underline{a}$ in T, and let T_b and T_c be the X-reduced supergroup trees of T at respectively $\underline{b}$ and $\underline{c}$;

  if T_c is an empty tree then the *X-reduced supergroup tree of T at $\underline{a}$* is T_b, if T_c is not an empty tree and T_b is an empty tree then the *X-reduced supergroup tree of T at $\underline{a}$* is T_c, and if T_b and T_c are not empty trees then the *X-reduced supergroup tree of T at* $\underline{a}$ is obtained from the disjoint union of T_b and T_c by adding $\underline{a}$ and making $\underline{a}$ adjacent to $\underline{b}$ and to $\underline{c}$ and by labelling $\underline{a}$ with $\{(A_1 \cap X), \ldots, (A_p \cap X)\} \setminus \{\emptyset\}$.

The *X-reduced supergroup tree of T* is the X-reduced supergroup tree of T at the root of T. The following lemma is easy and straightforward to prove, and we give it without an explicit proof. The proof mainly relies on Lemma 2.8 and the fact that the compatibility property extends to induced subgraphs.

Lemma 2.10. *Let t be an integer with $t \geq 1$. Let G be a graph. Let T be a t-supergroup tree for G. Let $X \subseteq V(G)$. The X-reduced supergroup tree of T is a t-supergroup tree for $G[X]$.*

We will apply Lemma 2.10 to prove lower bounds by arguing about structural properties of supergroup trees of induced subgraphs.

3 Proper interval graphs and the bubble model

We define proper interval graphs and the bubble model. We define classes of proper interval graphs that will be the studied graphs in this paper.

3.1 Proper interval graphs and bubble models

A *proper interval graph* is a graph whose vertices can be assigned closed intervals of the real line of unit length such that vertices are adjacent if and only if their assigned intervals have a non-empty intersection [21]. Proper interval graphs are interval graphs, chordal graphs and cocomparability graphs. Proper interval graphs are widely studied and they have several characterisations [1, 9]. One characterisation is by vertex orderings. A *proper interval ordering* for a graph G is a vertex ordering σ for G, where $\sigma = \langle x_1, \ldots, x_n \rangle$, satisfying for every index triple i, j, k with $1 \leq i < j < k \leq n$ that $x_i x_k \in E(G)$ implies $x_i x_j \in E(G)$ and $x_j x_k \in E(G)$. A graph is a proper interval graph if and only if it has a proper interval ordering [16].

Most characterisations of proper interval graphs reflect their linear structure, and this linear structure is often the foundation of efficient algorithms on proper interval graphs. In case of clique-width, a different, a 2-dimensional model seems more suitable. A *bubble model* for a graph G is a 2-dimensional structure $\mathcal{B} = \langle B_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$ that satisfies the following conditions:

- $\emptyset \subseteq B_{i,j} \subseteq V(G)$ for every i, j with $1 \leq j \leq s$ and $1 \leq i \leq r_j$
- $B_{1,1}, \ldots, B_{r_s,s}$ are pairwise disjoint and $(B_{1,1} \cup \cdots \cup B_{r_s,s}) = V(G)$
- for every vertex pair u, v of G, where $u \in B_{i,j}$ and $v \in B_{i',j'}$ with $1 \leq j \leq j' \leq s$: $uv \in E(G)$ if and only if either (1) $j = j'$ or (2) $j + 1 = j'$ and $i > i'$.

We call s the *size* of the bubble model $\mathcal{B}$, and the sets $B_{i,j}$ are the *bubbles*.

Bubble models fully represent graphs, since they contain all vertices, and the edges are completely characterised. A graph is a proper interval graph if and only if it has a bubble model

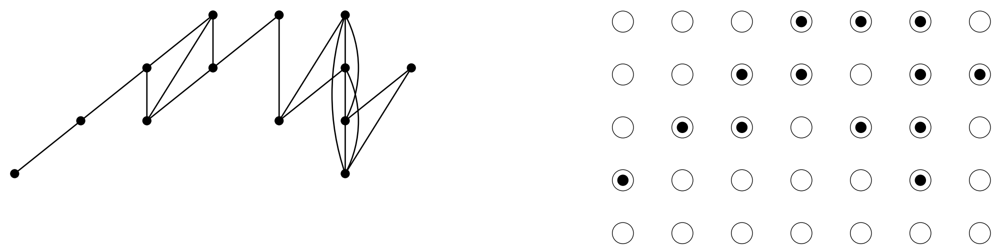

Figure 1: The left side shows a graph, that is represented by the bubble model on the right side. The vertices on the left are arranged as they appear in the bubble model on the right. The bubble model contains empty and non-empty bubbles, and every non-empty bubble contains exactly one vertex. The bubble model also satisfies additional technical assumptions except for the non-emptyness of $B_{r_j,j}$, that are all empty here.

[11], which means that the vertices of the graph can be placed in bubbles so that the edge set is exactly determined through the model. An example of a proper interval graph and a bubble model representation is given in Figure 1. Note that we will use the convention that $B_{i,j}$ is the bubble in row i and column j.

Let $\mathcal{B} = \langle B_{i,j} \rangle_{1 \le j \le s, 1 \le i \le r_j}$ be a bubble model for a graph G. As a technical observation, note that $s = 0$ means that G has no vertex. Also note that $r_j = 0$ is possible. For an index j with $1 \le j \le s$, the bubbles $B_{1,j}, \ldots, B_{r_j,j}$ form a *column* of $\mathcal{B}$, and each vertex from $B_{1,j} \cup \cdots \cup B_{r_j,j}$ appears in *column* j of $\mathcal{B}$, and each vertex of G has a corresponding *column index*. In our paper, we only consider non-empty graphs, so that we can henceforth assume $s \ge 1$ for each considered bubble model. Furthermore, it is no restriction to assume that the first and the last column of a bubble model contain vertices, which means that $r_1 \ge 1$ and $r_s \ge 1$. Finally, we will assume for every $1 \le j \le s$ that $r_j \ge 1$ implies $B_{r_j,j} \ne \emptyset$. If we represent a bubble model by drawing all empty and non-empty bubbles then the bubble model of Figure 1 does not satisfy the last technical assumption about $B_{r_j,j} \ne \emptyset$. We will allow such easy relaxations in our drawings for the sake of readability.

In this paper, we study proper interval graphs that have bubble models of special properties.

Definition 3.1. *Let G be a proper interval graph.*

1) Let $\mathcal{B} = \langle B_{i,j} \rangle_{1 \le j \le s, 1 \le i \le r_j}$ be a bubble model for G. We call $\mathcal{B}$ a full bubble model *if the following is satisfied for every index pair i, j with $1 \le j \le s$ and $1 < i \le r_j$:*
if $B_{i,j} \ne \emptyset$ then $B_{i-1,j} \ne \emptyset$.

2) A full bubble model graph *is a graph that has a full bubble model.*

Observe that the bubble model of Figure 1 is not a full bubble model, since it has empty bubbles “above” non-empty bubbles. We only mention here that there are proper interval graphs without full bubble models. In fact, the graph of Figure 1 has no full bubble model. By adding vertices to a proper interval graph and thereby filling empty bubbles, it is not difficult to see that every proper interval graph is an induced subgraph of a full bubble model graph. Full bubble model graphs can be recognised in linear time and a full bubble model for a full bubble model graph can be computed in linear time [15].

The objective of this paper is to compute the clique-width and linear clique-width of full bubble model graphs, where our main focus is on the clique-width. We can therefore restrict ourselves to connected full bubble model graphs without true twins, in accordance with Lemma 2.1. At the end of the paper, we will briefly discuss the extension of our linear clique-width results to disconnected graphs; the situation for true twins is complex [13]. The following lemma describes a necessary condition on full bubble models for connected full bubble model graphs without true twins.

Lemma 3.2. *Let G be a full bubble model graph with full bubble model $\mathcal{B} = \langle B_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$. If G is connected then $r_j \geq 2$ for every $1 \leq j < s$, and if G has no true twins then $|B_{i,j}| = 1$ for every $1 \leq j \leq s$ and $1 \leq i \leq r_j$.*

Proof. If $r_j \leq 1$ then no vertex from $B_{1,1} \cup \cdots \cup B_{r_1,1} \cup B_{1,2} \cup \cdots \cup B_{r_j,j}$ has a neighbour in $B_{1,j+1} \cup \cdots \cup B_{r_s,s}$. By our assumptions about bubble models, $B_{r_1,1}$ and $B_{r_s,s}$ are not empty, and thus, G is disconnected. If there is an index pair i, j with $|B_{i,j}| \geq 2$ then the vertices in $B_{i,j}$ are pairwise adjacent and have the same neighbours in G outside of $B_{i,j}$, so that the vertices in $B_{i,j}$ are pairwise true twins. ∎

Since we want to restrict to full bubble model graphs without true twins, we will henceforth assume $|B_{i,j}| = 1$ for every considered bubble model $\mathcal{B} = \langle B_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$. We will index the vertices as $b_{i,j}$, with the meaning of $B_{i,j} = \{b_{i,j}\}$. We will therefore also write $\mathcal{B}$ as $\langle b_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$. This indexation will be a major means to specify vertices. We will extend these notations and write $\langle b_{1,j}, \ldots, b_{r_j,j} \rangle$ to explicitly name column j of $\mathcal{B}$, or, analogously, $\langle b_{1,j}, \ldots, b_{p,j} \rangle$ with $1 \leq p \leq r_j$ for a "beginning" of column j of $\mathcal{B}$. Since bubble models are very visual, we believe that all used terminology about bubble models is straightforward understandable by the reader, even if not explicitly and fully defined.

As a final remark, we want to mention that Lemma 3.2 does not yield a characterisation of full bubble models without true twins. Graphs with bubble models satisfying the lemma may have true twins. Excluding these true twins and completing the necessary condition into a characterisation is possible but technical, and unnecessary here.

3.2 Special classes of full bubble model graphs

A basic class of full bubble model graphs are path powers. Let k and n be integers with $k \geq 1$ and $n \geq 1$, let G be a graph on n vertices, and let $\Lambda = \langle u_1, \ldots, u_n \rangle$ be a vertex ordering for G. We call Λ a *k-path layout* for G if for every $1 \leq i < j \leq n$, u_i and u_j are adjacent in G if and only if $j - i \leq k$. A *k-path power* is a graph that has a k-path layout, and a *path power* is an l-path power for some integer l. Path powers are full bubble model graphs; an example of a 4-path power on seventeen vertices and two representing bubble models are depicted in Figure 2. It will be important that a k-path power on at most $k + 1$ vertices is a complete graph, and each maximal clique of a k-path power on at least $k + 1$ vertices has size $k + 1$. Furthermore, the vertices of a maximal clique of a k-path power appear consecutively in each k-path layout for the graph. Observe that the 1-path powers are exactly the induced paths. We will consider graphs that are built from path powers for proving our lower and upper clique-width and linear clique-width bounds.

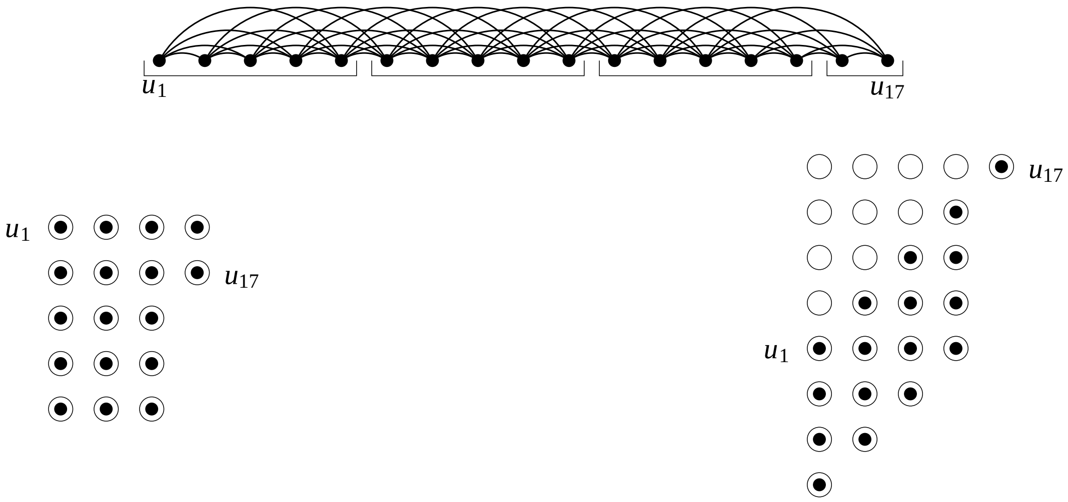

Figure 2: The top graph is a 4-path power on seventeen vertices. It has three pairwise disjoint maximal cliques, that have size 5, and two vertices do not belong to the three maximal cliques. The left below figure is a full bubble model for the graph. The right below figure is also a bubble model for the graph, however not a full bubble model.

Our upper clique-width and linear clique-width bounds will be obtained from considering "maximal" full bubble model graphs. We define these graphs in the following, through bubble models. Let $\mathcal{B} = \langle b_{i,j} \rangle_{1 \le j \le s, 1 \le i \le r_j}$ be a full bubble model. Let p, q be an index pair with $1 \le p \le q \le s$ and let d be an integer with $d \ge 1$. We say that $[p, q]$ is a *rectangle of* $\mathcal{B}$ *of depth* d if $r_p = \cdots = r_q = d$ and if either $p = 1$ or $r_{p-1} \neq d$ and if either $q = s$ or $r_{q+1} \neq d$; the *size* of a rectangle $[p, q]$ is $q - p + 1$, i.e., the number of comprised columns. Note that rectangles may have size 1. Observe that a rectangle of depth d is a $(d-1)$-path power. We may also understand rectangles as a part of $\mathcal{B}$. We will use $[p, q]$ and $\langle b_{i,j} \rangle_{p \le j \le q, 1 \le i \le r_j}$ as being equivalent; we shortly denote $\langle b_{i,j} \rangle_{p \le j \le q, 1 \le i \le r_j}$ as $\mathcal{B}[p, q]$.

Every bubble model admits a unique partition into rectangles. Let $[p_1, q_1], \ldots, [p_t, q_t]$ be the rectangles of the full bubble model $\mathcal{B}$. We assume that $p_1 = 1$ and $q_i + 1 = p_{i+1}$ for every $1 \le i < t$ and $q_t = s$, i.e., the rectangles are ordered from left to right according to their appearance in $\mathcal{B}$. Then, we can write $\mathcal{B}$ as $\langle \mathcal{B}[p_1, q_1], \ldots, \mathcal{B}[p_t, q_t] \rangle$. We call $\langle \mathcal{B}[p_1, q_1], \ldots, \mathcal{B}[p_t, q_t] \rangle$ the *rectangle partition* of $\mathcal{B}$. We will apply the rectangle partition notion to describe full bubble models, by specifying the rectangle partitions. Since our full bubble models should satisfy the conditions of Lemma 3.2, i.e., each bubble contains at most one vertex, the correspondence between full bubble models and their rectangle partitions is straightforward.

We define our graph classes by specifying rectangle partitions. Let k be an integer with $k \ge 3$. A *k-model* is a full bubble model $\mathcal{B}$ whose rectangle partition $\langle \mathcal{B}_1, \ldots, \mathcal{B}_t \rangle$ has at least two rectangles, i.e., $t \ge 2$, and there is an integer d with $d > k$ such that $\langle \mathcal{B}_1, \ldots, \mathcal{B}_t \rangle$ satisfies the following conditions:

- $\mathcal{B}_{t-1}$ is a rectangle of size k and depth d, and
  $\mathcal{B}_t$ is a rectangle of size 1 and depth 1

- for every index i with $1 \le i \le t-2$,
  $\mathcal{B}_i$ is a rectangle of depth k or is a rectangle of depth d and size at most $k-1$.

Note that the definition of rectangles implies that the depths of the rectangles $\mathcal{B}_1, \ldots, \mathcal{B}_{t-1}$ alternate between k and d. We will distinguish between these two types of rectangles. A rectangle of depth k is *shallow*, and a rectangle of depth d is *deep*. Then, the rectangles of a k-model alternate between deep and shallow rectangles, and the parity of t determines whether the first rectangle, $\mathcal{B}_1$, is deep or shallow. Recall that shallow rectangles are $(k-1)$-path powers and deep rectangles are $(d-1)$-path powers.

We use k-models as a base class and define further special full bubble models. Let $\mathcal{B}$ be a k-model with the rectangle partition $\langle \mathcal{B}_1, \ldots, \mathcal{B}_t \rangle$.

1) If the deep rectangles among $\mathcal{B}_1, \ldots, \mathcal{B}_{t-2}$ have size at most $k-2$ then $\mathcal{B}$ is an *open k-model*.

2) If $t \ge 6$ and $\mathcal{B}_1$ and $\mathcal{B}_{t-3}$ are deep rectangles of size $(k-1)$ and the deep rectangles among $\mathcal{B}_2, \ldots, \mathcal{B}_{t-4}$ have size at most $k-2$ then $\langle \mathcal{B}_1, \ldots, \mathcal{B}_{t-3} \rangle$ is a *short-end k-model*.

3) A *k-model with small separators* is obtained from $\mathcal{B}$ by deleting vertices from shallow rectangles in the following way:

   (1) for every $2 \le i \le t-2$, if $\mathcal{B}_i = \mathcal{B}[p,q]$ is a deep rectangle of size $(k-1)$ then delete $b_{k,p-1}$ and $b_{3,q+1}, \ldots, b_{k,q+1}$;

   (2) let a be with $2 \le a \le t-2$ such that $\mathcal{B}_a$ is a shallow rectangle and the deep rectangles among $\mathcal{B}_2, \ldots, \mathcal{B}_a$ have size at most $k-2$, let $\mathcal{B}_a = \mathcal{B}[p,q]$, and choose b with $p \le b \le q$ and k' with $k' \ge 1$ and $k' + \lfloor \frac{k'}{2} \rfloor \le k$: delete $b_{k'+2,b}, \ldots, b_{k,b}$.

When constructing a k-model with small separators, note that vertices that are about to be deleted may have already been deleted. Such can occur, for instance, when two deep rectangles of size $(k-1)$ are separated by a shallow rectangle of size 1. Schematic drawings of the four defined models are shown in Figure 3.

We will show that open k-models, short-end k-models and k-models with small separators represent the maximal full bubble model graphs of clique-width at most $k+1$ and open k-models and short-end k-models represent the maximal full bubble model graphs of linear clique-width at most $k+1$. The upper bounds are proved by defining expressions for shallow and deep rectangles, and these expressions will be combined into expressions for the whole graph. As a final remark, note that a rectangle of size 1 represents a complete graph, that contains true twins. Even though we excluded the existence of true twins, we allow deep rectangles of size 1 in our k-models, and therefore true twins in the represented graphs, in order to avoid unnecessary case distinctions.

4 Clique-width upper bounds for k-model graphs

In Subsection 3.2, we defined classes of bubble models. In this section, we show upper bounds on the clique-width and on the linear clique-width of graphs having such bubble models. These upper bounds will be obtained from special expressions for shallow and deep rectangles, that

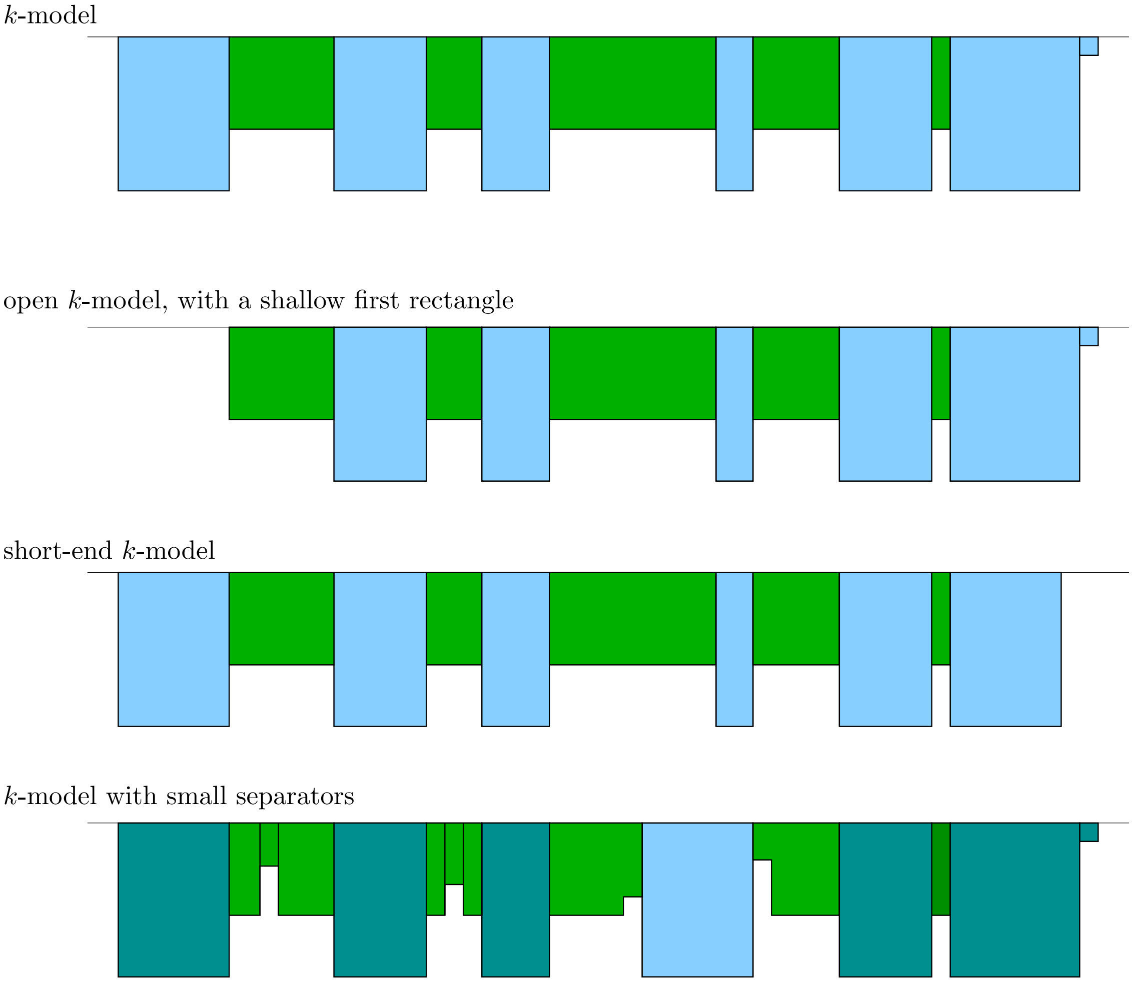

Figure 3: The figure illustrates the structure of k-models and the derived special classes. The rectangle partition of the k-model, on the top, has twelve rectangles, that alternate between shallow and deep. The vertex deletion for k-models with small separators is illustrated for a deep rectangle of size $(k-1)$, the fourth deep rectangle from the left, and deletion in shallow rectangles is applied to the left two shallow rectangles. For a k-model with small separators, only one of the two shallow rectangles may be selected.

are shown in the first part of this section, and combining these expressions, which is done in the second part of this section. The constructions repeat, apply and extend ideas from [15].

4.1 Particular linear expressions for shallow and deep rectangles

We consider shallow and deep rectangles separately. We begin with shallow rectangles. Recall that a shallow rectangle of a k-model is a $(k-1)$-path power. The expression is built on a $(k-1)$-path layout, by creating the vertices from right to left. For a brief description of the construction, consider the situation in Figure 4: the currently newly created vertex has label 2,

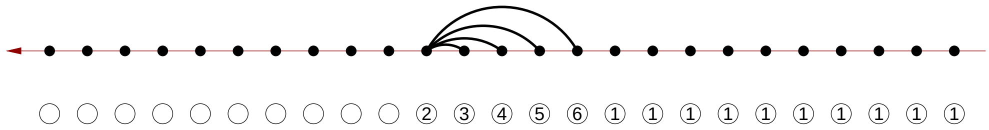

Figure 4: The figure illustrates the typical situation in a linear expression for a 4-path power. The vertices with label 1 have already received all neighbours, the vertex with label 2 is the newly created vertex, and the vertices with labels $3, 4, 5, 6$ are the already created neighbours of the new vertex.

its already created neighbours have labels $3, 4, 5, 6$, and the vertices with label 1 have received all their neighbours already. For the correspondence between path layouts and bubble models for path powers, re-consult Figure 2.

Lemma 4.1 (Shallow rectangles)**.** *Let k and n be integers with $k \geq 2$ and $n \geq k$. Let G be a $(k-1)$-path power on n vertices with $(k-1)$-path layout $\langle v_1, \ldots, v_n \rangle$. Then, G has a linear $(k+1)$-expression α with inactive label 1 such that the following two conditions are satisfied:*

- *α begins with $\alpha' = 3(v_{n-k+2}) \oplus (\cdots \oplus (k+1)(v_n) \cdots)$*
- *in $\text{val}(\alpha)$: v_i has label $i+2$ for every $1 \leq i < k$ and $v_k, \ldots, v_n$ have label 1.*

Proof. The desired expression first defines $G[\{v_{n-k+2}, \ldots, v_n\}]$. If $k = 2$ then $\alpha_{n-k+2} = \alpha_n =_{\text{def}} \alpha'$, if $k = 3$ then $\alpha_{n-k+2} = \alpha_{n-1} =_{\text{def}} \eta_{k,\{k+1\}}(\alpha')$, and if $k \geq 4$ then

$$\alpha_{n-k+2} \ =_{\text{def}} \ \eta_{3,\{4,\ldots,k+1\}}(\eta_{4,\{5,\ldots,k+1\}}(\cdots \eta_{k,\{k+1\}}(\alpha') \cdots)) .$$

Observe that label 1 is inactive in α_{n-k+2}. Next, we add the other vertices of G by following the given path layout. For i an index with $1 \leq i \leq n-k+1$, we define α_i by extending α_{i+1}:

$$\alpha_i \ =_{\text{def}} \ \rho_{2\to 3\to 4\to \cdots \to k \to (k+1) \to 1}(\eta_{2,\{3,\ldots,k+1\}}(2(v_i) \oplus \alpha_{i+1})) .$$

It is not difficult to verify that $[\text{val}(\alpha_i)]$ is equal to $G[\{v_i, \ldots, v_n\}]$ for every $1 \leq i \leq n-k+1$, particularly since $v_{i+1}, \ldots, v_{i+k-1}$ are the neighbours of v_i in $G[\{v_i, \ldots, v_n\}]$. So, $\alpha =_{\text{def}} \alpha_1$ is a desired expression for G. ■

The result of Lemma 4.1 is slightly more general than necessary for a result about shallow rectangles, since the path powers of Lemma 4.1 may have arbitrary numbers of vertices, and the number of vertices of a rectangle is a multiple of its depth. In fact, our application of the lemma to shallow rectangles will make use of this more general result. This completes the consideration of shallow rectangles.

We turn to the consideration of deep rectangles. Deep rectangles are also path powers, and we can therefore apply the construction of Lemma 4.1 to obtain linear expressions. However, the number of used labels will correspond to the depth of the rectangle. Instead, we want linear expressions using few labels, whose number is bounded in the size of the rectangle. We distinguish between three cases about deep rectangles, and the three cases are determined by the size of the rectangles: we consider deep rectangles of size k, of size $k-1$ and of size at most $k-2$. The expressions to be constructed have similarities and their specialities. The constructed expressions need to ensure their applicability in the construction of expressions for k-models.

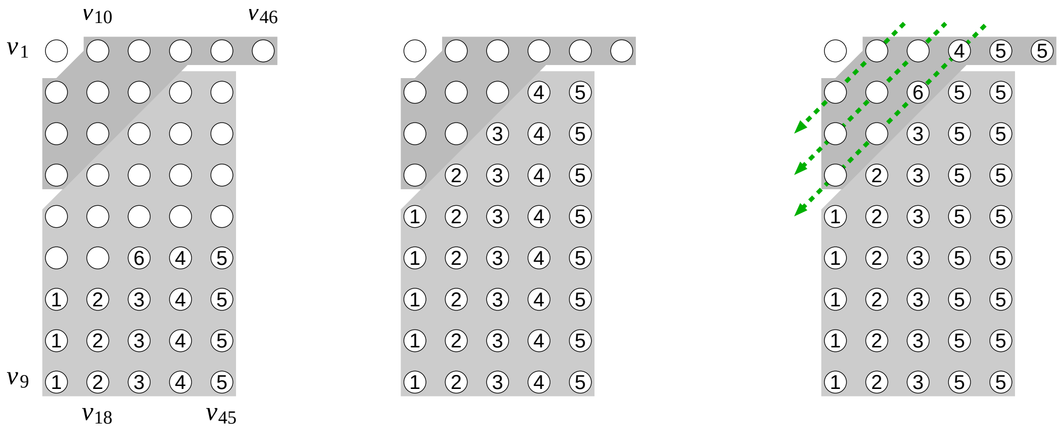

Figure 5: The three figures illustrate the construction of Lemma 4.2. With $d = 9$ and $k = 5$, the left-side figure illustrates the different indexations of the vertices: $b_{1,1} = v_1$, $b_{9,1} = v_9$, $b_{1,2} = v_{10}$ and $b_{1,6} = v_n$. The vertex set is partitioned into a lower- and an upper set, that are indicated by the two areas of different grey. The three figures illustrate three typical situations during the construction of a linear expression.

Lemma 4.2 (Deep rectangles of size k)**.** *Let d and k be integers with $d > k > 1$, and let $n =_{\text{def}} kd + 1$. Let G be a $(d-1)$-path power on n vertices with $(d-1)$-path layout $\langle v_1, \ldots, v_n \rangle$. Then, $G{-}v_1$ has a linear $(k+1)$-expression α such that in* $\text{val}(\alpha)$*: v_i has label $i+2$ for every $2 \leq i < k$ and $v_k, \ldots, v_d$ have label 3 and $v_{d+1}, \ldots, v_n$ have label 1.*

Proof. Let $\langle b_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$ be a full bubble model for G where $v_i = b_{i,1}$ for every $1 \leq i \leq d$ and $s = k+1$ and $r_1 = \cdots = r_{s-1} = d$ and $r_s = 1$; such a bubble model clearly exists and is straightforward to construct; consider the left-side figure of Figure 5 for an illustration. Using the vertex names of the bubble model, the given $(d-1)$-path layout for G may also be written as $\langle b_{1,1}, \ldots, b_{d,1}, b_{1,2}, \ldots, b_{d,k}, b_{1,k+1} \rangle$. The desired linear expression for $G{-}b_{1,1}$ is defined in two steps. We partition the vertex set of $G{-}b_{1,1}$ into a "lower" and an "upper" set. In the first step, we treat the lower-set vertices, and in the second step, we treat the upper-set vertices. The construction is illustrated by the three figures of Figure 5, and the two sets are indicated by the two differently shaded areas in the figures.

We treat the lower-set vertices. Let $A_1 =_{\text{def}} \{1\}$ and $A_j =_{\text{def}} \{j-1, j\}$ for $2 \leq j \leq k$. We use $A_1, \ldots, A_k$ as label sets, representing the neighbours of a new vertex. Let i, j be an index pair with $2 \leq i \leq d$ and $1 \leq j \leq k$ and $i + j \geq k+1$. Observe that $b_{i,j}$ is a lower-set vertex. Let $\alpha_{d,k} =_{\text{def}} k(b_{d,k})$ for the beginning, and let

$$\alpha_{i,j} \quad =_{\text{def}} \quad \rho_{(k+1)\to j}(\eta_{k+1,A_j}((k+1)(b_{i,j}) \oplus \begin{cases} \alpha_{i+1,1}))) & \text{, if } j = k \text{ and } k \leq i < d \\ \alpha_{i+1,k-i}))) & \text{, if } j = k \text{ and } i < k \\ \alpha_{i,j+1}))) & \text{, if } 1 \leq j < k \end{cases}$$

The vertices are added row-wise bottom-to-top, and within a row from right to left; this ordering is defined through the expression that is extended. The left-side figure of Figure 5 shows a typical situation during the construction: the vertex with label 6 is the one currently to be added, and its neighbours are exactly the vertices with labels 2 and 3, i.e., with labels from A_3. The middle figure of Figure 5 shows the situation after completing the first step, which means, after adding all lower-set vertices. The final obtained expression is $\alpha_{2,k-1}$. Recall that $k \geq 2$. This completes the first step.

We treat the upper-set vertices. We add the vertices by following the diagonals, as it is indicted in the right-side figure of Figure 5. First, we add the vertices $b_{1,k}$ and $b_{1,k+1}$, which is a special situation. Let

$$\alpha_{1,k} \quad =_{\text{def}} \quad \rho_{(k+1)\to k}(\eta_{k+1,k}((k+1)(b_{1,k+1}) \oplus \eta_{k+1,A_k}((k+1)(b_{1,k}) \oplus \alpha_{2,k-1})))\,.$$

It is important to note that no vertex of $\text{val}(\alpha_{1,k})$ with label k has a non-visible neighbour. Now, we define the diagonals. For $3 \leq t \leq k$, let $D(t) =_{\text{def}} \{b_{i,j} : i+j = t\} = \{b_{t-1,1}, \ldots, b_{1,t-1}\}$. Observe that the vertices from $D(3) \cup \cdots \cup D(k)$ are exactly the vertices of $G{-}b_{1,1}$ that are not of $\text{val}(\alpha_{1,k})$. We add these remaining vertices in this order: $D(k), \ldots, D(3)$. For every $3 \leq t \leq k$, we assume that there is a linear $(k+1)$-expression δ_t such that $\text{val}(\delta_t)$ has the following column properties:

- the vertices from $D(3) \cup \cdots \cup D(t)$ are exactly the vertices of $G{-}b_{1,1}$ that are not of $\text{val}(\delta_t)$
- the vertices of columns $t, \ldots, k+1$ have label t
- for every $1 < j < t$, $b_{t-j+1,j}, \ldots, b_{d,j}$ have label j
  (these are the vertices of column j that are not in $D(3) \cup \cdots \cup D(t)$), and

  $b_{k,1}, \ldots, b_{d,1}$ have label 1, and $b_{i,1}$ has label $i+2$ for every $t \leq i < k$
  (these are the vertices of column 1 that are not in $D(3) \cup \cdots \cup D(t)$).

Observe that no vertex of $\text{val}(\delta_t)$ has label $(t+1)$, and all other labels are assigned to some vertex of $\text{val}(\delta_t)$. Choosing $\delta_k =_{\text{def}} \alpha_{1,k}$, an expression having the column properties exists for the case of $t = k$.

The continuation of the construction begins with δ_t, and we add $b_{1,t-1}$. For $4 \leq t \leq k$, let:

$$\begin{aligned}
\alpha_{1,2} \quad &=_{\text{def}} \quad \rho_{4\to 3\to 2}(\eta_{4,\{1,2,5,\ldots,k+1\}}(4(b_{1,2}) \oplus \delta_3)) \\
\alpha_{1,t-1} \quad &=_{\text{def}} \quad \rho_{(t+1)\to t\to(t-1)}(\eta_{t+1,A_{t-1}}((t+1)(b_{1,t-1}) \oplus \delta_t))\,.
\end{aligned}$$

Observe about $\alpha_{1,2}, \ldots, \alpha_{1,k-1}$ that $b_{1,t-1}$ of $\text{val}(\alpha_{1,t-1})$ is the unique vertex with label t, and label $(t+1)$ is not assigned to any vertex of $\text{val}(\alpha_{1,t-1})$. We prove the existence of the claimed expressions $\delta_3, \ldots, \delta_{k-1}$ by adding the remaining diagonal vertices.

Let t be with $3 \leq t \leq k$, and let i, j be an index pair with $2 \leq i < k$ and $1 \leq j \leq k-1$ and $i + j = t$. Note that $b_{i,j} \in D(t)$. Assume that $\alpha_{i-1,j+1}$ is already defined and that $b_{i-1,j+1}$ is the unique vertex of $\text{val}(\alpha_{i-1,j+1})$ with label t, and no vertex of $\text{val}(\alpha_{i-1,j+1})$ has label $(t+1)$, and $\text{val}(\alpha_{i-1,j+1})$ has the column properties for all other vertices. Then, where $j \geq 3$, let:

$$\begin{aligned}
\alpha_{i,1} \quad &=_{\text{def}} \quad \rho_{t\to 2}(\eta_{t+1,A_1\cup\{t+2,\ldots,k+1\}\cup\{t\}}((t+1)(b_{i,1}) \oplus \alpha_{i-1,2})) \\
\alpha_{i,2} \quad &=_{\text{def}} \quad \rho_{(t+1)\to t\to 3}(\eta_{t+1,A_2\cup\{t+2,\ldots,k+1\}\cup\{t\}}((t+1)(b_{i,2}) \oplus \alpha_{i-1,3})) \\
\alpha_{i,j} \quad &=_{\text{def}} \quad \rho_{(t+1)\to t\to(j+1)}(\eta_{t+1,A_j\cup\{t\}}((t+1)(b_{i,j}) \oplus \alpha_{i-1,j+1}))\,.
\end{aligned}$$

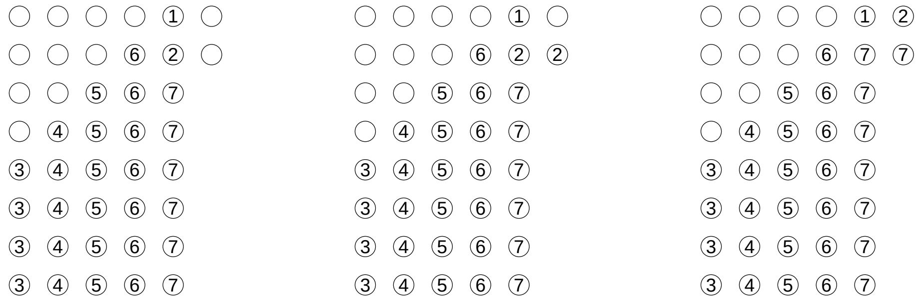

Figure 6: The three figures illustrate the construction of Lemma 4.3, using $d = 8$ and $k = 6$: before adding $b_{2,6}$ in the left-side figure, when adding $b_{2,6}$ in the middle figure, and when adding $b_{1,6}$ in the right-side figure.

Carefully analysing the defined expressions shows that $\text{val}(\alpha_{i,1})$, where $i = t-1$, has the column properties, and thus, we can choose δ_{t-1} as $\alpha_{i,1} = \alpha_{t-1,1}$.

The final constructed expression is $\alpha_{2,1}$. It remains to relabel vertices to satisfy the conditions of the lemma. So, we obtain the desired expression as $\alpha =_{\text{def}} \rho_{2\to 1\to 3}(\alpha_{2,1})$. This completes the construction and the proof. ■

Lemma 4.3 (Deep rectangles of size $k-1$). *Let d and k be integers with $d > k > 2$, and let $n =_{\text{def}} (k-1)d+2$. Let G be a $(d-1)$-path power on n vertices with $(d-1)$-path layout $\langle v_1, \ldots, v_n\rangle$. Then, $G - v_1$ has a linear $(k+1)$-expression α with inactive label 1 such that the following two conditions are satisfied:*

- *$\alpha = \alpha'(2(v_n) \oplus \alpha'')$, and label 1 is inactive in α'*
- *in $\text{val}(\alpha)$: v_i has label $i+2$ for every $2 \le i \le k-2$ and $v_{k-1}, \ldots, v_d$ have label 3 and $v_{d+1}, \ldots, v_n$ have label 1.*

Proof. The construction is very similar to the one from Lemma 4.2. Let $\langle b_{i,j}\rangle_{1\le j\le k, 1\le i\le r_j}$ be a full bubble model for G, where $b_{1,1} = v_1$ and $b_{1,k} = v_{n-1}$ and $b_{2,k} = v_n$. By a simple modification of the first-step construction about the lower-set vertices in the proof of Lemma 4.2, there is a linear $(k+1)$-expression α'' such that $G[V([\text{val}(\alpha'')])] = [\text{val}(\alpha'')]$, i.e., α'' defines the induced subgraph of $G - v_1$ on the lower-set vertices, and in $\text{val}(\alpha'')$:

- $b_{1,k-1}$ has label 1, and $b_{2,k-1}$ has label 2, and $b_{3,k-1}, \ldots, b_{d,k-1}$ have label $(k+1)$
- for every $1 \le j \le k-2$, $b_{k-j,j}, \ldots, b_{d,j}$ have label $(j+2)$.

The result of α'' is illustrated in the left-side figure of Figure 6. Next, let

$$\alpha_{1,k} \;=_{\text{def}}\; \rho_{(k+1)\to 1}(\rho_{2\to 1}(\eta_{2,k+1}(2(b_{1,k}) \oplus \rho_{2\to(k+1)}(\eta_{2,k+1}(2(b_{2,k}) \oplus \alpha''))))) \,,$$

and the result of $\alpha_{1,k}$, before applying the last two ρ-operations, is illustrated in the middle and right-side figure of Figure 6.

The final expression is obtained analogous to the construction for the upper-set vertices in the proof of Lemma 4.2. ■

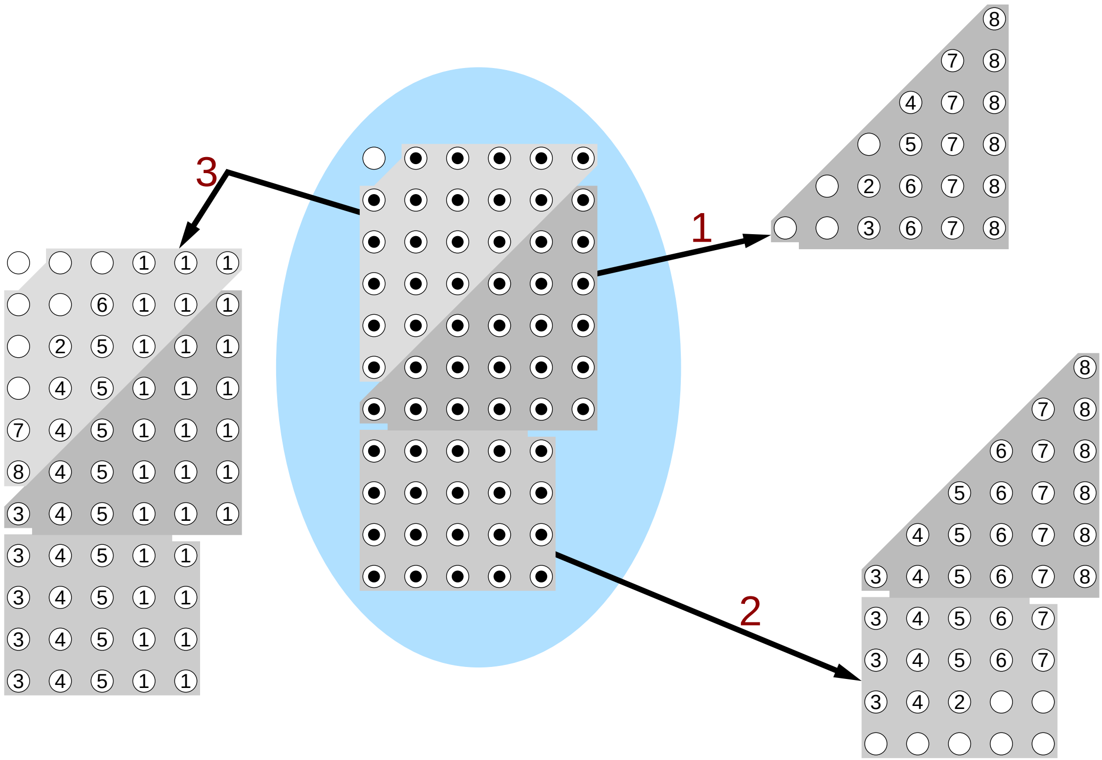

Figure 7: The figure illustrates the construction of a linear expression for deep rectangles, as it is defined in the proof of Lemma 4.4. We consider a rectangle of size 5 and of depth 11, and the value of the parameter k is 7. To construct the expression, the rectangle is partitioned into three areas, which are indicated in the central figure part by different shades of grey. In a first step, the vertices of the middle part are created, in a column-wise manner, the columns from right to left and within a column from bottom to top. A typical situation during the construction is depicted. In a second step, the vertices of the lower part are created, row-wise from top to bottom and within a row from left to right. In a third step, the upper-part vertices are created, by following the diagonals.

Lemma 4.4 (Deep rectangles of size at most $k-2$)**.** *Let d, k and q be integers with $d > k > k-2 \geq q \geq 1$, and let $n =_{\text{def}} qd+k$. Let G be a $(d-1)$-path power on n vertices with $(d-1)$-path layout $\langle v_1,\ldots,v_n\rangle$. Then, $G-v_1$ has a linear $(k+1)$-expression α with inactive label 1 such that the following two conditions are satisfied:*

- *α begins with $\delta' = 3(v_{n-k+2}) \oplus (\cdots \oplus (k+1)(v_n)\cdots)$*
- *in $\text{val}(\alpha)$: v_i has label $i+2$ for every $2 \leq i < k$ and $v_k,\ldots,v_d$ have label 3 and $v_{d+1},\ldots,v_n$ have label 1.*

Proof. The construction of the desired expression is similar to the construction of Lemma 4.2 in parts. It is nevertheless more complex due to the requirements about the beginning of the expression and the value of n. Let $\langle b_{i,j}\rangle_{1\leq j\leq s,1\leq i\leq r_j}$ be a full bubble model for G with $b_{i,1} = v_i$

for $1 \leq i < k$. Note that $s = q+1$ and $r_1 = \cdots = r_{s-1} = d$ and $r_s = k$. The major construction steps are illustrated in the accompanying Figure 7.

We partition the vertex set of $G - v_1$ into three parts: an upper set, a middle set and a lower set. The three sets are indicated by the three differently shaded areas in Figure 7. The construction will take the middle-set vertices first, then the lower-set vertices, and then the upper-set vertices. After the second step, when the middle- and lower-set vertices are added, the situation will resemble the situation after the first step of the construction for Lemma 4.2. Again, vertices from the same column will receive the same label, and we will call this label the "standard label" of the column. For every $1 \leq j \leq q+1$, let $a_j =_{\text{def}} (k-q)+j$. Since $k - q \geq 2$ due to the assumptions of the lemma, $a_1 \geq 3$ and $a_{q+1} = k+1$. Label a_j will be the standard label for column j.

The first step takes the middle-set vertices, column-wise, from right to left. We define expressions $\delta_{i,j}$. Let $\delta_{2,q+1} =_{\text{def}} \eta_{3,\{4,\ldots,k+1\}}(\eta_{4,\{5,\ldots,k+1\}} \cdots (\eta_{k,\{k+1\}}(\delta')) \cdots)$, and observe that $[\text{val}(\delta_{2,q+1})]$ is equal to $G[\{v_{n-k+2}, \ldots, v_n\}]$, and $b_{k,q+1} = v_n$ has label a_{q+1} in $\text{val}(\delta_{2,q+1})$. Let j be a column index with $1 \leq j \leq q$. Note that the vertices of column j in the middle set are $b_{q-j+3,j}, \ldots, b_{k,j}$. Let i be a row index with $q - j + 3 \leq i \leq k$. We want to add vertex $b_{i,j}$, and we assume that an expression $\delta_{i+1,j}$ for the case of $i < k$ or an expression $\delta_{q-j+2,j+1}$ for the case of $i = k$ is already defined. We assume about $\text{val}(\delta_{i+1,j})$ or $\text{val}(\delta_{q-j+2,j+1})$: the non-neighbours of $b_{i,j}$ have their respective standard label and the neighbours of $b_{i,j}$ have labels $3, \ldots, a_{j+1} - 1$. Then, let

$$\delta_{i,j} \;=_{\text{def}}\; \rho_{2\to3\to4\to\cdots\to a_j\to a_{j+1}}(\eta_{2,\{3,\ldots,a_j\}}(2(b_{i,j}) \oplus \begin{cases} \delta_{q-j+2,j+1})) \cdots)) & \text{, if } i = k \\ \delta_{i+1,j})) \cdots)) & \text{, if } i < k\,. \end{cases}$$

The final expression for the middle-set vertices is $\delta_{q+2,1}$. We list two column properties about $\text{val}(\delta_{q+2,1})$:

- for $2 \leq j \leq q+1$, the vertices of column j have their standard label a_j, and $b_{q+2,1}, \ldots, b_{k-1,1}$ have label respectively $3, \ldots, a_1 - 1$, and the other vertices from column 1 have label a_1
- no vertex has label 1 or 2.

In case of $\text{val}(\delta_{q+2,1})$, "the other vertices from column 1" only applies to $b_{k,1}$. This completes the construction of the first step and for the middle-set vertices.

For the second step, we consider the vertices from the lower set. The construction is like the first step of the construction in the proof of Lemma 4.2 in reverse order. It takes the vertices row-wise top-to-bottom, and within a row from left to right. Let $\beta_{k,q} =_{\text{def}} \delta_{q+2,1}$. Let i and j be indices with $k+1 \leq i \leq d$ and $1 < j \leq q$. Let

$$\begin{aligned} \beta_{i,1} \;&=_{\text{def}}\; \rho_{2\to a_1}(\eta_{2,\{3,\ldots,a_1\}\cup\{a_2\}}(2(b_{i,1}) \oplus \beta_{i-1,q})) \\ \beta_{i,j} \;&=_{\text{def}}\; \rho_{2\to a_j}(\eta_{2,\{a_j,a_{j+1}\}}(2(b_{i,j}) \oplus \beta_{i,j-1}))\,. \end{aligned}$$

The final expression for the middle-set and lower-set vertices is $\beta_{d,q}$. Observe that $[\text{val}(\beta_{d,q})]$ is equal to the subgraph of $G - v_1$ induced by the middle- and lower-set vertices. It is necessary and useful to observe that $\mathrm{val}(\beta_{d,q})$ also has the above listed two column properties. This completes the construction of the second step.

For the third step, we consider the upper-set vertices. The construction is reminiscent of the second step of the construction in the proof of Lemma 4.2. Let t be an integer with $3 \leq t \leq q+2$, and let i, j be an index pair with $1 \leq j \leq q+1$ and $1 \leq i \leq q+1$ and $i+j = t$. Let $A_t =_{\mathrm{def}} \{3, \ldots, a_1 - 1\} \cup \{a_t, \ldots, a_{q+1}\}$. Note that $A_t = \{k-q+t, \ldots, k+1\}$, for the case of $q = k-2$, or $A_t = \{3, \ldots, k-q, k-q+t, \ldots, k+1\}$, for the case of $q \leq k-3$. We define $\alpha_{i,j}$ so that $b_{i,j}$ in $\mathrm{val}(\alpha_{i,j})$ has label a_{t-1} and $\mathrm{val}(\alpha_{i,j})$ for all the other vertices has the following column properties:

- vertices from columns $t-1, \ldots, q+1$ except $b_{i,j}$ have label 1, and vertices from columns $2, \ldots, t-2$ have their respective standard labels $a_2, \ldots, a_{t-2}$
- $b_{k,1}, \ldots, b_{d,1}$ have label a_1, and $b_{t,1}, \ldots, b_{k-1,1}$ have pairwise different labels from A_t.

Let $\alpha_{q+2,1} =_{\mathrm{def}} \beta_{d,q}$. With $j \geq 3$, let

$$\begin{aligned} \alpha_{1,2} &=_{\mathrm{def}} \rho_{2\to a_2 \to 1}(\eta_{2,A_3 \cup \{a_1,a_2\}}(2(b_{1,2}) \oplus \alpha_{3,1})) \\ \alpha_{1,j} &=_{\mathrm{def}} \rho_{2\to a_j \to 1}(\eta_{2,\{a_{j-1},a_j\}}(2(b_{1,j}) \oplus \alpha_{j+1,1}))\,. \end{aligned}$$

If $q = 1$ then $\alpha_{3,1} = \beta_{d,q} = \beta_{d,1}$, and the construction is (almost) completed. Otherwise, $q \geq 2$, and we add the other vertices of a diagonal. With $j \geq 3$ and $i \geq 3$ and $t \geq 3$ for $\alpha_{t-1,1}$ and $t \geq 4$ for $\alpha_{t-2,2}$, let

$$\begin{aligned} \alpha_{2,j} &=_{\mathrm{def}} \rho_{2\to a_{t-1} \to 1}(\eta_{2,\{a_{j-1},a_j,a_{t-1}\}}(2(b_{2,j}) \oplus \alpha_{1,j+1})) \\ \alpha_{i,j} &=_{\mathrm{def}} \rho_{2\to a_{t-1} \to a_{j+1}}(\eta_{2,\{a_{j-1},a_j,a_{t-1}\}}(2(b_{i,j}) \oplus \alpha_{i-1,j+1})) \\ \alpha_{t-2,2} &=_{\mathrm{def}} \rho_{2\to a_{t-1} \to a_3}(\eta_{2,A_t \cup \{a_1,a_2,a_{t-1}\}}(2(b_{t-2,2}) \oplus \alpha_{t-3,3})) \\ \alpha_{t-1,1} &=_{\mathrm{def}} \rho_{2\to a_{t-1} \to a_2}(\eta_{2,A_t \cup \{a_1,a_{t-1}\}}((2(b_{t-1,1}) \oplus \alpha_{t-2,2}))\,. \end{aligned}$$

It is straightforward verified that the defined expressions have the requested column properties. This completes the construction of the third step.

The (almost) final expression for $G - v_1$ is $\alpha_{2,1}$. With the column properties for $\alpha_{2,1}$, it is not difficult to see that $\alpha_{2,1}$ is indeed a linear $(k+1)$-expression for $G - v_1$ with the desired properties, except for the labels of the vertices from the first column: the vertices $b_{2,1}, \ldots, b_{k-1,1}$ have pairwise different labels, but the assignment may differ from the requested one, and the vertices $b_{k,1}, \ldots, b_{d,k}$ have label a_1, which may not be label 3. So, it remains to change these labels, by using the unassigned label 2. This completes the construction. ∎

4.2 The composition lemma and the upper-bound results

In the preceding subsection, we showed the existence of special linear expressions for path powers, that relate to shallow and deep rectangles of k-models. In this subsection, we combine these expressions into expressions for the three types of full bubble models that we defined in Section 3.

We begin with a technical result that facilitates the composition of expressions for smaller graphs into expressions for larger graphs. Let F and H be two (arbitrary) graphs. The *union* of F and H is denoted as $F + H$ and is the graph on vertex set $V(F) \cup V(H)$ and with edge set $E(F) \cup E(H)$. The disjoint union of two graphs is a special case of the union, where the two graphs must be vertex-disjoint.

Lemma 4.5. *Let k be an integer with $k \geq 1$. Let G be a connected graph and let F and H be induced subgraphs of G such that $G = F + H$. Let $S =_{\text{def}} V(F) \cap V(H)$, and assume that $S = \{u_1, \ldots, u_r\}$.*

Let δ be a linear k-expression for H, let $\ell_1, \ldots, \ell_r$ be labels from $\{1, \ldots, k\}$ such that $u_1, \ldots, u_r$ have label respectively $\ell_1, \ldots, \ell_r$ in $\text{val}(\delta)$, and let β be a linear k-expression for F. Assume that β and δ satisfy the following conditions:

- $\beta = \beta'(\ell_1(u_1) \oplus \cdots \oplus (\ell_{r-1}(u_{r-1}) \oplus \ell_r(u_r)) \cdots)$ *or*
  $\beta = \beta'(\ell_1(u_1) \oplus \cdots \oplus (\ell_{r-1}(u_{r-1}) \oplus (\ell_r(u_r) \oplus \beta'')) \cdots)$
- *the vertices from $V(H) \setminus S$ have label 1 in* $\text{val}(\delta)$
- *label 1 is inactive in β'.*

Then, $\beta'(\delta)$ or $\beta'(\delta \oplus \beta'')$ is a k-expression for G, and $\beta'(\delta)$ is a linear k-expression.

Proof. Since δ is a linear k-expression and β' is an end of a linear k-expression, it is straightforward to see that $\beta'(\delta)$ is a linear k-expression. Let $\alpha =_{\text{def}} \beta'(\delta)$ or $\alpha =_{\text{def}} \beta'(\delta \oplus \beta'')$, depending on whether β'' is defined or not. For the following arguments, we will not need to properly distinguish between the two cases and simply assume the latter case with a possible "empty" expression β''. Observe the following simple observations:

- since the vertices from $V(H) \setminus S$ have label 1 in $\text{val}(\delta)$, and since label 1 is inactive in β', the vertices from $V(H) \setminus S$ have label 1 in $\text{val}(\alpha)$, and
  $[\text{val}(\alpha)][V(H) \setminus S] = [\text{val}(\delta)][V(H) \setminus S] = H \setminus S$
- the vertices of $F \setminus S$ are not vertices of H, so that
  $[\text{val}(\beta)][V(F) \setminus S] = F \setminus S$ and $[\text{val}(\alpha)][V(F) \setminus S] = F \setminus S$
- $[\text{val}(\delta)][S] = H[S]$ and $[\text{val}(\beta)][S] = F[S]$ and $H[S] = G[S] = F[S]$, and thus,
  $[\text{val}(\alpha)][S] = G[S]$.

So, $[\text{val}(\alpha)]$ coincides with G on the three induced subgraphs $G[S]$, $F \setminus S$ and $H \setminus S$. In particular, each vertex of G is a vertex of $[\text{val}(\alpha)]$. It remains to check the edges. It is important to note that no vertex of $F \setminus S$ is adjacent to a vertex of $H \setminus S$, which is a consequence of the assumptions of the lemma. So, let u, v be a vertex pair of G with $u \in S$ and $v \notin S$. If v is a vertex of H then $uv \in E(G)$ if and only if $uv \in E(H)$, if and only if $uv \in E([\text{val}(\delta)])$, if and only if $uv \in E([\text{val}(\alpha)])$. For the latter equivalence, it is most important to recall that the vertices of $H \setminus S$ have label 1 in $\text{val}(\delta)$ and label 1 is inactive in β'. So, non-adjacent vertex pairs of H cannot become adjacent through an operation of β'. Similarly, if v is a vertex of F then $uv \in E(G)$ if and only if $uv \in E(F)$, if and only if $uv \in E([\text{val}(\beta)])$, if and only if $uv \in E([\text{val}(\alpha)])$. Thus, $[\text{val}(\alpha)] = G$. ∎

The set S of Lemma 4.5 is a *separator* of G. We will prove the main result of this section, upper bounds on the clique-width and linear clique-width for the three special classes of proper interval graphs, by splitting the graphs on separators and combining expressions for smaller graphs into expressions for larger graphs.

We need to distinguish between linear expressions and expressions that may not be linear. We consider open k-models and analogous models obtained from k-models with small separators. An *open k-model with small separators* is obtained from a k-model $\mathcal{B}$ by first applying the vertex deletion about the deep rectangles of size $k-1$ and then deleting the first rectangle of $\mathcal{B}$. Alternatively, we can say that an open k-model with small separators is obtained from a k-model with small separators by deleting the first rectangle and re-inserting the deleted vertices $b_{k'+2,b}, \ldots, b_{k,b}$.

Lemma 4.6. *Let k be an integer with $k \geq 3$. Let G be a proper interval graph on at least two vertices, and let $\mathcal{B} = \langle b_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$ be a bubble model for G.*

1) Assume that $\mathcal{B}$ is an open k-model. Then, $G - b_{1,1}$ has a linear $(k+1)$-expression α such that in $\mathrm{val}(\alpha)$: $b_{2,1}, \ldots, b_{k-1,1}$ have label respectively $4, \ldots, k+1$ and $b_{k,1}, \ldots, b_{r_1,1}$ have label 3 and the other vertices have label 1.

2) Assume that $\mathcal{B}$ is an open k-model with small separators and the first rectangle of $\mathcal{B}$ is not a deep rectangle of size $k-1$. Then, $G - b_{1,1}$ has a $(k+1)$-expression α such that in $\mathrm{val}(\alpha)$: $b_{2,1}, \ldots, b_{k-1,1}$ have label respectively $4, \ldots, k+1$ and $b_{k,1}, \ldots, b_{r_1,1}$ have label 3 and the other vertices have label 1.

3) Assume that $\mathcal{B}$ is an open k-model with small separators and the first rectangle of $\mathcal{B}$ is a deep rectangle of size $k-1$. Then, $G - b_{1,1}$ has a $(k+1)$-expression α such that in $\mathrm{val}(\alpha)$: $b_{2,1}, \ldots, b_{k-2,1}$ have label respectively $4, \ldots, k$ and $b_{k-1,1}, \ldots, b_{r_1,1}$ have label 3 and the other vertices have label 1.

Proof. We prove the result by induction on the rectangle partition of $\mathcal{B}$. For the proof, we will distinguish between deep rectangles and rectangles that are not deep. In case of open k-models, the latter are clearly the shallow rectangles. In case of open k-models with small separators, also rectangles of size 1 and depth 2 or $k-1$ need to be considered. We simplify the proof and the cases by joining such rectangles. A *shallow pseudo-rectangle* is the combination of consecutive rectangles of $\mathcal{B}$ that are not deep. We can say that the rectangles between two consecutive deep rectangles form a shallow pseudo-rectangle. In case of open k-models, the shallow pseudo-rectangles are the shallow rectangles. In case of open k-models with small separators, a shallow pseudo-rectangle may be the combination of up to three rectangles of depth at most k. In the proof, we use some informal terminology, whose meanings should be clear without a proper definition.

Let $\langle \mathcal{B}_1, \ldots, \mathcal{B}_t \rangle$ be the pseudo-rectangle partition of $\mathcal{B}$. Since G has at least two vertices, $t \geq 2$ clearly holds. Let d be the depth of the deep rectangles. If $t = 2$ then G is a $(d-1)$-path power on $kd+1$ vertices, and the desired linear $(k+1)$-expression exists due to Lemma 4.2. Note that the expression is valid for the three cases, and the third case does not apply. We henceforth assume $t \geq 3$. We consider the three cases of the lemma simultaneously and discuss the differences when they are important.

We consider $\mathcal{B}_1$. Recall that $\mathcal{B}_1$ is a deep rectangle or a shallow pseudo-rectangle. Let q be such that $\mathcal{B}_1 = \mathcal{B}[1, q]$, and let M be the set of the vertices in $\mathcal{B}_1$, i.e., $M = \{b_{1,1}, \ldots, b_{r_q,q}\}$. We apply the induction hypothesis to $G \setminus M$ in the following way, by distinguishing between three situations. If $\mathcal{B}$ is an open k-model then $\langle \mathcal{B}_2, \ldots, \mathcal{B}_t \rangle$ is an open k-model for $G \setminus M$. If $\mathcal{B}$ is an open k-model with small separators and $\mathcal{B}_1$ is a deep rectangle of size at most $k - 2$ or a shallow pseudo-rectangle then $\langle \mathcal{B}_2, \ldots, \mathcal{B}_t \rangle$ is an open k-model with small separators for $G \setminus M$. And if $\mathcal{B}$ is an open k-model with small separators and $\mathcal{B}_1$ is a deep rectangle of size $k - 1$ then $r_{q+1} = 2$, and $\langle \mathcal{B}_2, \ldots, \mathcal{B}_t \rangle$ can be obtained from an open k-model with small separators by deleting vertices from the first column. In all situations, we can apply the induction hypothesis and obtain appropriate $(k + 1)$-expressions: there is a (linear) $(k + 1)$-expression δ for $G \setminus (M \cup \{b_{1,q+1}\})$ such that one of the following three is the case in val(δ):

1) $b_{2,q+1}$ has label 2 and the other vertices have label 1
2) $b_{2,q+1}, \ldots, b_{k-1,q+1}$ have label respectively $4, \ldots, k+1$ and $b_{k,q+1}, \ldots, b_{r_{q+1},q+1}$ have label 3 and the other vertices have label 1
3) $b_{2,q+1}, \ldots, b_{k-2,q+1}$ have label respectively $4, \ldots, k$ and $b_{k-1,q+1}, \ldots, b_{r_{q+1},q+1}$ have label 3 and the other vertices have label 1.

The second situation is the "standard" situation and directly corresponds to the induction hypothesis. The first situation is the case if $\mathcal{B}_1$ is a deep rectangle of size $k - 1$, and we obtain the expression by deleting the unnecessary vertices. The third situation is the case if $\mathcal{B}_2$ is a deep rectangle of size $k - 1$, in particular, $\mathcal{B}$ must be an open k-model with small separators. Note that $\mathcal{B}_1$ is a shallow pseudo-rectangle in the third situation. We distinguish between the two cases about $\mathcal{B}_1$, whether it is a deep rectangle or a shallow pseudo-rectangle.

First case: $\mathcal{B}_1$ is a shallow pseudo-rectangle
Note that $\mathcal{B}_2$ is a deep rectangle, and a $(k+1)$-expression δ for $G \setminus (M \cup \{b_{1,q+1})$ of the required properties exists due to the induction hypothesis. Recall that δ is a linear $(k + 1)$-expression if $G \setminus M$ has an open k-model. Let

$$\delta' =_{\text{def}} \rho_{2\to3\to1}(\eta_{2,\{3,\ldots,k+1\}}(2(b_{1,q+1}) \oplus \delta)) ,$$

which is a (linear) $(k + 1)$-expression for $G \setminus M$ such that in val(δ'): $b_{1,q+1}, \ldots, b_{k-2,q+1}$ have label respectively $3, \ldots, k$ and $b_{k-1,q+1}$ has label $(k+1)$ or 1 and the other vertices have label 1.

Assume that $\mathcal{B}_2$ is a deep rectangle of size k or of size at most $k - 2$. Note that the second above situation is the case. Let $S =_{\text{def}} \{b_{1,q+1}, \ldots, b_{k-1,q+1}\}$, and let $F =_{\text{def}} G[M \cup S]$. The vertices from S have pairwise different labels from $\{3, \ldots, k + 1\}$ in val(δ') and F is an induced subgraph of a $(k - 1)$-path power F' on $kq + (k - 1)$ vertices with the $(k - 1)$-path layout $\langle b_{1,1}, \ldots, b_{k,1}, b_{1,2}, \ldots, b_{k,q}, b_{1,q+1}, \ldots, b_{k-1,q+1} \rangle$. Observe that $V(F') \cap V(G \setminus M) = S$. We apply Lemma 4.1 to $F'-b_{1,1}$ and obtain a linear $(k + 1)$-expression for $F'-b_{1,1}$, and by applying Lemma 4.5 to F' and $G \setminus M$, we directly conclude the existence of a (linear) $(k + 1)$-expression for $(F'-b_{1,1}) + (G \setminus M)$ of the desired form. If $\mathcal{B}_1$ is a shallow rectangle then $F = F'$, and we can conclude. If $\mathcal{B}_1$ is not a shallow rectangle then $r_1 = 2$, and we obtain the desired $(k + 1)$-expression by deleting the additional vertices of F'.

Assume that $\mathcal{B}_2$ is a deep rectangle of size $k-1$. Then, the third above situation is the case. We proceed analogous to the previous case, by choosing $S' =_{\text{def}} \{b_{1,q+1}, \ldots, b_{k-2,q+1}\}$, and F' will be a $(k-1)$-path power on $qk + (k-2)$ vertices. ◻

Second case: $\mathcal{B}_1$ is a deep rectangle
Note that $\mathcal{B}_2$ is a shallow pseudo-rectangle, and $\mathcal{B}_1$ is a deep rectangle of size q. Assume $q \leq k-2$. Then, $r_{q+1} = k-1$ or $r_{q+1} = k$, and the former is only the case if $\mathcal{B}_2$ is a shallow pseudo-rectangle of size 1 and $\mathcal{B}_3$ is a deep rectangle of size $k-1$.

Let $S =_{\text{def}} \{b_{2,q+1}, \ldots, b_{r_{q+1},q+1}\}$. Let F' be a $(d-1)$-path power on $qd + k$ vertices such that $G[M \cup S]$ is an induced subgraph of F'. Observe that $F' = G[M \cup S]$ if $r_{q+1} = k$, and F' has $b_{k,q+1}$ as an additional vertex if $r_{q+1} = k-1$. We can apply Lemma 4.4 to F' and its $(d-1)$-path layout $\langle b_{1,1}, \ldots, b_{d,1}, b_{1,2}, \ldots, b_{d,q}, b_{1,q+1}, b_{2,q+1}, \ldots, b_{k,q+1} \rangle$, and by applying Lemma 4.5, we obtain the desired (linear) $(k+1)$-expression for $G - b_{1,1}$.

Assume that $\mathcal{B}_1$ is a deep rectangle of size $k-1$, i.e., $q = k-1$. Then, $r_{q+1} = 2$ must hold, and $G[M \cup \{b_{1,q+1}, b_{2,q+1}\}]$ satisfies the assumptions of Lemma 4.3. Let $\alpha = \alpha'(2(b_{2,q+1}) \oplus \alpha'')$ be the linear $(k+1)$-expression for $G[M \cup \{b_{1,q+1}, b_{2,q+1}\}] - b_{1,1}$ according to Lemma 4.3. Then, $\alpha'(\delta \oplus \alpha'')$ is a desired $(k+1)$-expression for $G - b_{1,1}$, that will not be linear. ◻

We have constructed desired $(k+1)$-expressions for $G - b_{1,1}$ from (linear) $(k+1)$-expressions for induced subgraphs and iteratively combining them through Lemma 4.5. This completes the proof. ■

The expressions of Lemma 4.6 will be central in the construction of appropriate expressions for our considered graph classes, to obtain the upper-bound results. To make the lemma applicable in further cases, we define a basic but useful index transformation on bubble models. We describe the operation on "right-side open" k-models; the transformation itself is general and is executable on every bubble model. Let k be an integer with $k \geq 3$, and let G be a graph with a k-model $\mathcal{B} = \langle b_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$. Let $\langle \mathcal{B}_1, \ldots, \mathcal{B}_t \rangle$ be the rectangle partition of $\mathcal{B}$. Note that $\mathcal{B}_{t-2}$ is a shallow rectangle of $\mathcal{B}$. Let p, q be such that $\mathcal{B}_{t-2} = \mathcal{B}[p, q]$. Note that $\langle b_{1,q}, \ldots, b_{k,q} \rangle$ is column q of $\mathcal{B}$. We consider $\mathcal{F} =_{\text{def}} \langle \mathcal{B}_1, \ldots, \mathcal{B}_{t-2} \rangle$. Let H be the induced subgraph of G with bubble model $\mathcal{F}$. We show that H is an induced subgraph of a graph with an open k-model whose first column begins as $\langle b_{k,q}, \ldots, b_{1,q} \rangle$. The index transformation on $\mathcal{F}$ can be described in four steps, and the four steps are illustrated in Figure 8. We begin with $\mathcal{F}$ and separate the lower parts of the deep rectangles (step 1) and move these parts above and shift them by one column to the right (step 2). It is important to observe and straightforward to verify that this is still a bubble model for H, however it may not be a full bubble model anymore. Next, we perform a half-circle rotation on the model (step 3), which is also equivalent to a horizontal and then a vertical flip, and finally, we add new vertices and fix the new rectangle partition, and obtain an open k-model for a graph that contains H as an induced subgraph (step 4).

Proposition 4.7. *Let k be an integer with $k \geq 3$. Let G be a proper interval graph.*

1) If G has an open k-model then $\text{lcwd}(G) \leq k+1$.

2) If G has a short-end k-model then $\text{lcwd}(G) \leq k+1$.

3) If G has a k-model with small separators then $\text{cwd}(G) \leq k+1$.

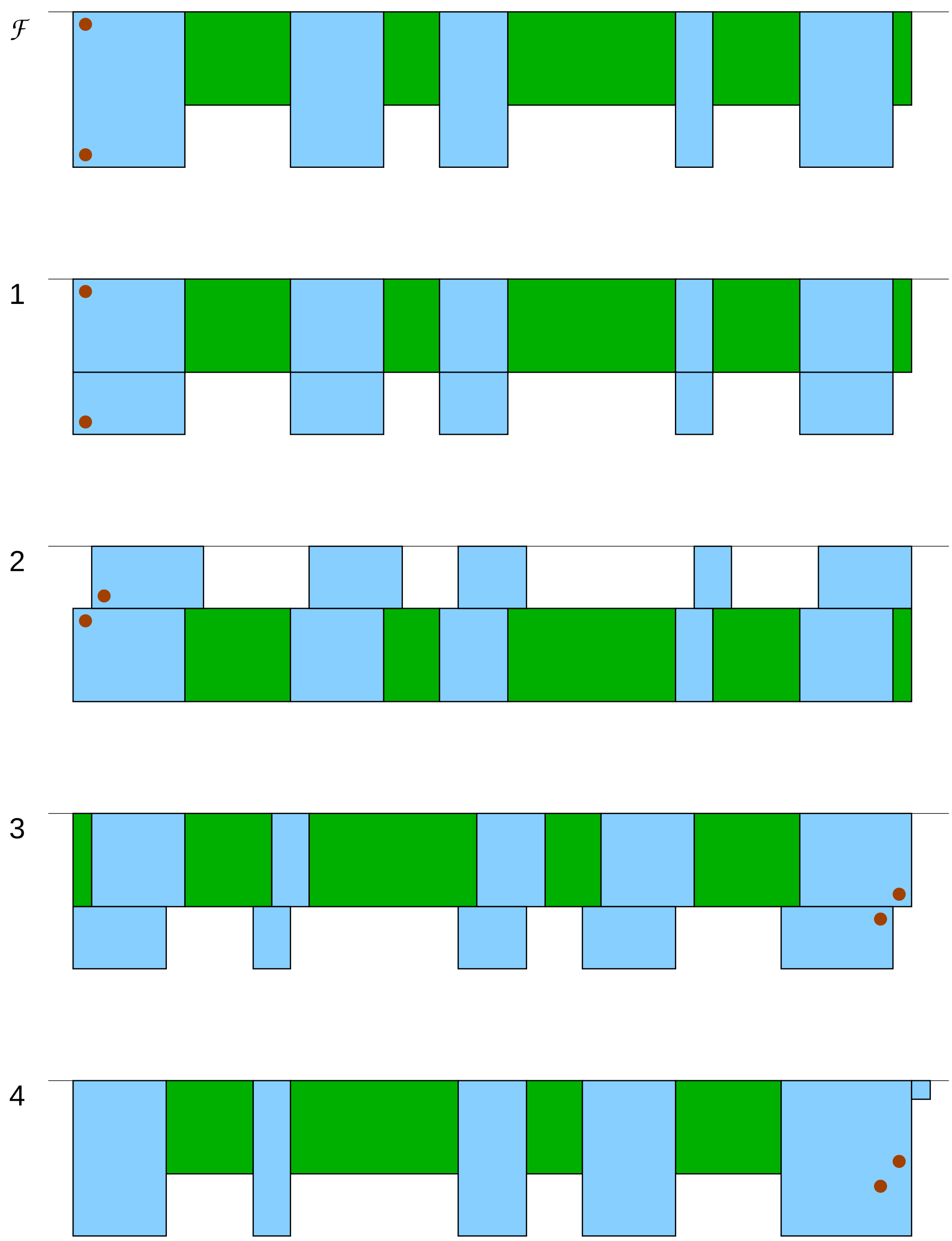

Figure 8: Taking a k-model and deleting the last two rectangles represents a graph that is an induced subgraph of a graph with an open k-model, as the four index transformation steps show. The two indicated vertices provide points of reference during the transformation.

Proof. We show that appropriate $(k+1)$-expressions for G exist, by separately considering the three different models. If G has an open k-model then the first statement of Lemma 4.6

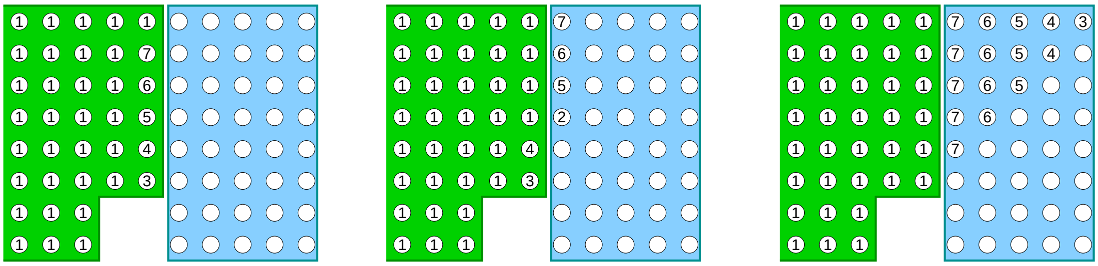

Figure 9: A linear $(k+1)$-expression for a graph with a short-end k-model is obtained by extending a linear $(k+1)$-expression for a graph with an open $(k+1)$-model. The figures illustrate three intermediate situations during the extension, for the special case of $k = 6$: the beginning in the left-side figure, adding the upper-set vertices in the middle figure, and before adding the lower-set vertices in the right-side figure.

is applicable, and by adding the remaining vertex to the obtained expression, analogous to the construction of δ' in the proof of Lemma 4.6, we obtain a linear $(k+1)$-expression for G, and $\mathrm{lcwd}(G) \leq k+1$ follows, which proves the first statement of the proposition.

We prove the second statement of the proposition. Let $\mathcal{B} = \langle b_{i,j}\rangle_{1\leq j\leq s, 1\leq i\leq r_j}$ be a short-end k-model for G, and let $\langle \mathcal{B}_1, \ldots, \mathcal{B}_t\rangle$ be the rectangle partition of $\mathcal{B}$. Recall that $\mathcal{B}_t$ is a deep rectangle of size $k-1$ and $\mathcal{B}_{t-1}$ is a shallow rectangle. Let H be the induced subgraph of G defined by the restriction of $\mathcal{B}$ to $\langle \mathcal{B}_1, \ldots, \mathcal{B}_{t-1}\rangle$, and let $q =_{\mathrm{def}} s-k+1$. Then, $\langle b_{1,q}, \ldots, b_{k,q}\rangle$ is the last column of $\mathcal{B}_{t-1}$. Now, recall from the discussion preceding the proposition statement that H is an induced subgraph of a graph F that has an open k-model $\mathcal{F} = \langle f_{i,j}\rangle_{1\leq j\leq s', 1\leq i\leq r'_j}$ such that $\langle f_{1,1}, \ldots, f_{k-1,1}, f_{k,1}\rangle = \langle b_{k,q}, \ldots, b_{2,q}, b_{1,q}\rangle$. So, as shown for the first statement of the proposition, F has a linear $(k+1)$-expression δ such that in $\mathrm{val}(\delta)$: $f_{1,1}, \ldots, f_{k-1,1}$ have label respectively $3, \ldots, k+1$ and the other vertices have label 1. Thus, H has a linear $(k+1)$-expression δ' such that in $\mathrm{val}(\delta')$: $b_{2,q}, \ldots, b_{k,q}$ have label respectively $k+1, \ldots, 3$ and the other vertices have label 1. Note that δ' is obtained from δ simply by deleting the vertices of F that are not vertices of H.

We extend δ' by adding the remaining vertices of G. The remaining vertices of G are the vertices that are not of H, which are the vertices in $\mathcal{B}_t$. It is not difficult to verify that δ' can be extended by adding the vertices analogous to the construction of Lemma 4.2 but in reverse order and neglecting the vertices from the two last columns k and $k+1$. This can be done using $k+1$ labels and keeping label 1 as an inactive label (see Figure 9). We can conclude that G has a linear $(k+1)$-expression, and $\mathrm{lcwd}(G) \leq k+1$ follows.

We prove the third statement of the proposition. Let $\mathcal{B} = \langle b_{i,j}\rangle_{1\leq j\leq s, 1\leq i\leq r_j}$ be a k-model with small separators for G. We choose a column to split $\mathcal{B}$ into two parts. If $r_j \geq k$ for every $1 \leq j < s$ then let q be smallest possible with $1 \leq q < s$ such that $r_q \leq k$. Otherwise, let q be smallest possible with $1 \leq q < s$ such that $r_q < k$. Recall from the definition of k-models with small separators that q does exist, especially since $\mathcal{B}$ has at least three rectangles. Let $\mathcal{C} =_{\mathrm{def}} \langle b_{i,j}\rangle_{1\leq j\leq q, 1\leq i\leq r_j}$ and $\mathcal{D} =_{\mathrm{def}} \langle b_{i,j}\rangle_{q<j\leq s, 1\leq i\leq r_j}$. We can say that we split $\mathcal{B}$ at column q into a left side and a right side, i.e., into $\mathcal{C}$ and $\mathcal{D}$. Let F and H be the induced subgraphs of G represented by respectively $\mathcal{C}$ and $\mathcal{D}$. It is important to note that $\mathcal{D}$ is an open k-model with

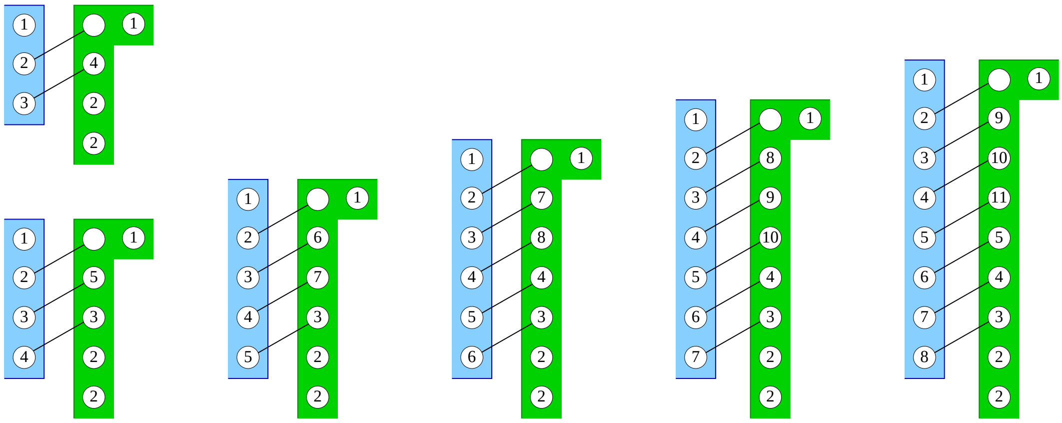

Figure 10: The six situations show the central idea of how to combine two graphs, as it is done for k-models with small separators in the proof of the third statement of Proposition 4.7. The numbers represent the labels of the vertices, and the line segments indicate the “closest” neighbours.

small separators. Thus, $H - b_{1,q+1}$ has a $(k+1)$-expression δ that satisfies the second or third statement of Lemma 4.6, depending on the size and depth of the first rectangle of $\mathcal{D}$. It is also important to note that F is an induced subgraph of a graph with an open k-model: since $k \geq 3$ and $r_j \geq k$ for every $1 \leq j < q$, the rectangle partition of $\mathcal{C}$ has at most one deep rectangle of size more than $k-2$, and this is the first rectangle, and it is of size $k-1$. Let $k' =_{\text{def}} r_q$. Recall that $\langle b_{1,q}, \ldots, b_{k',q} \rangle$ is the last column of $\mathcal{C}$, and column q of $\mathcal{B}$, and $k' \geq 2$. Analogous to the above construction for short-end k-models in the proof of the second statement of the proposition, F has a $(k+1)$-expression β such that in val(β): $b_{2,q}, \ldots, b_{k',q}$ have label respectively $k+1, \ldots, k-k'+3$ and the other vertices have label 1.

Observe that no vertex of val(β) or val(δ) has label 2. Using the available label 2, we assign new labels to the vertices of val(β) and val(δ). Let $a =_{\text{def}} \lfloor \frac{k'-1}{2} \rfloor$ and $a' =_{\text{def}} \lceil \frac{k'-1}{2} \rceil$. Clearly, $k' = a + a' + 1$. Then, F and $H - b_{1,q+1}$ have $(k+1)$-expressions respectively β' and δ' such that

- in val(β'):
  $b_{2,q}, \ldots, b_{k',q}$ have label respectively $2, \ldots, k'$ and the other vertices have label 1
- in val(δ'):
  $b_{2,q+1}, \ldots, b_{a+1,q+1}$ have label respectively $k'+1, \ldots, k'+a$, if $a \geq 1$, and
  $b_{a+2,q+1}, \ldots, b_{k'-1,q+1}$ have label respectively $a'+1, \ldots, 3$, and
  $b_{k',q+1}, \ldots, b_{r_{q+1},q+1}$ have label 2, and the other vertices have label 1.

Note here that $(k'-1) - (a+2) + 1 = k' - a - 2 = a' - 1$, so that the labels of the vertices of val(δ') are well-defined. Also note that $k' + a \leq (k'-1) + \lfloor \frac{k'-1}{2} \rfloor + 1 \leq k+1$ due to the definition of k-models with small separators. The obtained labellings for small values of k' are illustrated in Figure 10.

We show that $\beta' \oplus \delta'$ can be extended into a $(k+1)$-expression for G. It is important to note that $[\text{val}(\beta' \oplus \delta')]$ is equal to $F \oplus (H - b_{1,q+1})$, and so, it remains to add the missing edges between vertices of F and $H - b_{1,q+1}$, and the last vertex $b_{1,q+1}$. It is important to observe that the missing edges are exactly between vertices from $\{b_{2,q}, \ldots, b_{k',q}\}$ and $\{b_{2,q+1}, \ldots, b_{k'-1,q+1}\}$, and between $b_{1,q+1}$ and the vertices from $\{b_{2,q}, \ldots, b_{k',q}\} \cup \{b_{2,q+1}, \ldots, b_{k'-1,q+1}\} \cup \{b_{k',q+1}, \ldots, b_{r_{q+1},q+1}\}$. We consider the missing edges of the former type. For $2 \leq i < k'$, let d_i be the label of $b_{i,q+1}$ in $\mathrm{val}(\delta')$. Our final expression for $G - b_{1,q+1}$ is:

$$\alpha \ =_{\mathrm{def}} \ \eta_{d_2,\{3,\ldots,k'\}}(\eta_{d_3,\{4,\ldots,k'\}}(\cdots \eta_{d_{k'-1},\{k'\}}(\beta' \oplus \delta') \cdots)) \,.$$

We show that α is a $(k+1)$-expression for $G - b_{1,q+1}$. We first show that the operations are valid, which means that we need to show that $d_i \notin \{i+1, \ldots, k'\}$ for $2 \leq i \leq k'-1$. If $i \leq a+1$ then $d_i \geq k'+1$, and if $i \geq a+2$ then $d_i \leq i$, since: $d_i = k' - i + 2$ and $a \leq a' \leq a+1$ and $k' = (a+1) + a' \leq (i-1) + (a+1) \leq 2i - 2$. So, it remains to show that the added edges are the correct edges. Since $\{b_{2,q}, \ldots, b_{k',q}\}$ and $\{b_{2,q+1}, \ldots, b_{k'-1,q+1}\}$ are cliques of $F + (H - b_{1,q+1})$ and of $G - b_{1,q+1}$, it suffices to consider the added edges between vertices of F and of $H - b_{1,q+1}$. We consider i with $2 \leq i < k'$ and $\eta_{d_i,\{i+1,\ldots,k'\}}$. If $i \leq a+1$ then $b_{i,q+1}$ is the unique vertex of label d_i in $\mathrm{val}(\beta' \oplus \delta')$, and the added edges are exactly the missing edges for $b_{i,q+1}$. If $a+2 \leq i \leq k'-1$ then $\{i+1, \ldots, k'\} \subseteq \{a+3, \ldots, k'\}$, and since no vertex in $\mathrm{val}(\delta')$ has a label from $\{a+3, \ldots, k'\}$, which particularly follows from $a'+1 < a+3$, each added edge is incident to a vertex of F, and thus, the (newly) added edges are exactly the missing edges for $b_{i,q+1}$. We conclude that α is indeed a $(k+1)$-expression for $G - b_{1,1}$.

For the final desired expression for G, it remains to add $b_{1,q+1}$. ∎

Corollary 4.8. *Let k be an integer with $k \geq 3$. Let G be a proper interval graph.*

1) *If G is an induced subgraph of a graph with an open k-model or with a short-end k-model then* $\mathrm{lcwd}(G) \leq k+1$.
2) *If G is an induced subgraph of a graph with a k-model with small separators then* $\mathrm{cwd}(G) \leq k+1$.

5 First clique-width lower-bound result

Let k be an integer with $k \geq 3$, and let $n =_{\mathrm{def}} (k-1)(k+1)+2$. Observe that $(k-1)(k+1)+2 = k^2+1$. Let $\Lambda_k = \langle v_1, \ldots, v_n \rangle$ be a sequence of n pairwise different vertices. The k-path power on n vertices and with the k-path layout Λ_k is denoted as R_k. The graph S_k is obtained from R_k by adding the four new vertices w_1, w_2, w_3, w_4 such that $N_{S_k}(w_1) = \{w_2\}$ and $N_{S_k}(w_2) = \{w_1, v_1\}$ and $N_{S_k}(w_3) = \{v_n, w_4\}$ and $N_{S_k}(w_4) = \{w_3\}$. An example, of S_4, is depicted in Figure 11. We analyse the supergroup trees for S_k and identify the $(k+1)$-supergroup trees for S_k. As a consequence, we will conclude that the linear clique-width of S_k is at least $k+2$. We will also consider further graphs, that are obtained from S_k by adding a single vertex, and show that their clique-width is at least $k+2$.

Most of our results will focus on the vertices of R_k, and it will be convenient to identify the vertices of R_k by their names in a bubble model. Let $\langle b_{i,j} \rangle_{1 \leq j \leq k, 1 \leq i \leq r_j}$ be a full bubble model for R_k where $b_{1,1} = v_1$ and $b_{1,k} = v_{n-1}$ and $b_{2,k} = v_n$. As an example for such a bubble model, consider the left-side bubble model of Figure 2, that is a full bubble model for R_4. For

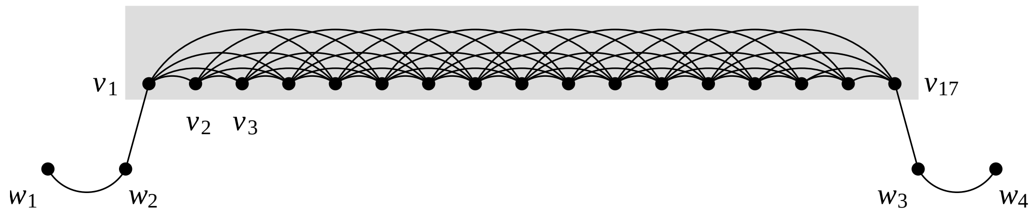

Figure 11: The graphs considered in Section 5 are obtained from k-path powers on k^2+1 vertices by attaching two paths of length 1. The figure shows the obtained graph for the case of $k = 4$. The induced subgraph in the shaded area is R_4.

$1 \leq j \leq k-1$, let $K_j =_{\text{def}} \{b_{1,j}, \ldots, b_{k+1,j}\}$ and $K_k =_{\text{def}} \{b_{1,k}, b_{2,k}\}$; these are the vertices in the columns of the bubble model. Throughout this section, we fix these and the above definitions. It will be important to observe that the reverse of Λ_k is also a k-path layout for R_k. This fact is often referred to by arguing on the "automorphic copy" of R_k or about "automorphically equivalent" cases.

The approach to determining lower clique-width bounds is by counting maximal groups and determining the size of supergroup partitions. We begin our analysis by showing a useful property about the structure of supergroups of subgraphs of S_k. This result slightly extends a similar result for path powers from [14]. Let $X, Y \subseteq V(S_k)$. We say that Y is *full in* X if $Y \subseteq X$, and we say that Y is *empty in* X if $Y \cap X = \emptyset$. We say that X *has a full maximal clique* of R_k if there is a maximal clique of R_k that is full in X, and we say that X *has an empty maximal clique* of R_k if there is a maximal clique of R_k that is empty in X. Recall that the maximal cliques of R_k are sets of $k+1$ vertices that appear consecutively in Λ_k.

Lemma 5.1 ([14]). *Let (B, C) be a partial partition of $V(S_k)$ such that neither B nor C has a full maximal clique of R_k. Let A be a supergroup of $S_k[B] \oplus S_k[C]$ containing a vertex of R_k. Then, $A \subseteq B$ or $A \subseteq C$, and A is a clique of R_k.*

Proof. We repeat and extend the corresponding proof from [14]. Let $H =_{\text{def}} S_k[B] \oplus S_k[C]$. Suppose for a contradiction that A contains two vertices v_p and v_q of R_k, where $1 \leq p < q \leq n$, that are non-adjacent in H. Due to the supergroup condition of Definition 2.2, v_p and v_q are non-adjacent also in S_k, and thus, $q - p > k$. If $1 \leq p < k+1 < n-k < q \leq n$ then $\{v_1, \ldots, v_{k+1}\} \subseteq B$ or $\{v_1, \ldots, v_{k+1}\} \subseteq C$ and $\{v_{n-k}, \ldots, v_n\} \subseteq B$ or $\{v_{n-k}, \ldots, v_n\} \subseteq C$, and B or C contains a full maximal clique of R_k. It is important to observe for this argument that v_{k+1} cannot be adjacent to v_q and v_{n-k} cannot be adjacent to v_p in S_k, which is the case, since $(n-k) - p \geq (n-k) - k > k$ and $q - (k+1) \geq (n-k+1) - (k+1) > k$. If $k+1 \leq p$ then $\{v_{p-k}, \ldots, v_p\} \subseteq B$ or $\{v_{p-k}, \ldots, v_p\} \subseteq C$, and analogously for $q \leq n-k$. So, $A \cap V(R_k)$ must be a clique of H, in particular, $A \cap V(R_k) \subseteq B$ or $A \cap V(R_k) \subseteq C$, and it remains to consider $A \cap \{w_1, w_2, w_3, w_4\}$.

Suppose that H has a supergroup $\{y, z\}$ with $y \in V(R_k)$ and $z \in \{w_1, w_2, w_3, w_4\}$. Assume that $y \in B$. Since every vertex in $N_{R_k}(y) \setminus (B \cup \{v_1, v_n\})$ s-distinguishes y and z in H, it follows that $N_{R_k}(y) \setminus \{v_1, v_n\} \subseteq B$, and B has a full maximal clique of R_k, a contradiction. Analogously for $y \in C$. ∎

The structural supergroup result of Lemma 5.1 is the starting point of our analysis. We aim at considering partial partitions without full maximal cliques of R_k, so that we precisely know the structure of the supergroups. This motivates the next definitions.

Let F be a k-path power on at least $k+1$ vertices, and let G be a graph that contains F as an induced subgraph. Let T be a supergroup tree for G. Let $\underline{a}$ be an inner node of T with $\underline{b}$ and $\underline{c}$ its children in T. We call $\underline{a}$ a *maximal F-clique split node* of T if $\Sigma_T(\underline{a})$ has a full maximal clique of F and neither $\Sigma_T(\underline{b})$ nor $\Sigma_T(\underline{c})$ has a full maximal clique of F. We can say that every maximal clique of F in $\Sigma_T(\underline{a})$ is split at $\underline{a}$, since each maximal clique of F that is full in $\Sigma_T(\underline{a})$ has a vertex in $\Sigma_T(\underline{b})$ and has a vertex in $\Sigma_T(\underline{c})$. The maximal R_k-clique split nodes of supergroup trees for S_k describe the situations that we are going to study.

We describe the central properties for maximal clique split nodes. Let T be a supergroup tree for S_k. Let $\underline{a}$ be a maximal R_k-clique split node of T with $\underline{b}$ and $\underline{c}$ its children in T. Observe that $(\Sigma_T(\underline{b}), \Sigma_T(\underline{c}))$ is a partial partition of $V(S_k)$ that satisfies the assumptions of Lemma 5.1. Let $H =_{\text{def}} S_k[\Sigma_T(\underline{b})] \oplus S_k[\Sigma_T(\underline{c})]$. Let A be a supergroup of H, and assume that A contains a vertex of R_k. Lemma 5.1 shows the following: A contains only vertices of R_k, i.e., $A \subseteq V(R_k)$, and A contains no vertex from $\{w_1, w_2, w_3, w_4\}$, and $A \subseteq \Sigma_T(\underline{b})$ or $A \subseteq \Sigma_T(\underline{c})$, and A is a clique of R_k. According to the second and third statement of Lemma 2.3, it follows that A is a group of $S_k[\Sigma_T(\underline{b})]$ or of $S_k[\Sigma_T(\underline{c})]$ and of H. So, for analysing T, which mainly means that we need to determine lower bounds on the size of supergroup partitions, we can focus on the maximal groups of H. Recall here that the maximal groups of H define a unique partition of $\Sigma_T(\underline{a})$.

The main result of this beginning provides a first specification of the situations to study. The following two results about supergroup trees and maximal groups of path powers are useful and give already a strong restriction on the cases to consider.

Lemma 5.2 ([14]). *Let G be a k-path power on at least $3k+1$ vertices. Let T be a t-supergroup tree for G with $t \geq 1$. Assume that T has a maximal G-clique split node $\underline{a}$ with $\underline{b}$ and $\underline{c}$ its children in T such that $\Sigma_T(\underline{b})$ and $\Sigma_T(\underline{c})$ have empty maximal cliques of G. Then, $t \geq k+2$.*

Lemma 5.3 ([14]). *Let $B \subseteq V(R_k)$ be such that B has no full and no empty maximal clique of R_k. For $1 \leq j \leq k-1$, let $L_j =_{\text{def}} K_j \setminus \{b_{1,j}, b_{2,j}\}$ and $M_j =_{\text{def}} K_j \setminus \{b_{1,j}\}$.*

1) If $R_k[B]$ has at most $k-1$ maximal groups then $B \subseteq L_1 \cup \cdots \cup L_{k-1}$ and $(B \cap L_1), \ldots, (B \cap L_{k-1})$ are the maximal groups of $R_k[B]$.

2) If $R_k[B] - v_n$ has at most $k-1$ maximal groups then $B \subseteq M_1 \cup \cdots \cup M_{k-1} \cup \{v_n\}$.

3) If B has a full clique of size k of R_k then $R_k[B]$ has at least k maximal groups.

Corollary 5.4. *Let T be a t-supergroup tree for S_k with $t \geq 1$. Then, $t \geq k+2$, or T has a maximal R_k-clique split node $\underline{a}$ with $\underline{b}$ and $\underline{c}$ its children in T such that $\Sigma_T(\underline{a}) \subseteq V(R_k)$ and $\Sigma_T(\underline{b})$ has no empty maximal clique of R_k and one of the following cases applies:*

1) $S_k[\Sigma_T(\underline{b})]$ has exactly k maximal groups and $|\Sigma_T(\underline{c})| = 1$

2) $\Sigma_T(\underline{b}) \subseteq \bigcup_{j=1}^{k-1} K_j \setminus \{b_{1,j}, b_{2,j}\}$, and $|\Sigma_T(\underline{c}) \cap \{b_{1,1}, \ldots, b_{1,k}\}| = 1$ and $|\Sigma_T(\underline{c}) \cap \{b_{2,1}, \ldots, b_{2,k}\}| = 1$.

Proof. Let T' be the $V(R_k)$-reduced supergroup tree of T. Due to Lemma 2.10, T' is a t-supergroup tree for R_k.

By descending from the root node of T, it is not difficult to verify that T has indeed a maximal R_k-clique split node $\underline{a}$. Let $\underline{b}$ and $\underline{c}$ be the children of $\underline{a}$ in T; let $B =_{\text{def}} \Sigma_T(\underline{b})$ and $C =_{\text{def}} \Sigma_T(\underline{c})$, and let $B' =_{\text{def}} B \cap V(R_k)$ and $C' =_{\text{def}} C \cap V(R_k)$. It is clear that B' and C' are non-empty, since B' and C' contain vertices from a maximal clique of R_k. Then, there are nodes $\underline{a'}$, $\underline{b'}$, $\underline{c'}$ of T' with $\underline{b'}$ and $\underline{c'}$ the children of $\underline{a'}$ in T' such that $\Sigma_{T'}(\underline{b'}) = B'$ and $\Sigma_{T'}(\underline{c'}) = C'$. Observe that $\underline{a'}$ is a maximal R_k-clique split node of T'. If B' and C' have empty maximal cliques of R_k then Lemma 5.2 is applicable to T', and we conclude $t \geq k+2$. As the other case, B' or C' has no empty maximal clique of R_k, and we assume that $t \leq k+1$.

Without loss of generality, we assume that B' has no empty maximal clique of R_k. This particularly implies that B has no empty maximal clique of R_k. We distinguish between three cases about the number of maximal groups of $S_k[B]$. Let $H =_{\text{def}} S_k[B] \oplus S_k[C]$. Due to Lemma 5.1, no supergroup of H with a vertex from B' contains a vertex from C, and no supergroup of H with a vertex from C' contains a vertex from B.

As the first case, assume that $S_k[B]$ has at least $k+1$ maximal groups: every supergroup partition for H has at least $k+1$ supergroups with vertices from B and at least one further supergroup with vertices from C', and thus, every supergroup partition for H has size at least $k+2$, which contradicts our assumptions about t.

As the second case, assume that $S_k[B]$ has exactly k maximal groups with vertices from B': every supergroup partition for H has at least $k+1$ supergroups with vertices from $B' \cup C'$. Since H has a supergroup partition of size at most $k+1$, we can apply Lemma 5.1, and $(B \cup C) \subseteq V(R_k)$ must hold. Furthermore, every supergroup partition for H has at most one supergroup with vertices from C, which means that C is the unique maximal group of $S_k[C]$. Due to Lemma 5.1, C is a clique of R_k, so that $C \subseteq \{v_p, \ldots, v_{p+k}\}$ for some suitable p. If $p \geq k+1$ then there is a vertex from $\{v_{p-k}, \ldots, v_{p-1}\}$ that is not in C and s-distinguishes two vertices from C, and if $p \leq k$ then $p+k \leq 2k < n-k$, and there is a vertex from $\{v_{p+k+1}, \ldots, v_{p+2k}\}$ that s-distinguishes two vertices from C, both cases yielding a contradicting to C being a supergroup of H by application of Lemma 2.5. Thus, $|C| = 1$, and the first case of the corollary applies.

As the third case, assume that $S_k[B]$ has at most $k-1$ maximal groups with vertices from B'. Then, $R_k[B']$ has at most $k-1$ maximal groups due to Corollary 2.9, and we can apply the first statement of Lemma 5.3: $B' \subseteq (K_1 \cup \cdots \cup K_{k-1}) \setminus \{b_{1,1}, b_{2,1}, \ldots, b_{1,k-1}, b_{2,k-1}\}$ and $R_k[B']$ has exactly $k-1$ maximal groups and every clique of $R_k[B']$ has size at most $k-1$. Since $B' \cup C'$ contains a maximal clique of R_k, it directly follows that C' contains at least two vertices, in particular, $|C \cap \{b_{1,1}, \ldots, b_{1,k}\}| \geq 1$ and $|C \cap \{b_{2,1}, \ldots, b_{2,k}\}| \geq 1$.

Analogous to the preceding second case, if $R_k[C']$ has exactly one maximal group, i.e., if C' is a group of $R_k[C']$, then $|C'| = 1$, which is a contradiction. Thus, $R_k[C']$ has at least two maximal groups, and it follows that $S_k[C]$ has at least two maximal groups with vertices of R_k. So, every supergroup partition for H has at least $k+1$ supergroups with vertices of R_k, and our assumptions about $t \leq k+1$ and Lemma 5.1 show that $S_k[C]$ has exactly two supergroups with vertices of R_k and $B \cup C \subseteq V(R_k)$.

Suppose for a contradiction that $|C \cap \{b_{1,1}, \ldots, b_{1,k}\}| \geq 2$ or $|C \cap \{b_{2,1}, \ldots, b_{2,k}\}| \geq 2$. So, $|C| \geq 3$, and there are $1 \leq p, q \leq k$ such that $\{b_{1,p}, b_{2,q}\}$ is a supergroup of H. Due to Lemma 5.1,

$\{b_{1,p}, b_{2,q}\}$ is a clique of S_k, so that $p-1 \leq q \leq p$. If $q = p-1$ then $b_{1,p-1}$ and $b_{2,p}$ may s-distinguish $v_{1,p}$ and $b_{2,q} = b_{2,p-1}$, so that $\{b_{1,p-1}, b_{2,p-1}, b_{1,p}, b_{2,p}\} \subseteq C$, and $\{b_{1,p-1}\}, \{b_{2,p-1}, b_{1,p}\}, \{b_{2,p}\}$ are contained in pairwise different maximal groups of $S_k[C]$, which yields a contradiction. If $q = p$ then either $p = 1$ and w_2 s-distinguishes $b_{1,p} = b_{1,1} = v_1$ and $b_{2,p} = b_{2,1} = v_2$ or $p = k$ and w_3 s-distinguishes $b_{1,k} = v_{n-1}$ and $b_{2,k} = v_n$ or $2 \leq p \leq k-1$ and $b_{2,p-1} \in C$ and $b_{1,p+1} \in C$ and $\{b_{1,p}, b_{2,p}\}, \{b_{2,p-1}\}, \{b_{1,p+1}\}$ are contained in pairwise different maximal groups of $S_k[C]$, yielding contradictions in each case. We conclude that the second case of the corollary applies. ■

The two cases of Corollary 5.4 are results of different kinds. The second case provides a strong description of the structure of $\Sigma_T(\underline{a})$ and the partial partition of $V(S_k)$ related to the maximal R_k-clique split node $\underline{a}$. Such a precise description of a situation can be used to prove lower-bound results. The first case is less descriptive, since it eludes immediate consequences about $\Sigma_T(\underline{b})$. The next subsection will deal with this case. As a particular result, we will prove a structural result about $\Sigma_T(\underline{b})$ and we will identify the vertex from $\Sigma_T(\underline{c})$.

Before we continue in the next subsection, we end this preliminary part by giving an example of the challenges and combinatorial complexity to deal with when proving lower clique-width bounds. We consider the particular case of R_3 and analyse supergroup trees for R_3 that have nodes of special properties.

Lemma 5.5. *Let T be a t-supergroup tree for R_3 with $t \geq 1$. Let $\underline{a}$ be an inner node of T with $\underline{b}$ and $\underline{c}$ its children in T.*

Assume that $\Sigma_T(\underline{a}) \subseteq \{v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and $\Sigma_T(\underline{b})$ has a full and no empty maximal clique of R_3. Then, $v_5 \in \Sigma_T(\underline{b})$ and $t \geq 5$.

Proof. We consider $\Sigma_T(\underline{b})$. Since $\Sigma_T(\underline{b})$ has a full maximal clique of R_3 and $\Sigma_T(\underline{b}) \subset \Sigma_T(\underline{a}) \subseteq \{v_2, v_3, \ldots, v_8\}$, there is a largest index d with $5 \leq d \leq 8$ such that $\{v_{d-3}, \ldots, v_d\}$ is a full maximal clique of R_3 in $\Sigma_T(\underline{b})$. Note that this particularly means $v_5 \in \Sigma_T(\underline{b})$. By the choice of d as being largest possible, $v_{d+1} \notin \Sigma_T(\underline{b})$ directly follows.

We consider five cases about $\Sigma_T(\underline{b})$, that we depict in Figure 12, by using the bubble model representation for R_3. The three left-side cases correspond to the values of d with $d = 5$, $d = 6$ and $d = 7$, and the two right-side cases correspond to $d = 8$ and whether v_4 is a vertex in $\Sigma_T(\underline{b})$. In each of the five cases, the maximal groups of $R_3[\Sigma_T(\underline{b})]$ are identified and indicated by the rectangles. It turns out that $R_3[\Sigma_T(\underline{b})]$ has five maximal groups in the second and forth case, so that every supergroup partition for $R_3[\Sigma_T(\underline{b})] \oplus R_3[\Sigma_T(\underline{c})]$ has size at least 5, and thus, $t \geq 5$ in these two cases.

We consider the other three cases. In each of these cases, $R_3[\Sigma_T(\underline{b})]$ has four maximal groups. Note about the first case that v_7 or v_8 is a vertex in $\Sigma_T(\underline{b})$, since otherwise, $\Sigma_T(\underline{b})$ would have an empty maximal clique of R_3. Let $H =_{\text{def}} R_3[\Sigma_T(\underline{b})] \oplus R_3[\Sigma_T(\underline{c})]$. We analyse the sizes of the supergroup partition for H in the three remaining cases. We suppose for a contradiction that $t \leq 4$ holds, so that H must have a supergroup partition of size at most 4. Recall that $R_3[\Sigma_T(\underline{b})]$ has four maximal groups, so that H must have a supergroup for each vertex from $\Sigma_T(\underline{c})$ that contains also a vertex from $\Sigma_T(\underline{b})$. We distinguish between two cases about the vertices from $\Sigma_T(\underline{c})$. Let x, y be a vertex pair with $x \in \Sigma_T(\underline{c})$ and $y \in \Sigma_T(\underline{b})$ such that $\{x, y\}$ is a supergroup of H.

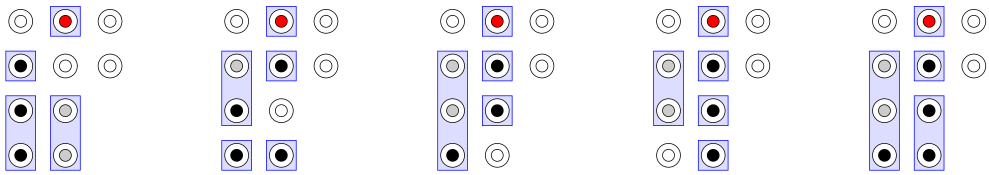

Figure 12: Depicted are the five situations, in a bubble model for R_3, about $R_3[B]$ such that $v_5 \in B$ and $v_1, v_9, v_{10} \notin B$ and B has a full and no empty maximal clique of R_3. Recall that $v_1 = b_{1,1}$ and $v_5 = b_{1,2}$ and $v_{10} = b_{2,3}$. The grey vertices may or may not be contained in B. The rectangles indicate the maximal groups in each case. These situations are analysed in Lemma 5.5.

- Assume $x \in \{v_2, v_3\}$.
  Since $v_1 \notin \Sigma_T(\underline{a})$, $y \in \{v_2, v_3, v_4\}$, and x and y are adjacent in R_3, a contradiction.
- Assume $x \in \{v_6, v_7, v_8\}$.
  Since $v_9 \notin \Sigma_T(\underline{a})$, $y \in \{v_6, v_7, v_8\}$, and x and y are adjacent in R_3, a contradiction.

We conclude that H has no supergroup partition of size at most 4, and $t \geq 5$ follows. ■

We briefly interpret the result of Lemma 5.5. Since $\Sigma_T(\underline{b})$ has a full maximal clique of R_3, $\underline{a}$ is not a maximal R_3-clique split node of T. So, if T is a 4-supergroup tree for R_3 and T has an inner node $\underline{a}$ with $\Sigma_T(\underline{a}) \subseteq \{v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and $\Sigma_T(\underline{a})$ has a full and no empty maximal clique of R_3 then $\underline{a}$ must be a maximal R_3-clique split node of T. As a final remark, observe that this latter 4-supergroup tree T for R_3 and $\underline{a}$ can only satisfy the first case of Corollary 5.4.

5.1 Structure and properties of maximal groups

We consider induced subgraphs of R_k. To be more precise, we consider sets B with $B \subseteq V(R_k)$ and the induced subgraph $R_k[B]$ and study the maximal groups of $R_k[B]$ with respect to R_k. The main contribution of this subsection will be a precise description of the maximal groups. As a consequence, we will be able to identify the vertex from $\Sigma_T(\underline{c})$ in the first case of Corollary 5.4.

We begin with an auxiliary result. Let $B, X \subseteq V(R_k)$, and assume that X is non-empty. If $B \cap X$ is non-empty then the *top vertex of* X *in* B is the vertex x from $B \cap X$ with $x = v_p$ for an appropriate index p with $1 \leq p \leq n$ such that $(B \cap X) \subseteq \{v_p, \ldots, v_n\}$. We can say that the top vertex of X in B is the vertex from $B \cap X$ with smallest index with respect to Λ_k. If $B \cap X$ is empty then X has no top vertex in B. Top vertices can be used to prove lower bounds on the number of maximal groups and to determine the structure of maximal groups. The auxiliary result about top vertices is an extention of an analogous result from [14], and we repeat and exend the proof from [14] for the sake of completeness.

Lemma 5.6 ([14])**.** *Let $B \subseteq V(R_k)$ be such that B has no full maximal clique of R_k. The top vertices of $K_1, \ldots, K_k$ in B appear in pairwise different maximal groups of $R_k[B]$.*

Proof. We apply Lemmas 2.5 and 2.3. Let $1 \leq i < j \leq k-1$, assume that $B \cap K_i$ and $B \cap K_j$ are non-empty, and we consider the top vertices of K_i and K_j in B. If $b_{1,i} \notin B$ then $b_{1,i}$

s-distinguishes the top vertices of K_i and K_j. If $b_{1,i} \in B$ then $b_{1,i}$ is the top vertex of K_i, and then, each vertex from $K_j \setminus B$ s-distinguishes $b_{1,i}$ and the top vertex of K_j. Recall that B has no full maximal clique of R_k, so that $K_j \not\subseteq B$.

We consider K_k. Let $1 \leq i \leq k-1$, and assume that $B \cap K_i$ and $B \cap K_k$ are non-empty. If $i \leq k-2$ then each vertex from $K_i \setminus B$ s-distinguishes the top vertices of K_i and K_k. If $i = k-1$ and $b_{1,k-1} \notin B$ then $b_{1,k-1}$ s-distinguishes the top vertices of K_{k-1} and K_k. Finally, if $i = k-1$ and $b_{1,k-1} \in B$ and $b_{1,k-1}$ and a vertex from $K_k \cap B$ form a group of $R_k[B]$ and $\{b_{2,k-2}, \ldots, b_{k+1,k-1}\} \subseteq B$ then B has a full maximal clique of R_k, a contradiction, so that $\{b_{2,k-2}, \ldots, b_{k+1,k-2}\} \setminus B$ is non-emtpy and contains a vertex that s-distinguishes $b_{1,k-1}$ and the vertex from $K_k \cap B$. ■

Our second result helps to describe the maximal groups of special induced subgraphs of R_k. Let $\langle d_1, \ldots, d_r \rangle$ be an index sequence with $1 \leq d_1 < \cdots < d_r \leq n$ such that the following three conditions are satisfied: (1) $d_1 \leq k$, and (2) $n-k+1 \leq d_r$, and (3) $d_i + k \leq d_{i+1} \leq d_i + k + 1$ for every $1 \leq i < r$. We call $\langle d_1, \ldots, d_r \rangle$ a *long step index sequence*. Observe that $r \leq k+1$, and if $r = k+1$ then $d_{i+1} = d_i + k$ for every $1 \leq i < r$. We call $\langle d_1, \ldots, d_r \rangle$ a *forward long step index sequence* if $d_1 = 1$, and we call $\langle d_1, \ldots, d_r \rangle$ a *backward long step index sequence* if $d_r = n$ For $B \subseteq V(R_k)$, an index sequence $\langle d_1, \ldots, d_r \rangle$ is *B-empty* if $B \cap \{v_{d_1}, \ldots, v_{d_r}\} = \emptyset$. We are interested in empty forward and backward long step index sequences.

Lemma 5.7. *Let $B \subseteq V(R_k)$. Let $\langle d_1, \ldots, d_r \rangle$ be a B-empty forward long step index sequence. For every $1 \leq i < r$, let $A_i =_{\text{def}} B \cap \{v_j : d_i \leq j \leq d_{i+1}\}$ and let $A_r =_{\text{def}} B \cap \{v_{d_r}, \ldots, v_n\}$.*

1) *Every group of $R_k[B]$ is a subset of one of $A_1, \ldots, A_r$.*

2) *Assume that $d_r = n-1$. Let $1 \leq l \leq r$. Assume that $l \leq r-2$ or $A_r = \emptyset$. Every group of $R_k[B \cup \{v_{d_l}\}]$ is a subset of one of $A_1, \ldots, A_r, \{v_{d_l}\}$.*

3) *Assume that $d_r = n$. Let $1 \leq l \leq r$.
If $1 < l < r$ then every group of $R_k[B \cup \{v_{d_l}\}]$ is a subset of one of $A_1, \ldots, A_{r-1}, \{v_{d_l}\}$.
If $l = 1$ then every group of $R_k[B \cup \{v_1\}]$ is a subset of one of $A_1 \cup \{v_1\}, A_2, \ldots, A_{r-1}$.
If $l = r$ then every group of $R_k[B \cup \{v_n\}]$ is a subset of one of $A_1, \ldots, A_{r-2}, A_{r-1} \cup \{v_n\}$.*

Proof. We prove the first statement. Observe that $A_1, \ldots, A_r$ are pairwise disjoint. Recall that $N_{R_k}(v_1) = \{v_2, \ldots, v_{k+1}\}$. Since $k+1 \leq d_2 \leq k+2$, it follows that $B \cap N_{R_k}(v_1) = A_1$. Analogously, $B \cap N_{R_k}(v_{d_i}) = A_{i-1} \cup A_i$ for every $1 < i \leq r$. Since $\{v_{d_1}, \ldots, v_{d_r}\}$ is empty in B, every group of $R_k[B]$ is a subset of $B \cap N_{R_k}(v_{d_i})$ or $B \setminus N_{R_k}(v_{d_i})$ for every $1 \leq i \leq r$. The overlap structure of $(B \cap N_{R_k}(v_{d_1})), (B \cap N_{R_k}(v_{d_2})), \ldots, (B \cap N_{R_k}(v_{d_r}))$ shows the result of the first statement.

For the proof of the second and third statement, we first consider the groups of $R_k[B \cup \{v_{d_l}\}]$ that do not contain v_{d_l}. Let u, v be a vertex pair from B, and assume that w is a vertex that s-distinguishes u and v in $R_k[B]$. If $w \neq v_{d_l}$ then w s-distinguishes u and v also in $R_k[B \cup \{v_{d_l}\}]$. So, let $w = v_{d_l}$, and we may assume $u \in N_{R_k}(v_{d_l})$ and $v \notin N_{R_k}(v_{d_l})$. Let p and q be such that $u \in A_p$ and $v \in A_q$. Note that $l-1 \leq p \leq l$ and that $1 \leq q \leq l-2$ or $l+1 \leq q \leq r$. We distinguish between three cases.

- $q \leq l - 2$
  Then, $q < p$, and v_{d_q} is adjacent to v and non-adjacent to u in R_k, and v_{d_q} s-distinguishes u and v in $R_k[B \cup \{v_{d_l}\}]$.

- $p < q - 1 < q$
  Then, $N_{R_k}(v_{d_q}) \cap B = A_{q-1} \cup A_q$, and $v \in N_{R_k}(v_{d_q})$ and $u \notin N_{R_k}(v_{d_q})$, and v_{d_q} also s-distinguishes u and v in $R_k[B \cup \{v_{d_l}\}]$.

- $l + 1 \leq q$ and $q \leq p + 1$, which means $p = l$ and $q = l + 1$
  If $q < r$ then $v_{d_{q+1}}$ s-distinguishes u and v in $R_k[B \cup \{v_{d_l}\}]$.
  If $q = r$ then $p = l = r-1$, and $u \in A_{r-1}$ and $v \in A_r$ must hold. In particular, $A_r \neq \emptyset$. This contradicts $d_r = n$ of the third statement and the assumptions of the second statement, so that $q = r$ is in fact not possible.

It follows that every group of $R_k[B \cup \{v_{d_l}\}]$ that does not contain v_{d_l} is a group of $R_k[B]$, and we conclude according to the first statement.

Let D be the maximal group of $R_k[B \cup \{v_{d_l}\}]$ containing v_{d_l}. For the second statement, observe that $d_r = n - 1$ implies $d_{i+1} = d_i + k + 1$ for every $1 \leq i < r$. It follows that v_{d_l} is the only vertex of $R_k[B \cup \{v_{d_l}\}]$ without a non-visible neighbour from $\{v_{d_1}, \ldots, v_{d_r}\} \setminus \{v_{d_l}\}$ in $R_k[B \cup \{v_{d_l}\}]$, so that $D = \{v_{d_l}\}$. For the third statement, observe that $d_r = n$ implies $d_{i+1} = d_i + k$ for every $1 \leq i < r$. If $1 < l < r$ then $\{v_{d_l}\} = N_{R_k}(v_{d_{l-1}}) \cap N_{R_k}(v_{d_{l+1}})$, and $D = \{v_{d_l}\}$. If $l = 1$ then $v_{d_2} = v_{k+1}$ is the unique non-visible neighbour of v_{d_1} from $\{v_{d_2}, \ldots, v_{d_r}\}$, and $D \subseteq A_1 \cup \{v_1\}$, and if $l = r$ then $D \subseteq A_{r-1} \cup \{v_n\}$. ∎

We will use empty long step index sequences to identify the maximal groups of induced subgraphs of R_k by applying Lemma 5.7. It remains to prove that such index sequences actually exist. And we even want to apply the second and third statement of Lemma 5.7, so that we need to prove the existence of "nice" empty long step index sequences. We do this by the two next results, that are strong results about the structure of maximal groups of special induced subgraphs of R_k.

Lemma 5.8. *Let $B \subseteq V(R_k)$ be such that B has no full and no empty maximal clique of R_k and $v_1, v_n \notin B$. If $R_k[B]$ has at most k maximal groups then there is a B-empty backward long step index sequence or $B \cap \{b_{1,j} : 1 \leq j \leq k\} = \emptyset$.*

Proof. For $1 \leq j \leq k$, let $\Delta_j =_{\text{def}} \{b_{1,j}, b_{2,j}\}$. If $B \cap (\Delta_2 \cup \cdots \cup \Delta_k) = \emptyset$ then the claim of the lemma directly follows. Recall here that $v_1 \notin B$ by the assumptions of the lemma, and $b_{1,1} = v_1$. Otherwise, there is a largest index t with $2 \leq t \leq k$ such that $B \cap \Delta_t \neq \emptyset$.

Let Φ be the set of the top vertices of $K_1, \ldots, K_k$ in B. Since B has no empty maximal clique of R_k, $(B \cap K_1), \ldots, (B \cap K_{k-1})$ are non-empty, and thus, $k - 1 \leq |\Phi| \leq k$, depending on whether $(B \cap K_k)$ is empty or not. Due to Lemma 5.6, the vertices in Φ appear in pairwise different maximal groups of $R_k[B]$. If $|\Phi| = k$ then each maximal group of $R_k[B]$ contains a vertex from Φ, and if $|\Phi| = k - 1$ then there is at most one maximal group of $R_k[B]$ without a vertex from Φ. We will use the vertices from Φ to identify the maximal groups of $R_k[B]$.

We are going to prove the lemma by induction. First, we prove two claims, for the induction base.

Claim 1. *One of the three cases applies:*

1) $B \cap \{b_{1,1}, \ldots, b_{1,k}\} = \emptyset$

2) $t = k$, *and there are* c_{k-1}, c_k *with* $c_k = 2$ *and* $2 \leq c_{k-1} \leq 3$ *such that* $B \cap \{b_{c_{k-1},k-1}, b_{c_k,k}\} = \emptyset$

3) $t < k$, *and there are* $c_t, \ldots, c_k$ *with* $c_t = \cdots = c_k = 2$ *such that* $B \cap \{b_{c_t,t}, \ldots, b_{c_k,k}\} = \emptyset$ *and* $(B \cap K_t) \setminus \Delta_t$ *is the unique maximal group of* $R_k[B]$ *without a vertex from* Φ.

Proof of claim. We distinguish between two cases. For the first case, we assume that $b_{1,k} \in B$. Note that this means $t = k$, and observe that $|\Phi| = k$. So, every maximal group of $R_k[B]$ contains a vertex from Φ. Suppose for a contradiction that $\{b_{2,k-1}, b_{3,k-1}\} \subseteq B$. Note that $b_{2,k}$ s-distinguishes $b_{2,k-1}$ and $b_{3,k-1}$. Let A be the maximal group of $R_k[B]$ containing $b_{3,k-1}$. Since $b_{2,k}$ is a non-visible neighbour of $b_{3,k-1}$ in $R_k[B]$, it directly follows that $A \subseteq N_{R_k}(b_{2,k})$. Since A must contain a vertex from Φ, this vertex can only be $b_{1,k}$. Therefore, $\{b_{3,k-1}, b_{1,k}\}$ is a group of $R_k[B]$. Thus, $\{b_{4,k-2}, \ldots, b_{k+1,k-2}, b_{1,k-1}, b_{2,k-1}\} \subseteq B$, so that B has a full maximal clique of R_k, a contradiction. Thus, c_{k-1} and c_k exist so that the second case applies.

Now, we assume that $b_{1,k} \notin B$. Note that this means $B \cap \Delta_k = \emptyset$, and so, $t < k$. Recall that $|\Phi| = k-1$. Also note that $B \cap (\Delta_{t+1} \cup \cdots \cup \Delta_k) = \emptyset$ implies that $B \cap K_j$ for every $t < j < k$ is the union of maximal groups of $R_k[B]$. If $B \cap \{b_{3,t}, \ldots, b_{k+1,t}\} = \emptyset$ then $\{b_{3,t}, \ldots, b_{k+1,t}, b_{1,t+1}, b_{2,t+1}\}$ is an empty maximal clique of R_k in B, which does not exist by our assumptions. Thus, $(B \cap K_t) \setminus \Delta_t \neq \emptyset$. According to the choice of t, $B \cap \Delta_t \neq \emptyset$, and thus, the top vertex of K_t in B is from Δ_t. And since $b_{2,t+1} \notin B$, it follows that $(B \cap K_t) \setminus \Delta_t$ is the union of maximal groups of $R_k[B]$, and none of these maximal groups contains a vertex from Φ, so that $(B \cap K_t) \setminus \Delta_t$ must in fact be the unique maximal group of $R_k[B]$ without a vertex from Φ. It directly follows that $(B \cap K_{t+1}), \ldots, (B \cap K_{k-1})$ are maximal groups of $R_k[B]$. If $b_{2,t} \notin B$ then we choose $c_t =_{\text{def}} \cdots =_{\text{def}} c_k =_{\text{def}} 2$, and the third case applies.

As the other case, we assume that $b_{2,t} \in B$. If $b_{1,t} \in B$ then $b_{1,t}$ is the top vertex of K_t in B, and since $B \cap \Delta_{t+1} = \emptyset$, $\{b_{2,t}\}$ would be a maximal group of $R_k[B]$ without a vertex from Φ, a contradiction. Thus, $b_{1,t} \notin B$, and $\{b_{2,t}\}$ is a maximal group of $R_k[B]$. If $t = 2$ then we can conclude that the first case of the claim applies. Otherwise, $t \geq 3$. Let A be a maximal group of $R_k[B]$ containing a vertex from $\{b_{2,t-1}, \ldots, b_{k+1,t-1}\}$. Recall that $B \cap \{b_{2,t-1}, \ldots, b_{k+1,t-1}\} = \emptyset$ would mean that $\{b_{2,t-1}, \ldots, b_{k+1,t-1}, b_{1,t}\}$ is an empty maximal clique of R_k in B. Observe that $b_{1,t}$ is a non-visible neighbour of some vertex in A, so in fact of all vertices in A, so that $A \subseteq N_{R_k}(b_{1,t})$, and therefore, $A \subseteq B \cap \{b_{2,t-1}, \ldots, b_{k+1,t-1}\}$. Also recall that A must contain a vertex from Φ, which will be the top vertex of K_{t-1} in B, and this means $b_{1,t-1} \notin B$. So, $A = B \cap K_{t-1}$. We can repeatedly apply this argument and see that $b_{1,t} \notin B$ implies $b_{1,1}, \ldots, b_{1,t} \notin B$, so that the first case of the claim applies. □

Claim 1 proves our induction base. If the first case of the claim applies then we can already conclude the lemma. So, we assume that the first case does not apply, and therefore, the second or third case of the claim applies. For the induction step, we assume that at least two indices of a suitable index sequence have already been determined, and we show that the next index in the sequence can be found.

Claim 2. *Let j be with $2 \leq j \leq k-1$ and $j \leq t$.*
Let c and c' be with $2 \leq c' \leq c \leq k$ and $c-1 \leq c'$. Assume that $j = t$ implies $c' = 2$. Also assume that $b_{c,j}, b_{c',j+1} \notin B$. Then, $\{b_{c,j-1}, b_{c+1,j-1}\} \not\subseteq B$.

Proof of claim. Suppose for a contradiction that $\{b_{c,j-1}, b_{c+1,j-1}\} \subseteq B$. Let A be the maximal group of $R_k[B]$ containing $b_{c+1,j-1}$. If $|\Phi| = k$ then A contains a vertex from Φ, and if $|\Phi| = k-1$ then A also contains a vertex from Φ, since $A \neq (B \cap K_t) \setminus \Delta_t$. Since $b_{c,j}$ is a non-visible neighbour of $b_{c+1,j-1}$ in $R_k[B]$, it follows that $A \subseteq N_{R_k}(b_{c,j})$. Thus, A contains the top vertex of K_j in B. If $b_{1,j-1} \notin B$ then $b_{1,j-1}$ s-distinguishes $b_{c+1,j-1}$ and the top vertex of K_j, a contradiction, so that $b_{1,j-1} \in B$ must hold. Since $b_{1,1} = v_1$ and $v_1 \notin B$, this implies $j-1 \geq 2$ and $k \geq 4$, so that Claim 2 is already proved for the case of $k = 3$. Note that $b_{1,j-1}$ is the top vertex of K_{j-1} in B.

If $b_{c-1,j} \notin B$ then $b_{c-1,j}, b_{c,j} \notin B$ and $\{b_{c,j-1}\}$ is a maximal group of $R_k[B]$, that contains no vertex from Φ, which is not possible, so that $b_{c-1,j} \in B$ must hold. Let A' be the maximal group of $R_k[B]$ containing $b_{c-1,j}$. Note that $A' \neq (B \cap K_t) \setminus \Delta_t$, which is obvious for $j < t$, and if $j = t$ then $c - 1 \leq 2$ and therefore $b_{c-1,j} \in \Delta_t$. Thus, A' contains a vertex from Φ.

Observe that $A' \subseteq N_{R_k}(b_{c,j})$, and since $b_{c-1,j}$ and $b_{c',j+1}$ are non-adjacent in R_k, A' cannot contain neighbours of $b_{c',j+1}$, and thus, $A' \subseteq K_{j-1} \cup K_j$. Furthermore, $b_{1,j-1}$ is the top vertex of K_{j-1} in B, and we conclude that A' contains the top vertex of K_j in B, and since also A contains the top vertex of K_j in B, we obtain $A = A'$. In particular, $\{b_{c+1,j-1}, b_{c-1,j}\}$ is a group of $R_k[B]$, and since $j \geq 3$,

$$\{b_{c+2,j-2}, \ldots, b_{k+1,j-2}\} \cup \{b_{1,j-1}, \ldots, b_{c-1,j-1}\} \cup \{b_{c,j-1}, b_{c+1,j-1}\} \ \subseteq \ B\,,$$

and B has a full maximal clique of R_k, a contradiction. □

So, by repeatedly applying Claim 2 to already defined c_{k-1}, c_k or $c_t, \ldots, c_k$ from Claim 1, we conclude that there are $c_k, \ldots, c_1$ with $2 \leq c_k \leq \cdots \leq c_1 \leq k+1$ such that $B \cap \{b_{c_1,1}, \ldots, b_{c_k,k}\} = \emptyset$. It remains to observe that there is a backward long step index sequence $\langle d_1, \ldots, d_k\rangle$ or $\langle d_0, d_1, \ldots, d_k\rangle$ with $v_{d_i} = b_{c_i,i}$ for every $1 \leq i \leq k$ and $d_0 = 1$. ■

Corollary 5.9. *Let $B \subseteq V(R_k)$ be such that B has no full and no empty maximal clique of R_k and B has a full clique of size k of R_k and $v_1, v_n \notin B$ and $R_k[B]$ has at most k maximal groups. Then, one of the following three cases applies:*

1) $B \cap \{b_{1,j} : 1 \leq j \leq k\} = \emptyset$

2) $B \cap \{b_{2,j} : 1 \leq j \leq k\} = \emptyset$

3) $B \cap \{v_{i \cdot k+1} : 0 \leq i \leq k\} = \emptyset$.

Proof. For $0 \leq i \leq k$, let $c_i =_{\text{def}} ik + 1$. If $\langle c_0, \ldots, c_k\rangle$ is a B-empty forward long step index sequence then the third case applies. For a contradiction, we suppose that none of the three cases applies. This particularly means for $\langle c_0, \ldots, c_k\rangle$ that there are a smallest and largest index p and q so that $v_{c_p} \in B$ and $v_{c_q} \in B$, and clearly, $0 < p \leq q < k$. We will obtain the contradiction by showing that B cannot have a full clique of size k of R_k under these assumptions.

We apply Lemma 5.8 to $R_k[B]$ and the automorphic copy of $R_k[B]$: since the first and second case of the corollary do not apply by our assumptions, there are B-empty forward and backward

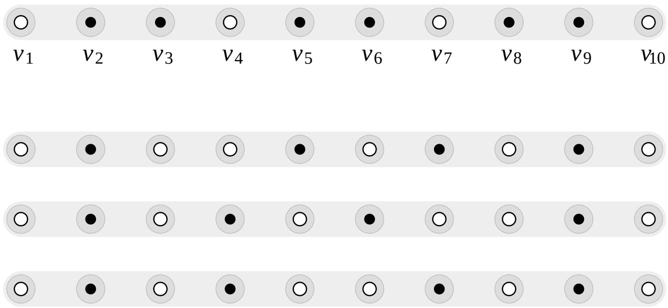

Figure 13: The four figures illustrate the result of Corollary 5.9 and the proof of it, when applied to $R_3[B]$. On the top, we consider $B = \{v_2, v_3, v_5, v_6, v_8, v_9\}$, and $\langle 1, 4, 7, 10\rangle$ is a B-empty forward long step index sequence. The three below situations consider sets B for $B \subseteq \{v_2, \ldots, v_9\}$ where $v_4 \notin B$ or $v_7 \notin B$. In these three situations, none of the three cases of Corollary 5.9 applies, and B does not have a full clique of size 3 of R_k.

long step index sequences $\langle e_0, \ldots, e_{k-1}\rangle$ and $\langle d_1, \ldots, d_k\rangle$, respectively. Note that $e_{k-1} < n$ and $d_1 > 1$. Also note that $c_i \leq e_i$ for every $0 \leq i \leq k-1$ and $d_i \leq c_i$ for every $1 \leq i \leq k$. And since $v_{c_p} \in B$ and $v_{c_q} \in B$, it follows that $c_p < e_p$ and $d_q < c_q$, and therefore $c_i < e_i$ for every $p \leq i \leq k-1$ and $d_i < c_i$ for every $1 \leq i \leq q$. Thus, $d_i < e_i$ for every $1 \leq i \leq k-1$. We extend this inequality to $e_{i-1} < d_i < e_i < d_{i+1}$ for every $1 \leq i \leq k-1$, by observing for $1 \leq i \leq k$:

$$c_{i-1} \ \leq\ e_{i-1} \ \leq\ c_{i-1} + (i-1) \quad \text{and} \quad c_{i-1} + i \ =\ c_i - (k-i) \ \leq\ d_i \ \leq\ c_i\,,$$

so that indeed $e_{i-1} < d_i$.

We prove the contradiction. Let M be a maximal clique of R_k. Then, there is an index a with $1 \leq a \leq n-k$ such that $M = \{v_i : a \leq i \leq a+k\}$. Recall that the vertices of maximal cliques of R_k appear consecutively in Λ_k. Since $e_0 = 1$ and $d_k = n$, there is an index s with $0 \leq s \leq k-2$ such that $e_s \leq a < d_{s+1}$ or $d_{s+1} \leq a < e_{s+1}$. Since $e_{k-1} \geq n-k+1$, $a < e_{k-1}$ must hold. If $e_s \leq a < d_{s+1}$ then, with $d_{s+1} < e_{s+1} \leq e_s + (k+1)$, it follows that $\{v_{e_s}, v_{d_{s+1}}\} \subseteq M$ or $\{v_{d_{s+1}}, v_{e_{s+1}}\} \subseteq M$, and thus, $|M \cap \{v_{d_1}, \ldots, v_{d_k}, v_{e_0}, \ldots, v_{e_{k-1}}\}| \geq 2$. If $d_{s+1} \leq a < e_{s+1}$ then $\{v_{d_{s+1}}, v_{e_{s+1}}\} \subseteq M$ or $\{v_{e_{s+1}}, v_{d_{s+2}}\} \subseteq M$, and $|M \cap \{v_{d_1}, \ldots, v_{d_k}, v_{e_0}, \ldots, v_{e_{k-1}}\}| \geq 2$. It directly follows with the assumption about B that $|M \cap B| \leq k-1$, and thus, B has no full clique of size k of R_k that is a subset of M. We conclude that B has no full clique of size k of R_k, the desired contradiction. ∎

The result of Corollary 5.9 is our major technical tool, that provides a good description of the maximal groups. To illustrate the deduction of the contradiction in the proof of Corollary 5.9, consider the four situations of Figure 13 for the smallest case of $R_3[B]$. The top situation illustrates the case when $\langle 1, 4, 7, 10\rangle$ is a B-empty forward long step index sequence. The three other cases illustrate the possible situations when $\langle 1, 4, 7, 10\rangle$ and $\langle 1, 5, 9\rangle$ and $\langle 2, 6, 10\rangle$ are not B-empty long step index sequences, the situations that are proved to imply the contradiction. For the three situations, it is easy to check that B does not have a full clique of size 3 of R_k.

Figure 14: The two figures illustrate the situations in S_5 when the bubbles corresponding to the vertices in respectively Ψ and Ψ' are empty. It is important to note that the two situations are automorphically equivalent.

5.2 First part of analysis of supergroup trees

We are going to analyse supergroup trees and classify them according to their maximal R_k-clique split nodes. In this subsection, we consider supergroup trees whose maximal R_k-clique split nodes are in accordance with the first or second case of Corollary 5.9. We define two sets of vertices:

$$\begin{aligned} \Psi &=_{\text{def}} \{b_{1,j} : 1 \leq j \leq k\} \cup \{w_3\} \\ \Psi' &=_{\text{def}} \{b_{2,j} : 1 \leq j \leq k\} \cup \{w_2\}\,. \end{aligned}$$

We are going to study partial partitions (B, C) of $V(S_k)$ such that $B \cap \Psi = \emptyset$ or $B \cap \Psi' = \emptyset$ and corresponding supergroup trees. For an illustration of the two situations, consider the two bubble models of Figure 14: the empty bubbles represent the vertices from respectively Ψ and Ψ', so that B will be a set of the remaining vertices. Since Ψ and Ψ' are automorphically equivalent, the cases of $B \cap \Psi = \emptyset$ and $B \cap \Psi' = \emptyset$ are automorphically equivalent, and it suffices to consider only one of the two situations in fact. We will therefore concentrate on Ψ. The main result of this section is a classification of the $(k+1)$-supergroup trees for S_k that have a maximal R_k-clique split node corresponding to Ψ.

We analyse the supergroup trees mainly by determining lower bounds on the size of supergroup partitions. For determining such lower bounds, we will study partial partitions (B, C) of $V(S_k)$ so that $S_k[B]$ has at least $k+1$ maximal groups. Since no supergroup of $S_k[B] \oplus S_k[C]$ can contain vertices from different maximal groups of $S_k[B]$ due to Lemma 2.3, the size of every supergroup partition for $S_k[B] \oplus S_k[C]$ is at least the number of maximal groups of $S_k[B]$. So, if $S_k[B]$ has $k+1$ maximal groups and $S_k[B] \oplus S_k[C]$ has a supergroup partition of size at most $k+1$ then every vertex from C must be in a supergroup with a vertex from B. In our analyses, we will use this observation to capture and catch the situations. Let y be a vertex from C. For z a vertex from B, we call $\{y, z\}$ a *y-cac supergroup* of $S_k[B] \oplus S_k[C]$ if $\{y, z\}$ is a supergroup of $S_k[B] \oplus S_k[C]$ and $z \in V(R_k)$.

Lemma 5.10. *Let (B, C) be a partial partition of $V(S_k)$, and assume that B has no empty maximal clique of R_k. Let $H =_{\text{def}} S_k[B] \oplus S_k[C]$. Let y be a vertex from $C \cap V(R_k)$. If H has a y-cac supergroup then $y \in \{v_1, \ldots, v_k\}$ or $y \in \{v_{n-k+1}, \ldots, v_n\}$.*

Proof. Let z be a vertex from $B \cap V(R_k)$ and assume that $\{y, z\}$ is a supergroup of H. Let p, q be the indices with $1 \leq p, q \leq n$ such that $y = v_p$ and $z = v_q$. Since y and z must be

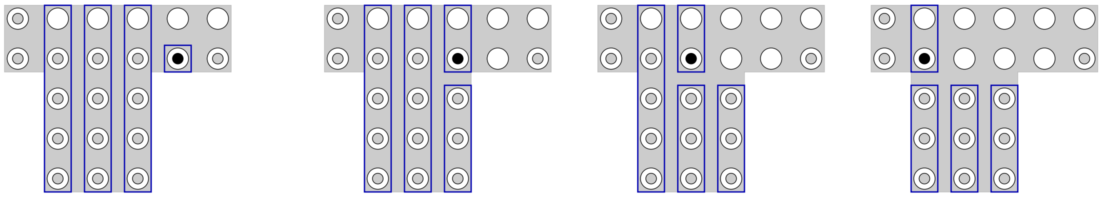

Figure 15: The four bubble models for S_4 represent the analysed situations. The analysed situations rely on properties about $S_4[B]$ for $B \subseteq V(S_4)$, and it is distinguished between $b_{2,k} \in B$, which is the case in the left-side figure, and $b_{2,k} \notin B$, which is the case in the three right-side figures. In all figures, an empty bubble for vertex x means $x \notin B$, if vertex x is black then $x \in B$, and if vertex x is grey then $x \in B$ or $x \notin B$. The rectangles indicate unions of maximal groups of $R_4[B \cap V(R_4)]$.

non-adjacent in S_k, it holds that $|p-q| \geq k+1$. Assume that $p < q$. If $k < p < p+k < q$ then each vertex from $\{v_{p-k}, \ldots, v_{p-1}\}$ may s-distinguish y and z in H, so that $\{v_{p-k}, \ldots, v_{p-1}\} \subseteq C$ must hold. However, $\{v_{p-k}, \ldots, v_p\}$ is a maximal clique of R_k, so that C has a full maximal clique of R_k, and therefore, B has an empty maximal clique of R_k, a contradiction. Thus, $p \leq k$, which means that $y \in \{v_1, \ldots, v_k\}$.

The other case, when $q < p$, analogously implies $y \in \{v_{n-k+1}, \ldots, v_n\}$. ■

We will apply Lemma 5.10 to learn about the vertices that are contained in C. Let (B, C) be a partial partition of $V(S_k)$ that satisfies the assumptions of the lemma. Let $H =_{\text{def}} S_k[B] \oplus S_k[C]$. We will apply the result only in case that every vertex from C has a cac supergroup in H. Recall that $\{v_1, \ldots, v_k\} = \{b_{1,1}, \ldots, b_{k,1}\}$ and $\{v_{n-k+1}, \ldots, v_n\} = \{b_{4,k-1}, \ldots, b_{2,k}\}$. It follows that C has this form: $C \subseteq \{b_{1,1}, \ldots, b_{k,1}\} \cup \{b_{4,k-1}, \ldots, b_{2,k}\} \cup \{w_1, w_2, w_3, w_4\}$. As a direct consequence for our coming proofs, it suffices to consider only very special vertices to obtain desired results.

We are going to analyse the supergroup trees for S_k that correspond to Ψ. We consider partial partitions of very restricted form and determine the sizes of compatible supergroup partitions. Before we begin with the formal analysis, we briefly describe the analysed situations with an example about S_4. Consider the bubble models for S_4 in Figure 15. The four figures describe different sets B, subsets of $V(S_4)$, where $B \cap \Psi = \emptyset$. The left-side figure represents sets B that contain vertex $b_{2,k}$. By the result of Lemma 5.7, the groups of $S_4[B]$ containing only vertices of R_4 respect the columns. The three other figures represent the possible situations when $b_{2,k}$ is not a vertex from B. We analyse these situations in Lemma 5.12, for the case when $b_{2,k} \notin B$, and Lemma 5.13, for the case when $b_{2,k} \in B$. The main consequence of these results about Ψ will be given in Corollary 5.14.

We begin with an auxiliary result about $b_{1,1}$-cac supergroups.

Lemma 5.11. *Let (B, C) be a partial partition of $V(S_k)$, and assume that B has no empty maximal clique of R_k and $|B \cap \Psi \cap V(R_k)| = 1$ and $b_{1,1} \in C$. Let $H =_{\text{def}} S_k[B] \oplus S_k[C]$. Assume that H has an x-cac supergroup for every vertex x from C. Then, $w_1 \in C$, and $\{b_{1,1}, b_{1,2}\}$ and $\{w_1, b_{1,2}\}$ are the unique $b_{1,1}$-cac and w_1-cac supergroups of H.*

Proof. Since B has no empty maximal clique of R_k, $B \cap K_1$ is non-empty, and the vertices in $B \cap K_1$ are non-visible neighbours of $b_{1,1}$ in H. Let $\{b_{1,1}, z\}$ be a $b_{1,1}$-cac supergroup of H.

Then, $z \in K_1 \cup K_2$, and since $b_{1,1}$ and z must be non-adjacent in S_k, $z \in K_2$ follows. If $b_{1,2} \notin B$ then $b_{1,2}$ s-distinguishes $b_{1,1}$ and z, and H has no $b_{1,1}$-cac supergroup, so that $b_{1,2} \in B$ must hold. Recall that $b_{1,2} \in \Psi$, so that $|B \cap \Psi \cap V(R_k)| = 1$ by the assumptions of the lemma implies $B \cap \Psi \cap V(R_k) = \{b_{1,2}\}$, and it follows that $b_{1,3} \notin B$, and thus, $b_{1,3}$ s-distinguishes $b_{1,1}$ and the vertices from $\{b_{2,2}, \ldots, b_{k+1,2}\}$, so that $z = b_{1,2}$ must hold. We conclude that $\{b_{1,1}, b_{1,2}\}$ is the only $b_{1,1}$-cac supergroup of H.

Next, observe that $w_2 \notin C$ means that w_2 s-distinguishes $b_{1,1}$ and $b_{1,2}$ in H, so that $w_2 \in C$. If $w_1 \notin C$ then w_1 s-distinguishes w_2 and each vertex from B, and H cannot have a w_2-cac supergroup, a contradiction, so that $w_1 \in C$. Let $\{w_1, z'\}$ be a w_1-cac supergroup of H. Since $B \cap \Psi \cap V(R_k) = \{b_{1,2}\}$, it follows that each vertex from $(B \setminus \{b_{1,2}\}) \cap V(R_k)$ has a non-visible neighbour from $V(R_k)$ in H, so that $z' \notin (B \setminus \{b_{1,2}\})$, and thus, $z' = b_{1,2}$ must hold, and we conclude that $\{w_1, b_{1,2}\}$ is the only w_1-cac supergroup of H. ∎

Let (B, C) be a partial partition of $V(S_k)$, and let $H =_{\text{def}} S_k[B] \oplus S_k[C]$. Assume that $S_k[B]$ has at least $k+1$ maximal groups. Then, every supergroup partition for H has size more than $k+1$ or every maximal group of $S_k[B]$ is contained in exactly one supergroup of the partition. In the latter case, we can say that the supergroups in the supergroup partition do not "split" the maximal groups of $S_k[B]$. We use this observation for the following notion, that we slightly extend to sets of vertices of H. Let $A \subseteq B \cup C$. We say that A is a *non-splittable supergroup of* H if for every supergroup partition $\{A_1, \ldots, A_r\}$ for H of size at most $k+1$, there is $1 \leq i \leq r$ such that $A \subseteq A_i$. Of course, if $S_k[B]$ has at least $k+1$ maximal groups and H has a supergroup partition of size at most $k+1$ then $S_k[B]$ has exactly $k+1$ maximal groups and each group of $S_k[B]$ is a non-splittable supergroup of H. The non-splittable supergroup notion is particularly interesting for supergroups containing vertices from B and from C.

As the first case, we consider sets that do not contain $b_{2,k}$. We show for a t-supergroup tree for S_k that $t \geq k+2$ or T has a node satisfying a very special condition, that we call the *limit*. Let (B, C) be a partial partition of $V(S_k)$. We say that (B, C) satisfies the *limit condition* if the following is satisfied:

- $R_k[B \cap V(R_k)]$ has exactly $k+1$ maximal groups
- $B \cap \Psi = \{b_{1,k-1}\}$ and $\{b_{2,k}, w_3, w_4\} \subseteq C$, and
  if $k = 3$ then $B \setminus \{w_1, w_2\} = \{b_{2,1}, b_{3,1}, b_{4,1}, b_{1,2}, b_{2,2}, b_{3,2}, b_{4,2}\}$
- $\{b_{k-1,1}, b_{k+1,1}\}$ and $\{b_{2,k-1}, b_{2,k}\}$ are non-splittable supergroups of $S_k[B] \oplus S_k[C]$.

We will later see that the limit condition precisely describes the $(k+1)$-supergroup trees for S_k.

Lemma 5.12. *Let h be an integer with $1 \leq h \leq k-1$ and let $\Phi =_{\text{def}} \{b_{2,j} : h < j \leq k\}$. Let $B \subseteq V(S_k)$, and assume that $B \cap (\Psi \cup \Phi) = \emptyset$ and $b_{2,h} \in B$ and B has no empty maximal clique of R_k. Let $x \in \Psi \cap V(R_k)$.*

Let T be a t-supergroup tree for S_k with $t \geq 1$. Assume that T has an inner node $\underline{a}$ with $\underline{b}$ and $\underline{c}$ its children in T such that $\Sigma_T(\underline{b}) = B \cup \{x\}$ and $\Sigma_T(\underline{c}) \cap (\Psi \cup \Phi) \neq \emptyset$. Then, $t \geq k+2$, or T is not a supergroup caterpillar tree and $(\Sigma_T(\underline{b}), \Sigma_T(\underline{c}))$ satisfies the limit condition.

Proof. Let $B' =_{\text{def}} B \cap V(R_k)$. We determine the maximal groups of $R_k[B' \cup \{x\}]$ and $S_k[B \cup \{x\}]$. For an illustration, simultaneously consider the three right-side figures of Figure 15.

Observe that $b_{1,k} \in \Psi$ and $b_{2,k} \in \Phi$ and therefore $b_{1,k}, b_{2,k} \notin B$. Let $A_j =_{\text{def}} B \cap K_j$ for every $1 \le j \le k-1$. Clearly, $B' = A_1 \cup \cdots \cup A_{k-1}$. Observe that the second statement of Lemma 5.7 is applicable to $R_k[B' \cup \{x\}]$, so that $A_1, \ldots, A_{k-1}, \{x\}$ are unions of maximal groups of $R_k[B' \cup \{x\}]$. And since $b_{2,h+1} \notin (B \cup \{x\})$, no group of $R_k[B' \cup \{x\}]$ contains vertices from $A_h \cap \{b_{1,h}, b_{2,h}\}$ and $A_h \cap \{b_{3,h}, \ldots, b_{k+1,h}\}$. Recall that $A_h \cap \{b_{1,h}, b_{2,h}\} = \{b_{2,h}\}$, and since B has no empty maximal clique of R_k and $b_{1,h+1}, b_{2,h+1} \notin B$, it follows that $A'_h =_{\text{def}} A_h \cap \{b_{3,h}, \ldots, b_{k+1,h}\}$ is non-empty. Thus, each group of $R_k[B' \cup \{x\}]$ is a subset of one of $A_1, \ldots, A_{h-1}, A'_h, A_{h+1}, \ldots, A_{k-1}, \{b_{2,h}\}, \{x\}$, and all these are non-empty sets. It follows that $R_k[B' \cup \{x\}]$ has at least $k+1$ maximal groups, and so, $S_k[B \cup \{x\}]$ has at least $k+1$ maximal groups due to Corollary 2.9.

Let $H =_{\text{def}} S_k[\Sigma_T(\underline{b})] \oplus S_k[\Sigma_T(\underline{c})]$. If every supergroup partition for H has size at least $k+2$ then $t \ge k+2$. Otherwise, H has a supergroup partition of size at most $k+1$. This particularly means that $S_k[\Sigma_T(\underline{b})]$ has exactly $k+1$ maximal groups, which also implies that $R_k[\Sigma_T(\underline{b}) \cap V(R_k)]$ has exactly $k+1$ maximal groups. Furthermore, each maximal group of $S_k[\Sigma_T(\underline{b})]$ contains one of $A_1, \ldots, A_{h-1}, A'_h, A_{h+1}, \ldots, A_{k-1}, \{b_{2,h}\}, \{x\}$ as a subset. It follows that H has a v-cac supergroup for each vertex v from $\Sigma_T(\underline{c})$, in particular, for each vertex v from $\Sigma_T(\underline{c}) \cap (\Psi \cup \Phi)$. We consider four cases and show a contradiction in each case or that the limit condition is satisfied.

As the first case, assume that $b_{1,1} \in \Sigma_T(\underline{c})$. Observe that Lemma 5.11 is applicable, and $w_1, b_{1,1} \in \Sigma_T(\underline{c})$ and $\{b_{1,1}, b_{1,2}\}$ and $\{w_1, b_{1,2}\}$ are the unique $b_{1,1}$-cac and w_1-cac supergroups of H. Since H has a supergroup partition $\mathcal{A}$ of size $k+1$, the supergroup in $\mathcal{A}$ containing $b_{1,2}$ must contain $b_{1,1}$ and w_1, so that $\{w_1, b_{1,1}, b_{1,2}\}$ is a supergroup of H. However, the vertices from A_1 s-distinguish w_1 and $b_{1,1}$ in H, a contradiction.

As the second case, assume that $b_{1,k} \in \Sigma_T(\underline{c})$ and $b_{2,k} \notin \Sigma_T(\underline{c})$. Let $\{b_{1,k}, z\}$ be a $b_{1,k}$-cac supergroup of H. Since $b_{2,k}$ is a non-visible neighbour of $b_{1,k}$, $z \in \{b_{3,k-1}, \ldots, b_{k+1,k-1}\}$, and then, $b_{1,k}$ and z are adjacent in S_k, a contradiction.

As the third case, assume that $w_3 \in \Sigma_T(\underline{c})$ and $b_{2,k} \notin \Sigma_T(\underline{c})$. If $w_4 \notin \Sigma_T(\underline{c})$ then w_4 s-distinguishes w_3 and each vertex from $B' \cup \{x\}$, and H cannot have a w_3-cac supergroup, so that $w_4 \in \Sigma_T(\underline{c})$. Let $\{w_3, z\}$ be a w_3-cac supergroup of H. Since $b_{2,k}$ is a non-visible neighbour of w_3 in H, $z \in \{b_{3,k-1}, \ldots, b_{k+1,k-1}, b_{1,k}\}$. If $b_{1,k} \notin B' \cup \{x\}$ then $b_{1,k}$ s-distinguishes w_3 and z, a contradiction, so that $b_{1,k} \in \Sigma_T(\underline{b})$ must hold, and therefore, $x = b_{1,k}$ and $b_{1,k-1} \notin \Sigma_T(\underline{b})$, and thus, $b_{1,k-1}$ s-distinguishes w_3 and each vertex from A_{k-1}, so that $z = b_{1,k}$. This means that $\{b_{1,k}, w_3\}$ is the only w_3-cac supergroup of H. Observe that each vertex from $A_1 \cup \cdots \cup A_{k-1}$ has a non-visible neighbour in H from $V(R_k)$. Since H has a w_4-cac supergroup, it follows that $\{b_{1,k}, w_4\}$ must be the only w_4-cac supergroup of H. Then, $\{b_{1,k}, w_3, w_4\}$ must be a supergroup of H, however, $b_{2,k}$ s-distinguishes w_3 and w_4 in H, a contradiction.

As the fourth case, assume that $b_{2,k} \in \Sigma_T(\underline{c})$. We show that $(\Sigma_T(\underline{b}), \Sigma_T(\underline{c}))$ indeed satisfies the limit condition, by verifying the satisfaction of the three items. The proof relies on the existence of cac supergroups.

- $\{b_{2,k}, w_3, w_4\} \subseteq \Sigma_T(\underline{c})$

  Analogous to the third case, if $w_3 \notin \Sigma_T(\underline{c})$ then H has no $b_{2,k}$-cac supergroup, so that $w_3 \in \Sigma_T(\underline{c})$, and if $w_4 \notin \Sigma_T(\underline{c})$ then H has no w_3-cac supergroup, so that $w_4 \in \Sigma_T(\underline{c})$.

- $\Sigma_T(\underline{b}) \cap \Psi = \{b_{1,k-1}\}$

  Recall that $\Sigma_T(\underline{b}) \cap \Psi = \{x\}$. If $x = b_{1,k}$ then each vertex from $A_1 \cup \cdots \cup A_{k-1}$ has a non-visible neighbour from $V(R_k)$ in H, and since $b_{2,k} \in \Sigma_T(\underline{c})$, $b_{2,k}$ is a non-visible neighbour of x in H, so that each vertex from $\Sigma_T(\underline{b})$ has a non-visible neighbour from $V(R_k)$ in H, and H cannot have a w_4-cac supergroup, a contradiction.

  If $x \in \{b_{1,1}, b_{1,2}, \ldots, b_{1,k-2}\}$ then $b_{1,k-1}, b_{1,k} \notin B' \cup \{x\}$ and H has no $b_{2,k}$-cac supergroup.

- $\{b_{k-1,1}, b_{k+1,1}\}$ and $\{b_{2,k-1}, b_{2,k}\}$ are non-splittable supergroups of H

  Since $x = b_{1,k-1}$, each vertex from B' has a non-visible neighbour from $\Psi \setminus \{b_{1,k-1}, w_3\} = \{b_{1,1}, \ldots, b_{1,k-2}, b_{1,k}\}$ in H. Since H has a w_3-cac and a w_4-cac supergroup, it follows that $\{b_{1,k-1}, w_3\}$ and $\{b_{1,k-1}, w_4\}$ are the only possible w_3-cac and w_4-cac supergroups of H, so that $\{b_{1,k-1}, w_3, w_4\}$ must be a supergroup of H. This particularly means that $b_{1,k-1}$ must not have non-visible neighbours in H, i.e.,

  $$\Big\{b_{2,k-2}, \ldots, b_{k+1,k-2}\Big\} \cup \Big\{b_{2,k-1}, \ldots, b_{k+1,k-1}\Big\} \ \subseteq\ B\,,$$

  and $h = k - 1$.

  Recall from the two introductory paragraphs of the proof that $A_1, \ldots, A_{k-2}$ are groups of $R_k[B' \cup \{x\}]$. Since $A_{k-2} = \{b_{2,k-2}, \ldots, b_{k+1,k-2}\}$, an easy induction shows for every $1 \leq j \leq k-2$ that $\{b_{k-j,j}, \ldots, b_{k+1,j}\} \subseteq A_j$, which particularly means that $\{b_{k-1,1}, b_{k,1}, b_{k+1,1}\} \subseteq A_1$. It directly follows that $\{b_{k-1,1}, b_{k+1,1}\}$ is a non-splittable supergroup of H.

  We consider $b_{2,k}$. Recall that $b_{1,k} \notin \Sigma_T(\underline{b})$ and $b_{2,k} \in \Sigma_T(\underline{c})$ and H has a $b_{2,k}$-cac supergroup. It follows that $\{b_{2,k-1}, b_{2,k}\}$ is the only possible $b_{2,k}$-cac supergroup of H, so that $\{b_{2,k-1}, b_{2,k}\}$ is a non-splittable supergroup of H.

- if $k = 3$ then $B \setminus \{w_1, w_2\} = \{b_{2,1}, b_{3,1}, b_{4,1}, b_{2,2}, b_{3,2}, b_{4,2}\}$

  This is observed in the preceding bullet point, by inserting $k = 3$. The equality follows from $B \cap \Psi = \emptyset$.

We have shown that the three items of the limit condition are satisfied, where the first item was already verified in the second paragraph of the proof. Note that $\Sigma_T(\underline{b})$ and $\Sigma_T(\underline{c})$ contain at least two vertices each, so that T is not a supergroup caterpillar tree.

We complete the proof of the lemma, based on the four above cases. If $b_{2,k} \in \Sigma_T(\underline{c})$ then the limit condition is satisfied, as it is shown in the forth case. If $b_{2,k} \notin \Sigma_T(\underline{c})$ then one of the first three cases must apply, since otherwise, $b_{1,1}, b_{1,k}, b_{2,k}, w_3 \notin \Sigma_T(\underline{c})$ means $\Sigma_T(\underline{c}) \cap (\Psi \cup \Phi) \subseteq \{b_{1,2}, \ldots, b_{1,k-1}\} \cup \{b_{2,h+1}, \ldots, b_{2,k-1}\}$, and this contradicts Lemma 5.10. ■

As the second case, we consider sets that contain $b_{2,k}$. The studied situations are as in the left-side case of Figure 15.

Lemma 5.13. *Let $B \subseteq V(S_k)$, and assume that $B \cap \Psi = \emptyset$ and $b_{2,k} \in B$ and B has no empty maximal clique of R_k. Let $x \in \Psi \cap V(R_k)$.*

Let T be a t-supergroup tree for S_k with $t \geq 1$. Assume that T has an inner node $\underline{a}$ with $\underline{b}$ and $\underline{c}$ its children in T such that $\Sigma_T(\underline{b}) = B \cup \{x\}$ and $\Sigma_T(\underline{c}) \cap \Psi \neq \emptyset$. Then, $t \geq k + 2$.

Proof. For every $1 \le j \le k$, let $A_j =_{\text{def}} B \cap K_j$. Let $B' =_{\text{def}} B \cap V(R_k)$. Observe that $B' = A_1 \cup \cdots \cup A_k$. Since B has no empty maximal clique of R_k, $A_1, \ldots, A_{k-1}$ are non-empty. Since $w_3 \notin \Sigma_T(\underline{b})$, every group X of $S_k[B \cup \{x\}]$ with $X \subseteq V(R_k)$ is a subset of one of $A_1, \ldots, A_{k-1}, \{b_{2,k}\}, \{x\}$, as can be seen as follows: if $x \notin \{b_{1,k-1}, b_{1,k}\}$ then this follows from the second statement of Lemma 5.7, and if $x \in \{b_{1,k-1}, b_{1,k}\}$ then this follows from the first statement of Lemma 5.7 and the fact that w_3 is a non-visible neighbour only of $b_{2,k}$. For an illustration of the situation, consider also the left-side figure of Figure 15. Let $H =_{\text{def}} S_k[\Sigma_T(\underline{b})] \oplus S_k[\Sigma_T(\underline{c})]$. We conclude that every supergroup partition for H has size at least $k+1$, and for a contradiction, we suppose that $t \le k+1$, and H has a v-cac supergroup for each vertex v from $\Sigma_T(\underline{c})$. We consider three cases and show a contradiction in each case.

As the first case, assume that $b_{1,1} \in \Sigma_T(\underline{c})$. As in the proof of Lemma 5.12, $\{w_1, b_{1,1}, b_{1,2}\}$ must be a supergroup of H, and the vertices from A_1 s-distinguish w_1 and $b_{1,1}$ in H, a contradiction.

As the second case, assume that $b_{1,k} \in \Sigma_T(\underline{c})$. Suppose for a contradiction that $\{b_{1,k}, z\}$ is a $b_{1,k}$-cac supergroup of H. Observe that $b_{2,k}$ is a non-visible neighbour of $b_{1,k}$ in H, so that z must be a neighbour of $b_{2,k}$ in R_k, and thus, $z \in \{b_{3,k-1}, \ldots, b_{k+1,k-1}\}$. Then, z is adjacent to $b_{1,k}$, a contradiction.

As the third case, assume that $w_3 \in \Sigma_T(\underline{c})$. If $w_4 \notin \Sigma_T(\underline{c})$ then w_4 s-distinguishes w_3 and each vertex from $\Sigma_T(\underline{b}) \cap V(R_k)$, and H has no w_3-cac supergroup. So, $w_4 \in \Sigma_T(\underline{c})$. Let $\{w_3, z\}$ be a w_3-cac supergroup of H. Since $b_{2,k}$ is a non-visible neighbour of w_3 in H, it follows that $z \in \{b_{3,k-1}, \ldots, b_{k+1,k-1}, b_{1,k}\}$. If $b_{1,k} \notin \Sigma_T(\underline{b})$ then $b_{1,k}$ s-distinguishes w_3 and z, so that $b_{1,k} \in \Sigma_T(\underline{b})$ must hold. In particular, $x = b_{1,k}$, and therefore, $b_{1,k-1} \notin \Sigma_T(\underline{b})$. Thus, $b_{1,k-1}$ s-distinguishes w_3 and each vertex from A_{k-1}, so that $z = x = b_{1,k}$, and $\{w_3, b_{1,k}\}$ is the only w_3-cac supergroup of H. Also note that $x = b_{1,k}$ implies that each vertex from $A_1 \cup \cdots \cup A_{k-1}$ has a non-visible neighbour from $V(R_k)$ in H. We consider w_4. Observe that $\{b_{1,k}, w_4\}$ and $\{b_{2,k}, w_4\}$ are w_4-cac supergroups of H, and these are the only possible w_4-cac supergroups of H. Let $\{M_1, \ldots, M_{k+1}\}$ be a compatible supergroup partition for H of size $k+1$. Then, there is $1 \le i \le k+1$ such that $\{b_{2,k}, w_4\} \subseteq M_i$ or $\{b_{1,k}, w_4\} \subseteq M_i$. If the latter holds then there are $1 \le i, j \le k+1$ such that $\{b_{1,k}, w_4\} \subseteq M_i$ and $\{w_3, b_{1,k}\} \subseteq M_j$, which clearly means that $i = j$ and $\{b_{1,k}, w_3, w_4\} \subseteq M_i$. However, $b_{2,k}$ s-distinguishes w_3 and w_4 in H. Thus, there are $1 \le i, j \le k+1$ with $i \ne j$ such that $\{b_{2,k}, w_4\} \subseteq M_i$ and $\{w_3, b_{1,k}\} \subseteq M_j$. Now, observe that $\{b_{2,k}, w_4\}$ and $\{w_3, b_{1,k}\}$ are not compatible in H, since $b_{1,k}$ and w_4 are non-adjacent in S_k, so that M_i and M_j are not compatible in H, and thus, $\{M_1, \ldots, M_{k+1}\}$ is not a compatible supergroup partition for H, a contradiction.

To complete the proof, it remains to see that $b_{1,1}, b_{1,k}, w_3 \notin \Sigma_T(\underline{c})$ means $\Sigma_T(\underline{c}) \cap \Psi \subseteq \{b_{1,2}, \ldots, b_{1,k-1}\}$. We directly conclude a contradiction by the application of Lemma 5.10. ▪

We summarise the results of this subsection about Ψ.

Corollary 5.14. *Let T be a t-supergroup tree for S_k with $t \geq 1$. Assume that T has a maximal R_k-clique split node $\underline{a}$ with $\underline{b}$ and $\underline{c}$ its children in T such that $\Sigma_T(\underline{a}) \subseteq V(R_k)$ and $\Sigma_T(\underline{b})$ has no empty maximal clique of R_k. Also assume that $\Sigma_T(\underline{a}) \cap \Psi \subseteq \Sigma_T(\underline{c})$ and $|\Sigma_T(\underline{c}) \cap \Psi| = 1$. If $t \leq k+1$ then T is not a supergroup caterpillar tree and the following two items apply:*

- *T has an inner node $\underline{a'}$ with $\underline{b'}$ and $\underline{c'}$ its children in T such that $(\Sigma_T(\underline{b'}), \Sigma_T(\underline{c'}))$ satisfies the limit condition*
- *if $k = 3$ then $\Sigma_T(\underline{a}) = \{v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$.*

Proof. Let x be the vertex with $\Sigma_T(\underline{c}) \cap \Psi = \{x\}$. Recall that $\Sigma_T(\underline{a}) \cap \Psi = \{x\}$. Let h be the largest integer with $1 \leq h \leq k$ such that $b_{2,h} \in \Sigma_T(\underline{a})$. Note that h indeed exists, since otherwise, $\Sigma_T(\underline{a}) \cap \Psi' = \emptyset$, so that $\Sigma_T(\underline{a})$ cannot have a full maximal clique of R_k, contradicting the choice of $\underline{a}$ as being a maximal R_k-clique split node of T. Also note that $b_{2,h} \notin \Psi$, so that $b_{2,h} \neq x$. We distinguish between two cases about h.

Case 1: $h = k$
Observe that this means $b_{2,k} \in \Sigma_T(\underline{a})$. Let $\underline{a'}$ be a node of T with $\underline{b'}$ and $\underline{c'}$ its children in T such that $\Sigma_T(\underline{a}) \subseteq \Sigma_T(\underline{b'})$ and $\Sigma_T(\underline{b'}) \cap \Psi = \{x\}$ and $\Sigma_T(\underline{c'}) \cap \Psi \neq \emptyset$. Observe that $\underline{a'}$ and $\underline{b'}$ and $\underline{c'}$ exist, where $\underline{a} = \underline{b'}$ is possible but not necessary, and $\underline{a'}, \underline{b'}, \underline{c'}$ satisfy the assumptions of Lemma 5.13, and we conclude $t \geq k+2$. □

Case 2: $h < k$
Let $\Phi =_{\text{def}} \{b_{2,j} : h < j \leq k\}$. According to the choice of h, it follows that $\Sigma_T(\underline{a}) \cap \Phi = \emptyset$, and thus, $\Sigma_T(\underline{a}) \cap (\Psi \cup \Phi) = \{x\}$. Let $\underline{a'}$ be a node of T with $\underline{b'}$ and $\underline{c'}$ its children in T such that $\Sigma_T(\underline{a}) \subseteq \Sigma_T(\underline{b'})$ and $\Sigma_T(\underline{b'}) \cap (\Psi \cup \Phi) = \{x\}$ and $\Sigma_T(\underline{c'}) \cap (\Psi \cup \Phi) \neq \emptyset$. Observe that $\underline{a'}$ and $\underline{b'}$ and $\underline{c'}$ exist and satisfy the assumptions of Lemma 5.12: if $t \leq k+1$ then T is not a supergroup caterpillar tree for S_k and $(\Sigma_T(\underline{b'}), \Sigma_T(\underline{c'}))$ satisfies the limit condition, and the first item applies.

Assume $k = 3$ and $t \leq 4$. Since $(\Sigma_T(\underline{b'}), \Sigma_T(\underline{c'}))$ satisfies the limit condition, in particular, the second item of the limit condition, $\Sigma_T(\underline{b'}) \setminus \{w_1, w_2\} = \{v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ holds. Suppose for a contradiction that $\Sigma_T(\underline{a}) \subset \Sigma_T(\underline{b'}) \cap V(R_3)$. This means that T has an inner node $\underline{a''}$ with $\underline{b''}$ and $\underline{c''}$ its children in T such that

$$\Sigma_T(\underline{a}) \;\subseteq\; \Sigma_T(\underline{b''}) \;\subset\; \Sigma_T(\underline{a''}) \;\subseteq\; \Sigma_T(\underline{b'}) \quad \text{and} \quad \Sigma_T(\underline{c''}) \cap \{v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \;\neq\; \emptyset\,.$$

Then, $\Sigma_T(\underline{b''})$ has a full and no empty maximal clique of R_3, and we can apply Lemma 5.5 to the $V(R_3)$-reduced supergroup tree of T, and it follows $t \geq 5$, a contradiction to the assumption of $t \leq 4$. Thus, $\Sigma_T(\underline{a}) = \Sigma_T(\underline{b'}) \cap V(R_3)$ must be the case, and the second item applies. □ ■

The result of Corollary 5.14 gives strong properties about supergroup trees for S_k that are in accordance with Ψ. Therefore, we also conclude strong properties about supergroup trees for S_k that are in accordance with Ψ', through the automorphic equivalent of Corollary 5.14.

5.3 Second part of analysis of supergroup trees and conclusion

In the previous subsection, we considered properties of supergroup trees for S_k that are in accordance with Ψ and Ψ'. Following Corollary 5.9, one case remains. Let

$$\Psi'' \;=_{\text{def}}\; \{v_{i\cdot k+1} : 0 \leq i \leq k\} \cup \{w_2, w_3\}\,.$$

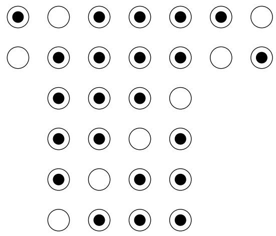

Figure 16: The situation for S_5 when the bubbles for the vertices in Ψ'' are empty. Note that this situation is automorphically equivalent to itself.

The described situation is depicted in Figure 16. This is an easy and nice case, when compared to the analysis of the situations in the preceding subsection, since $B \cap \Psi'' = \emptyset$ implies that $S_k[B]$ has many maximal groups already.

Lemma 5.15. *Let $B \subseteq V(S_k)$, and assume that $B \cap \Psi'' = \emptyset$ and B has no empty maximal clique of R_k. Let $x \in \Psi'' \cap V(R_k)$.*

Let T be a t-supergroup tree for S_k with $t \geq 1$. Assume that T has an inner node $\underline{a}$ with $\underline{b}$ and $\underline{c}$ its children in T such that $\Sigma_T(\underline{b}) = B \cup \{x\}$ and $\Sigma_T(\underline{c}) \cap \Psi'' \neq \emptyset$. Then, $t \geq k+2$.

Proof. Let $d_i =_{\text{def}} i \cdot k + 1$ for $0 \leq i \leq k$. Observe that $\langle d_0, \dots, d_k \rangle$ is a B-empty forward long step index sequence, where $d_k = k \cdot k + 1 = n$, and $x = v_{d_l}$ for some l with $0 \leq l \leq k$. For $1 \leq i \leq k$, let $A_i =_{\text{def}} B \cap \{v_{d_{i-1}}, \dots, v_{d_i}\}$. Since $d_i - d_{i-1} = k$ and B has no empty maximal clique of R_k, $A_1, \dots, A_k$ are non-empty. Also observe that each vertex in $\Sigma_T(\underline{b}) \cap V(R_k)$ has a non-visible neighbour from $\Psi'' \cap V(R_k)$ in H.

Let X be a group of $S_k[B \cup \{x\}]$ with $X \subseteq V(R_k)$. We show that X is a subset of one of $A_1, \dots, A_k, \{x\}$: if $0 < l < k$ then this is the case directly due to the third statement of Lemma 5.7, and if $l = 0$ or $l = k$ then this is the case due to the third statement of Lemma 5.7 and the fact that w_2 or w_3 s-distinguishes x and each other vertex from $B \cap V(R_k)$. It follows that $S_k[B \cup \{x\}]$ has at least $k+1$ maximal groups with vertices of R_k. Let $H =_{\text{def}} S_k[\Sigma_T(\underline{b})] \oplus S_k[\Sigma_T(\underline{c})]$. It follows that each supergroup partition for H has at least $k+1$ supergroups with vertices of R_k.

For a contradiction, suppose that $t \leq k+1$. Then, H has a v-cac supergroup for each vertex v from $\Sigma_T(\underline{c})$. Let y be a vertex from $\Sigma_T(\underline{c}) \cap \Psi''$, that exists by the assumptions of the lemma. Since H has a y-cac supergroup, Lemma 5.10 shows that $y \in \{v_1, v_n, w_2, w_3\}$. Observe that the cases $y = w_2$ and $y = w_3$ are automorphically equivalent. So, suppose for a contradiction that $y \in \{w_2, w_3\}$, and we can assume $y = w_2$. If $w_1 \notin \Sigma_T(\underline{c})$ then w_1 s-distinguishes w_2 and each vertex from $\Sigma_T(\underline{b}) \cap V(R_k)$, so that H cannot have a y-cac supergroup, a contradiction, and thus, $w_1 \in \Sigma_T(\underline{c})$, and H has a w_1-cac supergroup $\{w_1, z\}$. However, as observed above, z has a non-visible neighbour from $V(R_k)$ in H, that s-distinguishes w_1 and z, a contradiction. Thus, $w_2, w_3 \notin \Sigma_T(\underline{c})$, and $y \in \{v_1, v_n\}$ must hold. Since the two cases are automorphically equivalent, we may restrict to $y = v_1$. Note that w_2 s-distinguishes v_1 and each vertex from $\Sigma_T(\underline{b}) \cap V(R_k)$, and H has no y-cac supergroup, a contradiction. ■

We are ready to prove the main results about S_k. We do this in two steps.

Corollary 5.16. *Let T be a t-supergroup tree for S_k with $t \geq 1$. If $t \leq k+1$ then T is not a supergroup caterpillar tree and the following two items apply to T and S_k or to T and the automorphic copy of S_k:*

- *T has an inner node $\underline{a}'$ with $\underline{b}'$ and $\underline{c}'$ its children in T such that $(\Sigma_T(\underline{b}'), \Sigma_T(\underline{c}'))$ satisfies the limit condition*
- *if $k = 3$ then T has a maximal R_3-clique split node $\underline{a}$ with $\Sigma_T(\underline{a}) = \{v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$.*

Proof. We assume that $t \leq k+1$, and we apply Corollary 5.4: T has a maximal R_k-clique split node $\underline{a}$ with $\underline{b}$ and $\underline{c}$ its children in T such that $\Sigma_T(\underline{a}) \subseteq V(R_k)$ and $\Sigma_T(\underline{b})$ has no empty maximal clique of R_k and one of the two stated statements applies. Let $B =_{\text{def}} \Sigma_T(\underline{b})$ and $C =_{\text{def}} \Sigma_T(\underline{c})$. So, $B \cup C$ has a full maximal clique of R_k and B has no full and no empty maximal clique of R_k, and either $|C| = 1$ or $|(B \cup C) \cap \Psi| = 1$ and $|(B \cup C) \cap \Psi'| = 1$.

Assume $(B \cup C) \cap \Psi \subseteq C$. Then, $(B \cup C) \cap \Psi = C \cap \Psi$. Note that $(B \cup C) \cap \Psi \neq \emptyset$, since $B \cup C$ has a full maximal clique of R_k. So, $|C \cap \Psi| = 1$ also in case of $|C| = 1$. We can apply Corollary 5.14, and the two items apply. If $(B \cup C) \cap \Psi' \subseteq C$ then $|C \cap \Psi'| = 1$, and the two items apply to T and the automorphic copy of S_k according to Corollary 5.14. Observe that this fully captures the second case of Corollary 5.4.

We henceforth assume that the first case of Corollary 5.4 applies, so that $S_k[B]$ has exactly k maximal groups and $|C| = 1$, and we assume $(B \cup C) \cap \Psi \not\subseteq C$ and $(B \cup C) \cap \Psi' \not\subseteq C$. Observe that this means $B \cap \Psi \neq \emptyset$ and $B \cap \Psi' \neq \emptyset$. Since $B \cup C$ has a full maximal clique of R_k and since $|C| = 1$, B has a full clique of size k of R_k. Due to Corollary 2.9, applied to $S_k[B]$ and $R_k[B]$, $R_k[B]$ has at most k maximal groups. We distinguish between two situations about B.

Situation 1: $v_1 \notin B$ and $v_n \notin B$
Corollary 5.9 is applicable, and $B \cap \Psi'' = \emptyset$, and thus, $\Sigma_T(\underline{a}) \cap \Psi'' = C \cap \Psi'' = C$. Let $\underline{a}'$ be an inner node of T with $\underline{b}'$ and $\underline{c}'$ its children in T such that $\Sigma_T(\underline{a}) \subseteq \Sigma_T(\underline{b}')$ and $\Sigma_T(\underline{b}') \cap \Psi'' = C$ and $\Sigma_T(\underline{c}') \cap \Psi'' \neq \emptyset$. Observe that $\underline{a}'$ and $\underline{b}'$ and $\underline{c}'$ exist. Then, we can apply Lemma 5.15, which shows $t \geq k+2$. □

Situation 2: $v_1 \in B$ or $v_n \in B$
Since the case of $v_1 \in B$ is automorphically equivalent to the case of $v_n \in B$, we assume $v_n \in B$. We show that this yields a contradiction. Since $\Sigma_T(\underline{a}) \subseteq V(R_k)$, w_3 is a non-visible neighbour of v_n in $S_k[B]$, that s-distinguishes v_n and every other vertex of $S_k[B]$, so that $\{v_n\}$ is a maximal group of $S_k[B]$. Since $S_k[B]$ has at most k maximal groups, it follows from Lemma 2.8 and Corollary 2.9 that $S_k[B]{-}v_n$ and $R_k[B]{-}v_n$ have at most $k-1$ maximal groups each. We can apply the second statement of Lemma 5.3, which shows $B \cap \Psi = \emptyset$, a contradiction to our above assumptions. □

This completes the proof of the corollary. ■

We are finally in the position to prove the desired lower clique-width bounds. We define four further graphs, that we use a single name for. A graph S_k^+ is obtained from S_k by adding a new vertex w^+ whose neighbourhood is one out of the following four:

- $N_{S_k^+}(w^+) = \{v_1, \ldots, v_k, w_1, w_2\}$ or $N_{S_k^+}(w^+) = \{v_{n-k+1}, \ldots, v_n, w_3, w_4\}$
- $N_{S_k^+}(w^+) = \{v_1, \ldots, v_{k-1}, w_1, w_2\}$ or $N_{S_k^+}(w^+) = \{v_{n-k+2}, \ldots, v_n, w_3, w_4\}$.

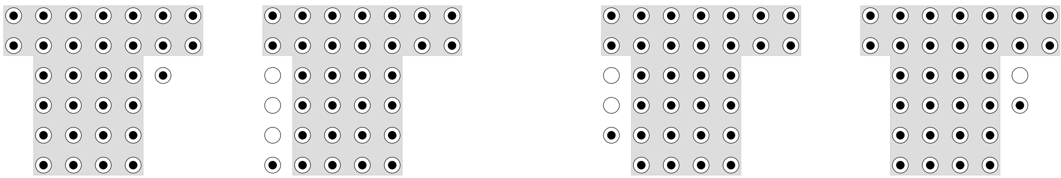

Figure 17: The four bubble models represent the four graphs that we obtained from adding a new vertex to S_k and that we denote by S_k^+, depicted for the special case of $k = 5$. The shaded area collects the vertices of S_5. The two bubble models to the left represent isomorphic graphs, which are therefore full bubble model graphs. The two bubble models to the right also represent isomorphic graphs, and these graphs are not full bubble model graphs.

The four possible bubble models for S_k^+ that are based on a full bubble model for S_k are depicted in Figure 17. Observe that S_k^+ may not be a full bubble model graph. It will be important in the coming lower-bound proof that two graphs are isomorphic to the other two graphs.

Proposition 5.17. *For $k \geq 3$, $\text{lcwd}(S_k) \geq k+2$ and $\text{cwd}(S_k^+) \geq k+2$.*

Proof. Let T be a t-supergroup caterpillar tree for S_k. Since T is also a t-supergroup tree for S_k, we can apply Corollary 5.16, so that $t \geq k+2$ directly follows, and we conclude $\text{lcwd}(S_k) \geq k+2$.

For the second lower bound, on the clique-width of S_k^+, we consider the supergroup trees for S_k^+. Let G be a graph S_k^+, and let w^+ be the new vertex. Let T^* be an arbitrary t-supergroup tree for G. We suppose for a contradiction that $t \leq k+1$.

Let T be the $V(S_k)$-reduced supergroup tree of T^*. Due to Lemma 2.10, T is a t-supergroup tree for S_k, and we can apply Corollary 5.16 to T and its automorphic equivalent. We restrict to the exact case. So, T has an inner node $\underline{a'}$ with $\underline{b'}$ and $\underline{c'}$ its children in T such that $(\Sigma_T(\underline{b'}), \Sigma_T(\underline{c'}))$ satisfies the limit condition and if $k = 3$ then T has a maximal R_3-clique split node $\underline{a}$ with $\Sigma_T(\underline{a}) = \{v_2, \dots, v_8\}$. We begin by exploring the implications of the limit condition, and we end by concluding for $k = 3$ in a remaining special situation.

We study the situation in T^*. Let $\underline{a^*}$ and $\underline{b^*}$ and $\underline{c^*}$ be nodes of T^*, where $\underline{b^*}$ and $\underline{c^*}$ are the children of $\underline{a^*}$ in T^*, that correspond to respectively $\underline{a'}, \underline{b'}, \underline{c'}$ of T. Note that

$$\begin{aligned}
\Sigma_{T^*}(\underline{a^*}) \setminus \{w^+\} &= \Sigma_T(\underline{a'}) \\
\Sigma_{T^*}(\underline{b^*}) \setminus \{w^+\} &= \Sigma_T(\underline{b'}) \\
\Sigma_{T^*}(\underline{c^*}) \setminus \{w^+\} &= \Sigma_T(\underline{c'})\,.
\end{aligned}$$

Let $H =_{\text{def}} S_k[\Sigma_T(\underline{b'})] \oplus S_k[\Sigma_T(\underline{c'})]$ and $H^* =_{\text{def}} G[\Sigma_{T^*}(\underline{b^*})] \oplus G[\Sigma_{T^*}(\underline{c^*})]$. It is important to observe that H is an induced subgraph of H^*, in fact, $H = H^*$ or $H = H^* - w^+$, depending on whether w^+ is a vertex of H^*.

Let $\{A_1, \dots, A_p\}$ be the supergroup partition label of $\underline{a^*}$ in T^*. Since $R_k[\Sigma_T(\underline{b'}) \cap V(R_k)]$ has $k+1$ maximal groups due to the first item of the limit condition, $p \geq k+1$ directly follows due to Corollary 2.9, so that $t = p = k+1$ due to the initial assumption about $t \leq k+1$ and $p \leq t$ due to the definition of supergroup trees. This particularly means that each of $A_1, \dots, A_p$ contains a vertex of R_k. We use this to show that $w^+ \in \Sigma_{T^*}(\underline{b^*})$ must hold.

As an auxiliary intermediate result, we show that $\{b_{k-1,1}, b_{k+1,1}\}$ and $\{b_{2,k-1}, b_{2,k}\}$ are contained in supergroups of $\{A_1, \ldots, A_p\}$. For every $1 \leq i \leq p$, $A_i \setminus \{w^+\}$ is a supergroup of H with respect to S_k due to Lemma 2.8. Thus,

$$\Big\{(A_1 \setminus \{w^+\}), \ldots, (A_p \setminus \{w^+\})\Big\}$$

is a supergroup partition for H with respect to S_k. If there are $1 \leq i < j \leq p$ such that

$$\begin{array}{lll} b_{k-1,1} \in (A_i \setminus \{w^+\}) & \text{and} & b_{k+1,1} \in (A_j \setminus \{w^+\}) \quad \text{or} \\ b_{k+1,1} \in (A_i \setminus \{w^+\}) & \text{and} & b_{k-1,1} \in (A_j \setminus \{w^+\})\,, \end{array}$$

i.e., if $b_{k-1,1}$ and $b_{k+1,1}$ are not contained in the same supergroup, then we obtain a contradiction to the fact that $\{b_{k-1,1}, b_{k+1,1}\}$ is a non-splittable supergroup of H according to the third item of the limit condition. Thus, there is $1 \leq i \leq p$ with $\{b_{k-1,1}, b_{k+1,1}\} \subseteq A_i$. Analogously, there is $1 \leq j \leq p$ with $\{b_{2,k-1}, b_{2,k}\} \subseteq A_j$. To complete: if $w^+ \notin \Sigma_{T^*}(\underline{b^*})$ then w^+ s-distinguishes $b_{k-1,1}$ and $b_{k+1,1}$ in H^* and A_i is not a supergroup of H^* or w^+ s-distinguishes $b_{2,k-1}$ and $b_{2,k}$ in H^* and A_j is not a supergroup of H^*. This yields a contradiction in each case. We conclude that $w^+ \in \Sigma_{T^*}(\underline{b^*})$ must hold, and, without loss of generality, we may assume $w^+ \in A_1$.

Recall that A_1 contains a vertex z from $\Sigma_{T^*}(\underline{b^*}) \cap V(R_k)$. So, $\{w^+, z\}$ is a supergroup of H^*. We distinguish between the two situations about the neighbourhood of w^+, whether w^+ is a neighbour of w_1 or of w_4.

- *w^+ is adjacent to w_4 in G*

  Observe that w^+ is adjacent to w_3 in G. Due to the second item of the limit condition, $b_{2,k}, w_3 \notin \Sigma_{T^*}(\underline{b^*})$. So, $z \neq b_{2,k}$, and w_3 s-distinguishes w^+ and z in H^*, a contradiction.

- *w^+ is adjacent to w_1 in G and $k \geq 4$*

  Observe that w^+ is adjacent to $b_{1,1}$ in G. Due to the second item of the limit condition, $b_{1,1}, b_{1,2} \notin \Sigma_{T^*}(\underline{b^*})$. So, $b_{1,1}$ is a non-visible neighbour of w^+ and z in H^*, and therefore, $z \in \{b_{2,1}, \ldots, b_{k+1,1}\}$, and then, $b_{1,2}$ s-distinguishes w^+ and z in H^*, a contradiction.

- *w^+ is adjacent to w_1 in G and $k = 3$*

  Recall that T has a maximal R_3-clique split node $\underline{a}$ with $\Sigma_T(\underline{a}) = \{v_2, \ldots, v_8\}$. Observe that $\Sigma_T(\underline{a})$ describes the situation depicted in the right-side bubble model of Figure 12. Let $\underline{d^*}$ be a node of T^* corresponding to $\underline{a}$, which means that $\Sigma_{T^*}(\underline{d^*}) \setminus \{w^+\} = \Sigma_T(\underline{a})$. If $w^+ \in \Sigma_{T^*}(\underline{d^*})$ then w_1 s-distinguishes w^+ and each vertex from $\Sigma_T(\underline{a})$, and $G[\Sigma_{T^*}(\underline{d^*})]$ has five maximal groups. If $w^+ \notin \Sigma_{T^*}(\underline{d^*})$ then w^+ s-distinguishes v_2 and v_4, and $G[\Sigma_{T^*}(\underline{d^*})]$ has five maximal groups. In both cases, $t \geq 5$.

We conclude: $t \geq k + 2$ must hold, and therefore, $\text{cwd}(S_k^+) \geq k + 2$. ∎

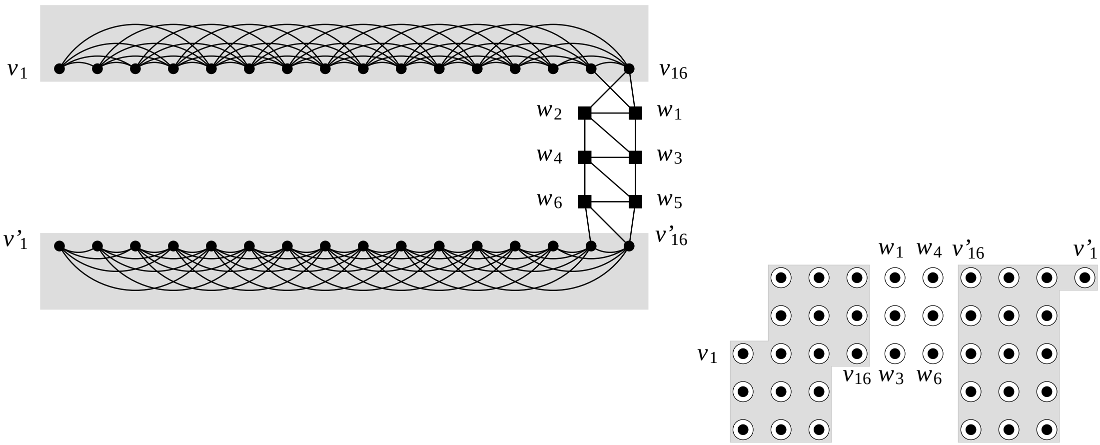

Figure 18: Depicted is the graph $M_{4,2,6}$. It is composed of two 4-path powers on $3 \cdot 5 + 1 = 16$ vertices each and a 2-path power on ten vertices, containing $w_1, \ldots, w_6$. The left side shows a graph drawing, and the right side shows a bubble model representation. The graphs F_4 and F'_4 are highlighted through shaded backgrounds.

6 Second clique-width lower-bound result

Let k be an integer with $k \geq 3$, and let $n =_{\text{def}} (k-1)(k+1)+1 = k^2$. Let $\Lambda_k = \langle v_1, \ldots, v_n \rangle$ and $\Lambda'_k = \langle v'_1, \ldots, v'_n \rangle$ be two sequences of pairwise different vertices. The k-path power on n vertices and with k-path layout Λ_k is denoted as F_k, and the k-path power on n vertices and with k-path layout Λ'_k is denoted as F'_k. Observe that F_k and F'_k have exactly one vertex less than R_k from Section 5. Let k' and l be two integers with $k > k' \geq 1$ and $l \geq 0$. The graph $M_{k,k',l}$ is obtained from the disjoint union of F_k and F'_k and l new vertices $w_1, \ldots, w_l$ such that $\{v_{n-k'+1}, \ldots, v_n, w_1, \ldots, w_l, v'_n, \ldots, v'_{n-k'+1}\}$ induces a k'-path power with k'-path layout $\Lambda''_{k',l} = \langle v_{n-k'+1}, \ldots, v_n, w_1, \ldots, w_l, v'_n, \ldots, v'_{n-k'+1} \rangle$:

$$\begin{aligned} V(M_{k,k',l}) &= V(F_k) \cup V(F'_k) \cup \{w_1, \ldots, w_l\} \\ E(M_{k,k',l}) &= E(F_k) \cup E(F'_k) \cup \{xy : x \text{ and } y \text{ are at distance at most } k' \text{ in } \Lambda''_{k',l}\}. \end{aligned}$$

We can say that $M_{k,k',l}$ is obtained as the union of a k'-path power with k'-path layout $\Lambda''_{k',l}$ and F_k and F'_k. If l is small then vertices of F_k and F'_k may be adjacent in $M_{k,k',l}$. An example, of $M_{4,2,6}$, is depicted in Figure 18. Throughout this section, we fix these definitions, and we necessarily require $k \geq 4$.

We show that the clique-width of $M_{k,k',l}$ is at least $k+2$ for large k' and the linear clique-width of $M_{k,k',l}$ is at least $k+2$ for every $k' \geq 1$. Our approach to proving these two lower bounds partially resembles the approach taken in Section 5. We will analyse the supergroup trees for $M_{k,k',l}$ and study supergroup partitions. The following technical lemma is a helpful observation about supergroups.

Lemma 6.1. *Let (B,C) be a partial partition of $V(M_{k,k',l})$ such that B and C have no full maximal clique of F_k. Let $H =_{\text{def}} M_{k,k',l}[B] \oplus M_{k,k',l}[C]$.*

1) Let y,z be a vertex pair of H with $y \in B \cap V(F_k)$. Assume that $\{y,z\}$ is a supergroup of H. Then, $\{y,z\}$ is a clique of H.

2) Assume that $B \cup C$ has a full maximal clique of F'_k and $v_n \notin B$. Let $D =_{\text{def}} (B \cup C) \setminus V(F_k)$. Assume that D is a supergroup of H. Then, $V(F'_k) \cup \{w_1, \ldots, w_l\} \cup \{v_{n-k'+1}, \ldots, v_n\} \subseteq C$.

Proof. We prove the first statement. Suppose for a contradiction that y and z are non-adjacent in H. Then, y and z are non-adjacent also in $M_{k,k',l}$. Let $y = v_p$. If $z \notin V(F_k)$ then either $1 \leq p \leq k+1$ and $\{v_1, \ldots, v_{k+1}\} \subseteq B$ or $k+2 \leq p \leq n$ and $\{v_{p-k}, \ldots, v_p\} \subseteq B$, both cases yielding a contradiction. So, z is a vertex of F_k, and $z = v_q$. Thus, $\{v_p, v_q\}$ is a supergroup of H and v_p and v_q are non-adjacent in $M_{k,k',l}$. We can henceforth assume $p < p+k < q$ without loss of generality. If $p \geq k+1$ then $\{v_{p-k}, \ldots, v_p\}$ is a full maximal clique of F_k in B, if $p \leq k$ and $q > n-k$ then $\{v_1, \ldots, v_{k+1}\}$ is a full maximal clique of F_k in B, and if $q \leq n-k$ then $\{v_q, \ldots, v_{q+k}\} \subseteq B$ or $\{v_q, \ldots, v_{q+k}\} \subseteq C$. So, each of the three possible cases yields a contradiction. For the second case, it is necessary to observe that $(k+1)+k = 2k+1 \leq n-k < q$.

We prove the second statement. Let a and b be indices with $1 \leq a \leq b \leq n$ such that $\{v'_a, \ldots, v'_b\} \subseteq D$ and $b-a$ is largest possible. Observe that $b-a \geq k$, since D has a full maximal clique of F'_k. If $a \geq 2$ then $v'_{a-1} \notin B \cup C$, and v'_{a-1} s-distinguishes v'_a and v'_{a+k} in H, and if $b \leq n-1$ then v'_{b+1} s-distinguishes v'_b and v'_{b-k} in H, so that $a = 1$ and $b = n$ and therefore $V(F'_k) \subseteq D$.

If there is a largest index c with $1 \leq c \leq l$ and $w_c \notin D$: if $c < l$ then w_c s-distinguishes w_{c+1} and v'_1 in H, and if $c = l$ then w_c s-distinguishes v'_n and v'_1 in H, so that $V(F'_k) \cup \{w_1, \ldots, w_l\} \subseteq D$ must hold.

Due to Lemma 2.4, $H[D] = M_{k,k',l}[D]$, and since $M_{k,k',l}[D]$ is a connected graph, $D \subseteq B$ or $D \subseteq C$ must hold. If $D \subseteq B$ then v_n s-distinguishes v'_1 and one of w_1 and v'_n, mainly depending on whether $l = 0$ or $l \geq 1$. So, $D \subseteq C$ must hold.

Finally, observe that each vertex from $\{v_{n-k'+1}, \ldots, v_n\}$ may s-distinguish v'_1 and one of w_1 and v'_n in H, so that $\{v_{n-k'+1}, \ldots, v_n\} \subseteq C$ must hold. ∎

For $1 \leq j \leq k-1$, let $K_j =_{\text{def}} \{v_{(j-1)(k+1)+1}, \ldots, v_{j(k+1)}\}$, and let $K_k =_{\text{def}} \{v_n\}$. The following lemma about top vertices in induced subgraphs of F_k mainly re-states the result of Lemma 5.6. Recall that R_k and F_k differ by a single vertex. The lemma can be proved analogous to the proof of Lemma 5.6, or it suffices to observe that it follows from Lemma 5.6 by considering sets B of vertices of R_k with either $b_{1,k}, b_{2,k} \notin B$ or $b_{1,k}, b_{2,k} \in B$.

Lemma 6.2. *Let $B \subseteq V(F_k)$ be such that B has no full maximal clique of F_k. The top vertices of $K_1, \ldots, K_k$ in B appear in pairwise different maximal groups of $F_k[B]$.*

Let T be a supergroup tree for $M_{k,k',l}$. Let $\underline{a}$ be an inner node of T and let $\underline{b}$ and $\underline{c}$ be the children of $\underline{a}$ in T. We call $\underline{a}$ a *two maximal clique split node of* T if $\Sigma_T(\underline{a})$ has a full maximal clique of F_k and a full maximal clique of F'_k and neither $\Sigma_T(\underline{b})$ nor $\Sigma_T(\underline{c})$ has a full maximal clique of F_k and of F'_k. We will often assume, without loss of generality, that $\Sigma_T(\underline{b})$ has no full maximal clique of F_k, and $\Sigma_T(\underline{c})$ may or may not have a full maximal clique of F_k. It will turn out that there are two major cases to consider.

The first case, that we consider in the next lemma, shows that first constructing F'_k and then constructing F_k requires many labels.

Lemma 6.3. *Let T be a t-supergroup tree for $M_{k,k',l}$ with $t \geq 1$. Let $\underline{a}$ be a two maximal clique split node of T with $\underline{b}$ and $\underline{c}$ its children in T such that $\Sigma_T(\underline{b})$ has no full and no empty maximal clique of F_k. Then, $t \geq k+2$, or $k' = 1$ and T is not a supergroup caterpillar tree for $M_{k,k',l}$.*

Proof. Let $H =_{\text{def}} M_{k,k',l}[\Sigma_T(\underline{b})] \oplus M_{k,k',l}[\Sigma_T(\underline{c})]$. Let $B =_{\text{def}} \Sigma_T(\underline{b}) \cap V(F_k)$ and $C =_{\text{def}} \Sigma_T(\underline{c}) \cap V(F_k)$, and let $D =_{\text{def}} \Sigma_T(\underline{a}) \setminus V(F_k)$. Let a be smallest such that $v'_a \in D$. Recall here that $D \cap V(F'_k)$ is non-empty and $a \leq n-k$, since $\Sigma_T(\underline{a})$ has a full maximal clique of F'_k. In particular, v'_a has no neighbour from $V(F_k)$.

Let Φ and Φ' be the sets of the top vertices of $K_1, \ldots, K_{k-1}, K_k$ in B and C, respectively. Since B has no empty maximal clique of F_k, $(B \cap K_1), \ldots, (B \cap K_{k-1})$ are non-empty, and since C is non-empty, also Φ' is non-empty. Thus, $k-1 \leq |\Phi| \leq k$ and $1 \leq |\Phi'|$. Since B and C have no full maximal clique of F_k and v'_a has no neighbour in $\Phi \cup \Phi'$: no supergroup of H contains two or more vertices from Φ or from Φ' due to Lemma 6.2, and no supergroup of H contains vertices from more than one of the sets $\Phi, \Phi', \{v'_a\}$ due to the first statement of Lemma 6.1. Thus, no supergroup of H contains two or more vertices from $\Phi \cup \Phi' \cup \{v'_a\}$, so that $|\Phi \cup \Phi' \cup \{v'_a\}| \geq k+2$ implies $t \geq k+2$.

Assume $|\Phi \cup \Phi' \cup \{v'_a\}| \leq k+1$. This means: $|\Phi| = k-1$ and $|\Phi'| = 1$ and $v_n \notin B$ and $C \subseteq K_p$ for some $1 \leq p \leq k$. Observe that the vertices from K_{p-1} or K_{p+1} s-distinguish the vertices from $C \cap K_p$. Thus, no supergroup of H contains more than one vertex from $\Phi \cup C \cup \{v'_a\}$, so that $|\Phi \cup C \cup \{v'_a\}| \geq k+2$ implies $t \geq k+2$.

Assume $|\Phi \cup C \cup \{v'_a\}| \leq k+1$. This means $|C| = 1$; let $C = \{v_c\}$. No supergroup of H contains vertices from $\Phi \cup \{v_c\}$ and D, according to the first statement of Lemma 6.1: if $c \leq n-k'$ then v_c has no neighbour in D, and if $c \geq n-k'+1$ then v_{c-k} s-distinguishes v_c and each vertex from D in H, and if Φ contains a vertex with a neighbour in D then this is the top vertex of K_{k-1} in B, which cannot be $v_{n-(k+1)}$, so that $v_{n-(k+1)} \notin B$ s-distinguishes the top vertex of K_{k-1} in B and each vertex from D in H. Thus, if D is not a supergroup of H then each supergroup partition for H has at least two supergroups containing vertices from D, and thus, each supergroup partition for H has size at least $k+2$, and thus, $t \geq k+2$.

Assume that D is a supergroup of H. Recall from the introductory definitions that $D = \Sigma_T(\underline{a}) \setminus V(F_k)$, and we have already seen $v_n \notin B$. So, the second statement of Lemma 6.1 is applicable: $V(F'_k) \cup \{w_1, \ldots, w_l\} \cup \{v_{n-k'+1}, \ldots, v_n\} \subseteq \Sigma_T(\underline{c})$. Since $|C| = 1$, it follows that $|\{v_{n-k'+1}, \ldots, v_n\}| \leq 1$, which means $k' = 1$. And T is not a supergroup caterpillar tree for $M_{k,k',l}$, since $|\Sigma_T(\underline{b})| \geq |B| \geq k$ and $|\Sigma_T(\underline{c})| \geq |V(F'_k)| + |\{v_{n-k'+1}, \ldots, v_n\}| \geq (k+1)+1$. ∎

We consider the second major case. Let G be a k'-path power and let $\langle x_1, \ldots, x_m \rangle$ be a k'-path layout for G. Let $B \subseteq V(G)$, and let x_g be a vertex of G. The *close left vertex of x_g in B* is the vertex with largest index from this set, if it is non-empty:

$$B \cap \left\{ x_{g-i(k'+1)} : i \geq 1 \text{ where } 1 \leq g - i(k'+1) \right\}.$$

Informally, using the bubble model notions, the close left vertex of x_g in B is the vertex from B "nearest" to the left of x_g in the same row of the bubble model. Equivalently, we can define *close right vertices*, that are the close left vertices in the reverse of $\langle x_1, \ldots, x_m \rangle$.

Lemma 6.4 ([14])**.** *Let G be a k'-path power on m vertices with $m \geq k'+3$, and let $\langle x_1, \ldots, x_m\rangle$ be a k'-path layout for G. Let (B, C) be a partial partition of $V(G)$. Let $1 \leq g < g' \leq m$ be such that $x_g, x_{g'} \notin B$, and assume that x_g and $x_{g'}$ have close left vertices x_h and $x_{h'}$, respectively, in B, where $h \neq h'$ and $h' < g$. Then, $\{x_h, x_{h'}\}$ is not a supergroup of $G[B] \oplus G[C]$.*

In an informal sense and using the bubble model, the result of Lemma 6.4 can be seen as the "rows equivalent" of Lemma 6.2 about top vertices in columns. We apply this result to prove a lower bound for an arbitrary path power, a result, that we use as the technical key result for our second lower-bound situation for $M_{k,k',l}$.

Lemma 6.5. *Let m be an integer with $m \geq 2(k'+1)$. Let Q be a k'-path power on m vertices with k'-path layout $\langle x_1, \ldots, x_m\rangle$. Let (B, C) be a partial partition for $V(Q)$, and let $H =_{\text{def}} Q[B] \oplus Q[C]$. Assume that B and C satisfy the following conditions:*

- *$\{x_1, \ldots, x_{k'+1}\} \subseteq B$ and $\{x_{m-k'}, \ldots, x_m\} \subseteq C$*
- *$B \setminus \{x_1\}$ and $C \setminus \{x_m\}$ have no full maximal clique of Q.*

Then, each compatible supergroup partition for H has size at least $k' + 1 + \lfloor \frac{k'}{2} \rfloor$.

Proof. Since $k' \geq 1$, it directly follows $m \geq 2(k'+1) \geq k'+2+1$, and Lemma 6.4 is formally applicable.

Before we begin the analysis, observe the following auxiliary result, that will be of importance later. Assume H has a supergroup $\{u, v\}$ with $u \in B$ and $v \in C$, where $u = x_p$ and $v = x_q$. If $q < p$ then $k'+1 < q < q+k' < p < m-k'$, and $\{x_{q-k'}, \ldots, x_q\}$ would be full in $C \setminus \{x_m\}$ and $\{x_p, \ldots, x_{p+k'}\}$ would be full in $B \setminus \{x_m\}$, contradicting the assumptions about Q. Thus, $p < q$. If $k'+1 < p < p+k' < q$ then $\{x_{p-k'}, \ldots, x_p\} \subseteq B \setminus \{x_1\}$, contradicting the assumptions about Q. Thus, $p \leq k'+1$.

Also note that the second condition on B implies $x_{k'+2} \notin B$, since otherwise, $B \setminus \{x_1\}$ has a full maximal clique of Q.

We analyse the size of the compatible supergroup partitions for Q. Let Φ be the set of the close left vertices of $x_{m-k'}, \ldots, x_m$ in B. Recall that $\{x_{m-k'}, \ldots, x_m\}$ is full in C and thus empty in B. Since $\{x_1, \ldots, x_{k'+1}\}$ is full in B, every vertex has a close left vertex, and thus, $|\Phi| = k'+1$. Note that $\Phi \subseteq \{x_1, \ldots, x_{m-k'-1}\}$. So, Lemma 6.4 is applicable, and the vertices in Φ appear in pairwise different supergroups of every supergroup partition for H. Analogously, let Φ' be the set of the close right vertices of $x_1, \ldots, x_{k'+1}$ in C. Then, $|\Phi'| = k'+1$, and the vertices in Φ' appear in pairwise different supergroups of every supergroup partition for H. Let $\{M_1, \ldots, M_r\}$ be a compatible supergroup partition for H. If the vertices from $(\Phi \setminus \{x_1\}) \cup \Phi'$ appear in pairwise different supergroups from $\{M_1, \ldots, M_r\}$ then $r \geq |\Phi \setminus \{x_1\}| + |\Phi'| \geq 2k'+1$.

As the other case, there are a largest index q with $2 \leq q \leq m$ and an index i with $1 \leq i \leq r$ and a vertex z from Φ' such that $x_q \in \Phi$ and $\{x_q, z\} \subseteq M_i$. The above auxiliary result shows that $q \leq k'+1$ must hold. This particularly means that $x_{k'+2}$ is a non-visible neighbour of x_q in H, and thus, $x_{k'+2}$ must be adjacent to z in Q, so that $z \in \{x_{k'+3}, \ldots, x_{2k'+2}\}$. And since x_q and z are non-adjacent in Q, we conclude $z \in \{x_{q+k'+1}, \ldots, x_{2k'+2}\}$.

Next, assume that there are vertices x_a, x_b with $x_a \in \Phi$ and $2 \leq a < q$ and $x_b \in \Phi'$ and an index j with $1 \leq j \leq r$ such that $\{x_a, x_b\} \subseteq M_j$. Observe that $i \neq j$ must hold, since otherwise, two vertices from Φ would make a supergroup of H, contradicting Lemma 6.4. Observe that $\{x_q, z\}$ and $\{x_a, x_b\}$ are compatible in H, since $\{M_1, \ldots, M_r\}$ is a compatible supergroup partition for H. We show that $q + k' + 1 \leq b \leq 2k' + 2$. First, suppose for a contradiction that $b \leq q + k'$, which means that x_q and x_b are adjacent in Q. Observe that $x_q x_b$ is a non-visible edge of H, since $x_q \in B$ and $x_b \in C$, and the compatibility condition implies that x_a and z must be adjacent in Q, meaning that $z \in \{x_1, \ldots, x_{a+k'}\}$, and since $a < q \leq a + k' < q + k'$, we obtain a contradiction to the above, and therefore, $q + k' + 1 \leq b$. Second, observe that $x_{k'+2}$ is a non-visible neighbour of x_a in H, and thus, $b \leq 2k' + 2$ directly follows.

We summarise the results. Let u, v be a vertex pair of H with $u \in \Phi \setminus \{x_1\}$ and $v \in \Phi'$ and assume that there is an index i with $1 \leq i \leq r$ such that $\{u, v\} \subseteq M_i$. Then, $u \in \{x_2, \ldots, x_q\} \cap \Phi$ and $v \in \{x_{q+k'+1}, \ldots, x_{2k'+2}\} \cap \Phi'$. Let f be the number of supergroups from $\{M_1, \ldots, M_r\}$ containing a vertex from $\Phi \setminus \{x_1\}$ and from Φ'. Observe that $r \geq |\Phi| + |\Phi'| - f - 1 = 2k' + 1 - f$. To see the usefulness of the inequality, note that x_1 may be a vertex from Φ and may also be in a supergroup with a vertex from Φ', and this supergroup does not contribute to f. If $f \leq \lceil \frac{k'}{2} \rceil$ then

$$r \;\geq\; |\Phi| + |\Phi'| - f - 1 \;\geq\; 2k' + 1 - \left\lceil \frac{k'}{2} \right\rceil \;=\; k' + 1 + \left\lfloor \frac{k'}{2} \right\rfloor ,$$

and the claimed lower bound on the size of $\{M_1, \ldots, M_r\}$ follows, and the proof is completed. It remains to verify the assumed assumption about the value of f. Applying the above obtained results, this follows:

$$\begin{aligned} f &\leq \min\left\{\Big|\{x_2, \ldots, x_q\} \cap \Phi\Big|, \Big|\{x_{q+k'+1}, \ldots, x_{2k'+2}\} \cap \Phi'\Big|\right\} \\ &\leq \min\left\{q - 1, (2k' + 2) - (q + k' + 1) + 1\right\} \\ &= \min\left\{q - 1, k' - q + 2\right\} \;\leq\; \left\lfloor \frac{q - 1 + k' - q + 2}{2} \right\rfloor \;=\; \left\lfloor \frac{k' + 1}{2} \right\rfloor \;=\; \left\lceil \frac{k'}{2} \right\rceil . \end{aligned}$$

This completes the proof of the lemma. ■

Corollary 6.6. *Let T be a t-supergroup tree for $M_{k,k',l}$ with $t \geq 1$. Let $\underline{a}$ be a two maximal clique split node of T with $\underline{b}$ and $\underline{c}$ its children in T such that $\Sigma_T(\underline{b})$ has no full maximal clique of F_k and has a full maximal clique of F'_k and $\Sigma_T(\underline{c})$ has no full maximal clique of F'_k and has a full maximal clique of F_k. Then, $t \geq k' + 1 + \lfloor \frac{k'}{2} \rfloor$.*

Proof. Since $\Sigma_T(\underline{c})$ has a full maximal clique of F_k, $\Sigma_T(\underline{b})$ has an empty maximal clique of F_k, and since $\Sigma_T(\underline{b})$ has a full maximal clique of F'_k, $\Sigma_T(\underline{c})$ has an empty maximal clique of F'_k.

We define a specific induced subgraph of $M_{k,k',l}$. Let Q' be the induced subgraph of $M_{k,k',l}$ on the following vertices:

$$\begin{aligned} \{w_1, \ldots, w_l\} \;\cup\; & \{v_p : n - k' \leq p + i(k+1) \leq n \text{ for some } i \geq 0\} \\ \cup\; & \{v'_p : n - k' \leq p + i(k+1) \leq n \text{ for some } i \geq 0\} . \end{aligned}$$

The definition of Q' is illustrated in Figure 19. It is important to observe that Q' is a k'-path power, and each of $\Sigma_T(\underline{b})$ and $\Sigma_T(\underline{c})$ contains a full and an empty maximal clique of Q', simply by restricting the full and empty maximal cliques of F_k and F'_k to the vertices of Q'.

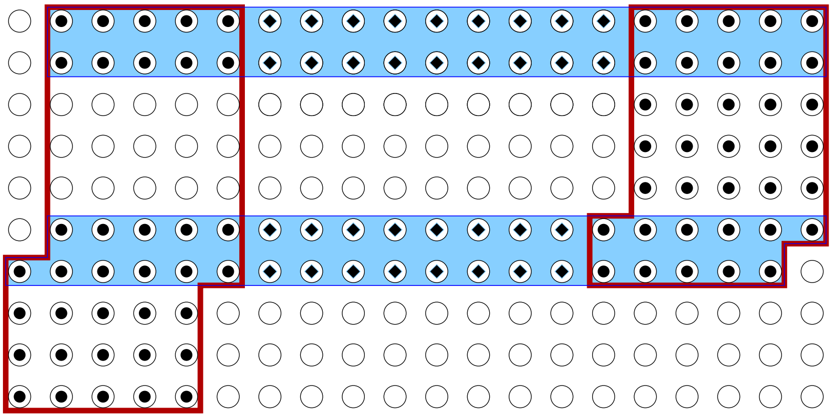

Figure 19: Depicted is a bubble model for $M_{6,3,34}$. The bubble model is not full. We see two types of vertices: the circles represent the vertices of F_6 and F_6', and the squares represent the added vertices $w_1, \ldots, w_{34}$. The vertices of F_6 and F_6' are additionally collected by the thick frames, and the vertices of the 3-path power, extending from $\{w_1, \ldots, w_{34}\}$ according to the construction in the proof of Corollary 6.6, are indicated by the shaded area. A 3-path layout can be read off the bubble model from left to right.

We define an induced subgraph of Q'. Let m' be the number of vertices of Q', and let $\langle y_1, \ldots, y_{m'} \rangle$ be a k'-path layout for Q'. Let $B' =_{\text{def}} \Sigma_T(\underline{b}) \cap V(Q')$ and $C' =_{\text{def}} \Sigma_T(\underline{c}) \cap V(Q')$. Recall from the preceding paragraph that B' and C' have full maximal cliques of Q'. We define Q by iteratively applying the following procedure: if B' and C' have full maximal cliques of Q' in $Q'{-}y_1$ then restrict Q' to $Q'{-}y_1$, and if B' and C' have full maximal cliques of Q' in $Q'{-}y_{m'}$ then restrict Q' to $Q'{-}y_{m'}$. Note that $\langle y_2, \ldots, y_{m'} \rangle$ is a k'-path layout for $Q'{-}y_1$ and $\langle y_1, \ldots, y_{m'-1} \rangle$ is a k'-path layout for $Q'{-}y_{m'}$. We repeat this procedure as long as possible. Then, Q is the resulting graph.

Observe that Q is a k'-path power. Let $\langle x_1, \ldots, x_m \rangle$ be a k'-path layout for Q; we can assume that $\langle x_1, \ldots, x_m \rangle$ is the remaining sublayout of $\langle y_1, \ldots, y_{m'} \rangle$. Let B and C be the restrictions of B' and C' to the vertices of Q, i.e., $B = B' \cap V(Q)$ and $C = C' \cap V(Q)$. The construction of Q yields the following properties: $\{x_1, \ldots, x_{k'+1}\}$ is a full maximal clique of B or C; without loss of generality, we may assume $\{x_1, \ldots, x_{k'+1}\} \subseteq B$. It follows that $\{x_{m-k'}, \ldots, x_m\}$ is a full maximal clique of Q in C.

We conclude the desired lower bound on t. Let $H =_{\text{def}} Q[B] \oplus Q[C]$. Observe that Q and H and (B, C) and $\langle x_1, \ldots, x_m \rangle$ satisfy the assumptions of Lemma 6.5, and we conclude that every compatible supergroup partition for H has size at least $k'+1+\lfloor \frac{k'}{2} \rfloor$. Recall that H is an induced subgraph of $H' =_{\text{def}} Q'[B'] \oplus Q'[C']$, and thus, each compatible supergroup partition for H' has size at least $k'+1+\lfloor \frac{k'}{2} \rfloor$. Next, H' is an induced subgraph of $M_{k,k',l}[\Sigma_T(\underline{b})] \oplus M_{k,k',l}[\Sigma_T(\underline{c})]$, and hereby, $t \geq k'+1+\lfloor \frac{k'}{2} \rfloor$ directly follows. For the arguments, recall the results of Subsection 2.2.3. ∎

We combine the obtained results and finish with the desired lower bounds on the clique-width and linear clique-width of $M_{k,k',l}$.

Proposition 6.7. *For $k \geq 4$ and $k > k' \geq 1$ and $l \geq 0$,*

1) $\mathrm{lcwd}(M_{k,k',l}) \geq k+2$, *and*

2) if $k' + \lfloor \frac{k'}{2} \rfloor \geq k+1$ *then* $\mathrm{cwd}(M_{k,k',l}) \geq k+2$.

Proof. Let T be a t-supergroup tree for $M_{k,k',l}$. By descending in T, it is clear that T has a two maximal clique split node $\underline{a}$. Let $\underline{b}$ and $\underline{c}$ be the children of $\underline{a}$ in T, and let $B =_{\mathrm{def}} \Sigma_T(\underline{b})$ and $C =_{\mathrm{def}} \Sigma_T(\underline{c})$. We begin by listing three cases.

1) *B and C have empty maximal cliques of F_k*

   Let T^* be the $V(F_k)$-reduced supergroup tree of T. Observe that T^* has a maximal F_k-clique split node $\underline{a^*}$ with $\underline{b^*}$ and $\underline{c^*}$ its children in T^* such that $\Sigma_{T^*}(\underline{b^*})$ and $\Sigma_{T^*}(\underline{c^*})$ have an empty maximal clique of F_k: if neither B nor C has a full maximal clique of F_k, we choose $\underline{a^*}$ as the node of T^* that corresponds to $\underline{a}$ of T, and if B or C has a full maximal clique of F_k, which means that $\underline{a}$ cannot serve as a maximal F_k-clique split node, then we find $\underline{a^*}$ by descending further from $\underline{a}$. We can apply Lemma 5.2 to T^*, and $t \geq k+2$ follows.

2) *B and C have empty maximal cliques of F'_k*

   This case is analogous to Case 1, and $t \geq k+2$ follows.

3) *B or C has no full and no empty maximal clique of F_k, or*
   B or C has no full and no empty maximal clique of F'_k

   Lemma 6.3 is applicable, and $t \geq k+2$ or the special case of the lemma applies.

Recall the special case of Lemma 6.3: $k' = 1$ and T is not a supergroup caterpillar tree. So, if T is a supergroup caterpillar tree, for the first statement of the proposition, then $t \geq k+2$ must hold. And if T is not a supergroup caterpillar tree, for the second statement of the proposition, then $k' = 1$ does not satisfy the inequality condition of the statement.

We summarise the situations for B and C that we have already considered by applying the three above cases. We list the sixteen possible situations, about B and C having empty maximal cliques, in the below table: a "+" entry for $B(F_k)$, for example, means that B has an empty maximal clique of F_k, and a "-" entry for $B(F_k)$ analogously means that B has no empty maximal clique of F_k. We give a reference to a case that considers this particular situation. It is to note that serveral cases may be applicable to the same situation.

$B(F_k)$	-	-	-	-	-	-	-	-	+	+	+	+	+	+	+	+
$B(F'_k)$	-	-	-	-	+	+	+	+	-	-	-	-	+	+	+	+
$C(F_k)$	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+	+
$C(F'_k)$	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+
case	3	3	3		3	2		2	3		1	1		2	1	1
				A			B			C			D			

We specially argue about the four remaining cases. We begin with Case A. If B has no full maximal clique of F_k or of F'_k then Case 3 is applicable, and if B has a full maximal clique of

F_k and of F'_k then $\underline{a}$ is not a two maximal clique split node. Case D is analogous to Case A, with the meanings of B and C interchanged.

We consider Case B: B has no empty maximal clique of F_k and C has no empty maximal clique of F'_k, so that B has no full maximal clique of F'_k and C has no full maximal clique of F_k. If B has no full maximal clique of F_k or if C has no full maximal clique of F'_k then Case 3 is applicable. If B has a full maximal clique of F_k and C has a full maximal clique of F'_k then Corollary 6.6 is applicable, with the meanings of B and C, or, equivalently, of F_k and F'_k interchanged, and $t \geq k' + 1 + \lfloor \frac{k'}{2} \rfloor \geq k + 2$. Note here that T is not a supergroup caterpillar tree. Case C is analogous to Case B. ∎

We remark that the second statement of Proposition 6.7 is interesting only for $k \geq 5$ and $k' \geq 4$, since $\lfloor \frac{k'}{2} \rfloor \leq 1$ for $k' \leq 3$.

7 Computation and characterisation

We are ready to obtain the full characterisation of the clique-width and linear clique-width of full bubble model graphs. We obtain the characterisation by combining the lower- and upper-bound results from the preceding sections. We proceed in two stages: we first summarise and complete the lower-bound results, and then, we prove the final characterisation and give the efficient computation algorithms.

7.1 The completed lower-bound results

We summarise the obtained lower-bound results from Sections 5 and 6. We also extend these results to capture the remaining few cases for obtaining a complete list of forbidden induced subgraphs of bounded clique-width and linear clique-width. We consider clique-width and linear clique-width separately.

We begin with clique-width. For k, k', l integers with $k \geq 3$, $k' \geq 1$, $k > k'$ and $l \geq 0$, the graphs S_k^+ are defined in Section 5 and the graphs $M_{k,k',l}$ are defined in Section 6. Let k be an integer with $k \geq 0$. A k-path power on $k(k+1)+2$ vertices is denoted by Z_k. The clique-width of such path powers was completely determined in [14]. The graph S_2 is the left-side graph of Figure 20.

Proposition 7.1. *Let k, k', l be integers with $k \geq 0$, $k' \geq 1$ and $l \geq 0$.*

1) $\text{cwd}(Z_k) \geq k + 2$ *[14]*

2) $\text{cwd}(S_2) \geq 4$, *and* $\text{cwd}(S_k^+) \geq k + 2$ *for* $k \geq 3$

3) $\text{cwd}(M_{k,k',l}) \geq k + 2$ *for* $k \geq 5$ *and where* k' *satisfies* $k > k'$ *and* $k' + \lfloor \frac{k'}{2} \rfloor \geq k + 1$.

Proof. The result of the first statement is proved in [14], the result of the third statement is proved in the second statement of Proposition 6.7, and Proposition 5.17 proves $\text{cwd}(S_k^+) \geq k+2$ for $k \geq 3$ of the second statement. It remains to show $\text{cwd}(S_2) \geq 4$.

For the used names of the vertices of S_2, we refer to Figure 20. Let F be a subgraph of S_2, and let X be a set of vertices of F. For $\mathcal{A}$ a supergroup partition for F, we say that X is a

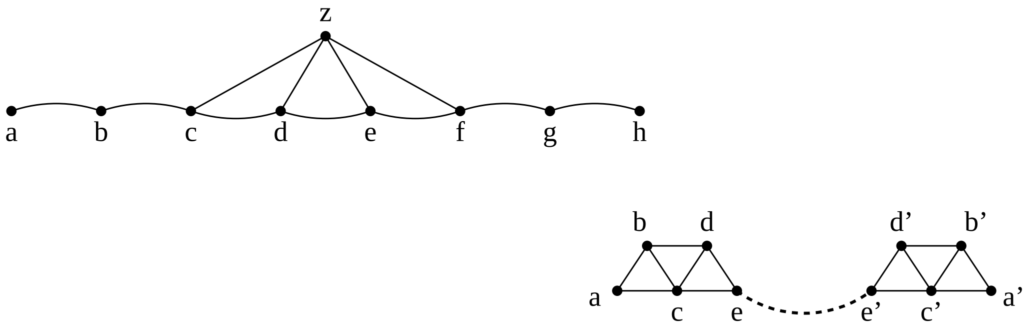

Figure 20: Depicted are the graphs S_2 to the left and M_2^+ and M_2^- to the right, which are forbidden induced subgraphs for graphs of linear clique-width at most 3. Note that M_2^+ has ee' as an edge and M_2^- does not have ee' as an edge.

witness set for $\mathcal{A}$ if the vertices from X are in pairwise different supergroups from $\mathcal{A}$, and we say that X is a *witness set* for F if X is a witness set for every compatible supergroup partition for F. A witness set shows a lower bound on the size of supergroup partitions.

Let T be a t-supergroup tree for S_2. We show $t \geq 4$ by identifying witness sets of size 4. We focus on the closed neighbourhood of z. Let N_z be short for the closed neighbourhood of z in S_2, i.e., $N_z = N_{S_2}[z] = \{c, d, e, f, z\}$. Let $\underline{a}$ be an inner node of T with $\underline{b}$ and $\underline{c}$ its children in T, and let $B =_{\text{def}} \Sigma_T(\underline{b})$ and $C =_{\text{def}} \Sigma_T(\underline{c})$ and $H =_{\text{def}} S_2[B] \oplus S_2[C]$. Assume that $1 \leq |B \cap N_z| \leq 2$ and $1 \leq |C \cap N_z| \leq 2$ and $3 \leq |(B \cup C) \cap N_z| \leq 4$. We consider three particular situations.

Situation 1: $|B \cup C| \geq 4$ and no supergroup of H contains two vertices from N_z

Observe that $(B \cup C) \cap N_z$ is a witness set for H already by the assumption. If $B \cup C \subseteq N_z$ then $B \cup C$ is a witness set of size 4 for H. Otherwise, $(B \cup C) \cap \{a, b, g, h\} \neq \emptyset$. Let x be a vertex of H with $x \in \{a, b, g, h\}$. If $\{x, y\}$ for some $y \in N_z$ is a supergroup of H: either $y \in B$ and $|B \cap N_z| \geq 3$, or $y \in C$ and $|C \cap N_z| \geq 3$, both cases yielding a contradiction. Thus, $(B \cup C) \cap (N_z \cup \{x\})$ is a witness set for H. □

Situation 2: H has a supergroup with two vertices from N_z

Let $\{u, v\}$ with $u, v \in N_z$ be a supergroup of H. We may assume $u \in B$. If $v \in B$ then $\{u, v\} = B \cap N_z$, and u or v has a non-visible neighbour from N_z that s-distinguishes u and v in H, a contradiction, so that $v \in C$ must hold. Note that u and v are non-adjacent in S_2, so that $\{u, v\} = \{c, e\}$ or $\{u, v\} = \{d, f\}$ or $\{u, v\} = \{c, f\}$. By symmetry arguments, it suffices to consider $\{u, v\} = \{c, e\}$ and $\{u, v\} = \{c, f\}$, and we can assume $u = c$.

Consider this special case: $\{b, c, d\} \subseteq B$ and $\{e, f\} \subseteq C$. Then, $z \notin B \cup C$, and z and e s-distinguish b, c, d from each other in H. Thus, for every compatible supergroup partition $\mathcal{A}$ for H, $\{b, c, d, e\}$ or $\{b, c, d, f\}$ is a witness set for $\mathcal{A}$. If $\{c, f\}$ is a supergroup of H, which means $\{b, c, d\} \subseteq B$ and $\{e, f, g\} \subseteq C$, or if $\{c, e\}$ is a supergroup of H and $d \in B$, which means $\{b, c\} \subseteq B$ and $\{e, f\} \subseteq C$, then the special case occurs.

To complete the proof of the situation, assume that $\{c, e\}$ is a supergroup of H and $d \notin B$. Observe that $b \in B$ and $f \in C$, so that $C \cap N_z = \{e, f\}$. If $z \in B$ then $\{b, c, f, z\}$ is a witness set for H. Otherwise, $d, z \notin B \cup C$. Let $\underline{a'}$ be the inner node of T with $\underline{b'}$ and $\underline{c'}$ its children in T such that $B \cup C \subseteq \Sigma_T(\underline{b'})$ and $\Sigma_T(\underline{b'}) \cap \{d, z\} = \emptyset$ and $\Sigma_T(\underline{c'}) \cap \{d, z\} \neq \emptyset$. Note that b, c, f appear in pairwise different maximal groups of $S_2[\{a, b, c, e, f, g, h\}]$. So: if $z \in \Sigma_T(\underline{c'})$ then $\{b, c, f, z\}$ is a witness set and if $z \notin \Sigma_T(\underline{c'})$ and $d \in \Sigma_T(\underline{c'})$ then $\{b, c, d, f\}$ is a witness set for $S_2[\Sigma_T(\underline{b'})] \oplus S_2[\Sigma_T(\underline{c'})]$. □

Situation 3: $|B \cup C| = 3$

Observe that the assumptions about B and C directly imply $B \cup C \subseteq N_z$. Let $\underline{a'}$ be the inner node of T with $\underline{b'}$ and $\underline{c'}$ its children in T, where $B' =_{\text{def}} \Sigma_T(\underline{b'})$ and $C' =_{\text{def}} \Sigma_T(\underline{c'})$ and $H' =_{\text{def}} S_2[B'] \oplus S_2[C']$, such that $B \cup C \subseteq B' \subseteq B \cup C \cup \{a, h\}$ and $C' \not\subseteq \{a, h\}$. Observe that $\underline{a'}$ indeed exists and $B \cup C$ is a witness set for H'. If $z \in C'$ then $B \cup C \cup \{z\}$ is a witness set for H', since z is a non-visible neighbour of each vertex from $B \cup C$. If $e \in C'$ and $\{e, u\}$ for some $u \in B \cup C$ is a supergroup of H' then $u = c$, but $b \notin B'$ s-distinguishes c and e in H', and $B \cup C \cup \{e\}$ is a witness set for H'; analogously for the case of $d \in C'$.

Assume that $d, e, z \notin C'$. If $f \in C'$ and $\{f, u\}$ for some $u \in B \cup C$ is a supergroup of H' then e and z are non-visible neighbours of f in H' and $u = d$ must hold, and thus, $\{c, d\} \subseteq B \cup C \subseteq \{c, d, e, z\}$ and $g \in C'$, and $B \cup C \cup \{g\}$ is a witness set for H'. Recall here that $e \notin B \cup C$ or $z \notin B \cup C$. The case of $c \in C'$ analogously follows by symmetry.

Assume that $c, d, e, f, z \notin C'$. Thus, $C' \subseteq \{a, b, g, h\}$ and $C' \cap \{b, g\} \neq \emptyset$. If $b \in C'$ and $\{b, u\}$ for some $u \in B \cup C$ is a supergroup of H' then c is a non-visible neighbour of b in H', and $u = d$ follows, so that $B \cup C = \{d, e, z\}$ and $a \in C'$, and $\{a, d, e, z\}$ is a witness set for H'; analogously for the case of $g \in C'$. □

It remains to see that $\underline{a}$ with the required properties indeed exists, by descending from the root node of T, and if Situation 2 does not occur then Situation 1 or Situation 3 occurs. We conclude $t \geq 4$, and thus, $\text{cwd}(S_2) \geq 4$. ■

We continue with linear clique-width. The graphs Z_k are the k-path powers on $k(k+1)+2$ vertices, the graphs S_k are defined in Section 5 and Figure 20, and the graphs $M_{k,1,l}$ are defined in Section 6. The graphs M_2^+ and M_2^- are shown as the right-side graph of Figure 20, where the dotted edge ee' is an edge of M_2^+ and is no edge of M_2^-. If we do not properly distinguish between M_2^+ and M_2^-, we simply write $M_2^\pm$.

Proposition 7.2. *Let k and l be integers with $k \geq 0$ and $l \geq 0$.*

1) $\text{lcwd}(Z_k) \geq k + 2$ *[14]*

2) $\text{lcwd}(S_k) \geq k + 2$ *for* $k \geq 2$

3) $\text{lcwd}(M_2^\pm) \geq 4$ *and* $\text{lcwd}(M_{k,1,l}) \geq k + 2$ *for* $k \geq 3$.

Proof. The result of the first statement is proved in [14], the result of the second statement for $k \geq 3$ is proved in Proposition 5.17 and $\text{lcwd}(S_2) \geq 4$ is proved in the second statement of Proposition 7.1, and the first statement of Proposition 6.7 proves $\text{lcwd}(M_{k,1,l}) \geq k+2$ for $k \geq 4$. It remains to show $\text{lcwd}(M_2^\pm) \geq 4$ and $\text{lcwd}(M_{3,1,l}) \geq 5$, by identifying appropriate witness sets (for the definitions, we refer to the beginning of the proof of Proposition 7.1).

Proof of $\mathrm{lcwd}(M_2^{\pm}) \geq 4$
Let $G =_{\text{def}} M_2^{\pm}$. For the names of the vertices of G, we refer to Figure 20. Let T be a t-supergroup caterpillar tree for G. By a symmetry argument, we can assume that T has an inner node $\underline{a}$ with $\underline{b}$ and $\underline{c}$ its children in T, where $B =_{\text{def}} \Sigma_T(\underline{b})$ and $\Sigma_T(\underline{c}) = \{x\}$, such that $|B \cap \{e, a', b', c', d', e'\}| \geq 2$ and $|B \cap \{a, b, c, d\}| = 1$ and $x \in \{a, b, c, d\}$. Observe that $|N_G(x) \cap \{a, b, c, d, e\}| \geq 2$. Let $H =_{\text{def}} G[B] \oplus G[\{x\}]$.

Assume that $\{x, u\}$ for $u \in B$ is a supergroup of H. Since x has at least two non-visible neighbours from $\{a, b, c, d, e\}$ in H, it directly follows that $\{x, u\} \subseteq \{a, b, c, d, e\}$ must hold. Note here that $\{u\} = B \cap \{a, b, c, d, e\}$. Since x and u must be non-adjacent in G, only three situations about $\{u, x\}$ must be considered: (1) $\{x, u\} = \{a, e\}$ is not possible, since b or d s-distinguishes a and e, and (2) $\{x, u\} = \{b, e\}$ is not possible, since a s-distinguishes b and e. Thus, $\{x, u\} = \{a, d\}$ is the only possible situation. If $x = d$ then e s-distinguishes x and u in H, so that $x = a$ and $u = d$ must hold, and $e \in B$, and $b, c \notin B$. Let $\underline{a'}$ be the inner node of T with $\underline{b'}$ and $\underline{c'}$ its children in T such that $B \cup \{x\} \subseteq \Sigma_T(\underline{b'})$ and $\Sigma_T(\underline{b'}) \cap \{b, c\} = \emptyset$ and $\Sigma_T(\underline{c'}) \subseteq \{b, c\}$. Let y be an arbitrary vertex from $\Sigma_T(\underline{b'}) \cap \{a', b', c', d', e'\}$. Observe that a, e, y are s-distinguished from each other by b and c in $G[\Sigma_T(\underline{b'})]$, so that a, e, y appear in pairwise different maximal groups of $G[\Sigma_T(\underline{b'})]$. If $\Sigma_T(\underline{c'}) = \{c\}$ then $\{a, c, e, y\}$ is a witness set for $G[\Sigma_T(\underline{b'})] \oplus G[\{c\}]$, and if $\Sigma_T(\underline{c'}) = \{b\}$ then $\{a, b, e, y\}$ is a witness set for $G[\Sigma_T(\underline{b'})] \oplus G[\{b\}]$. Thus, $t \geq 4$.

As the other case, assume that there is no vertex $u \in B$ such that $\{x, u\}$ is a supergroup of H. If $G[B]$ has at least three maximal groups then every supergroup partition for H has size at least 4. If $G[B]$ has at most two maximal groups then this is only possible as follows: $G = M_2^-$ and $\{a', b', c', d', e'\} \subseteq B$. We ascend in T from $\underline{a}$ toward the root of T. Let u, v, w be a vertex triple from $\{a, b, c, d, e\}$, let $B' =_{\text{def}} \{u, v, a', b', c', d', e'\}$, and let $H' =_{\text{def}} G[B'] \oplus G[\{w\}]$. Observe that the parent node of $\underline{a}$ in T defines such a situation. We show that H' has a witness set of size 4 or u, v, w is a very special vertex triple. Observe about H' that each of u, v, w has a non-visible neighbour and $\{u, v\}$ is not a supergroup of H'. It follows that $\{u, v, a'\}$ is a witness set for H' and $\{w, a'\}$ is not a supergroup of H'. It therefore suffices to show that $\{u, v, w\}$ is a witness set for H', since this implies that also $\{u, v, w, a'\}$ is a witness set for H'. It is easy to verify that the following three graphs have exactly three maximal groups with respect to G: $G[\{a, b, c\}], G[\{b, c, d\}], G[\{a, c, e\}]$, so that $\{u, v, w\}$ is a witness set for H' in particular. In case of $\{u, v, w\} = \{a, b, d\}$ and $\{u, v, w\} = \{a, c, d\}$, $G[\{u, v, w\}]$ is an induced path of length 2 and e s-distinguishes a and d, and $\{u, v, w\}$ is a witness set for H', independent of the actual choice of w. As the final case, consider $\{u, v, w\} = \{a, b, e\}$. This case is special, since $\{a, b, e\}$ is not a witness set for H'. We ascend further in T and consider $H'' =_{\text{def}} G[\{a, b, e\} \cup \{a', \ldots, e'\}] \oplus G[\{x\}]$, where $x \in \{c, d\}$. If $x = c$ then $\{a, b, c, a'\}$ is a witness set for H'', and if $x = d$ then $\{a, b, d, a'\}$ is a witness set for H''.

We conclude $t \geq 4$. □

Proof of $\mathrm{lcwd}(M_{3,1,l}) \geq 5$
Let $G =_{\text{def}} M_{3,1,l}$. For the names of the vertices and induced subgraphs of G, we refer to the beginning of Section 6: G is composed of F_3 and F_3' and the connecting vertices $w_1, \ldots, w_l$. Let T be a t-supergroup caterpillar tree for G. Let $\underline{a}$ be a two maximal clique split node of T with $\underline{b}$ and $\underline{c}$ its children in T. By a symmetry argument, we can assume that $\Sigma_T(\underline{c}) = \{x\}$ for some

vertex x of F_3. Let $B =_{\text{def}} \Sigma_T(\underline{b})$, and let $H =_{\text{def}} G[B] \oplus G[\{x\}]$. Let p be the smallest index with $1 \leq p \leq 9$ such that $v'_p \in B$. Since B has a full maximal clique of F'_3, $p \leq 6$, and v'_p has no (non-visible) neighbour from $\{v_1, \ldots, v_9\}$ in H. We show that H has a witness set of size at least 5.

Suppose for a contradiction that $\{x, u\}$ for $u \in B$ is a supergroup of H. Then, all (non-visible) neighbours of x are neighbours of u in G, i.e., $N_G(x) \subseteq N_G(u)$. This is only possible if $x = v_1$ and $u = v_5$, and then, $\{v_5, v_6, v_7, v_8\}$ is a full maximal clique of F_3 in B, the claimed contradiction. Thus, no supergroup of H contains x and a vertex from B. We show that B contains a witness set of size 4. We distinguish between two cases.

Assume that B has an empty maximal clique $\{v_a, \ldots, v_{a+3}\}$ of F_3. Recall that $B \cup \{x\}$ has a full maximal clique of F_3. It follows that $|B \cap \{v_1, \ldots, v_{a-1}\}| \geq 3$ or $|B \cap \{v_{a+4}, \ldots, v_9\}| \geq 3$. Let Φ and Φ' be the sets of the respectively close left and close right vertices of $v_a, \ldots, v_{a+3}$ in $B \cap V(F_3)$. Due to Lemma 6.4, Φ and Φ' are witness sets for H, and since B has no full maximal clique of F_3, each vertex from $\Phi \cup \Phi'$ has a non-visible neighbour from $V(F_3)$ in H, so that in fact $\Phi \cup \{v'_p\}$ and $\Phi' \cup \{v'_p\}$ are witness sets for H. It follows that $\Phi \cup \{v'_p, x\}$ and $\Phi' \cup \{v'_p, x\}$ are witness sets for H, and one of the two sets is of size at least 5.

Assume that B has no empty maximal clique of F_3. Let Ψ be the set of the top vertices of K_1, K_2, K_3 in B. Note that $B \cap K_1$ and $B \cap K_2$ are non-empty, so that $2 \leq |\Psi| \leq 3$, and each vertex in Ψ has a non-visible neighbour from $V(F_3)$. Thus, due to Lemma 6.2 and the above, $\Psi \cup \{v'_p, x\}$ is a witness set for H. Note here that the result of Lemma 6.2 is indeed applicable, since the proof does not require $k \geq 4$ and is valid also for $k = 3$. If $|\Psi| = 3$ then $|\Psi \cup \{v'_p, x\}| = 5$, and $\Psi \cup \{v'_p, x\}$ is a witness set of size 5 for H. If $|\Psi| = 2$ then $v_9 \notin B$, and H has a vertex y such that $\Psi \cup \{y, v'_p, x\}$ is a witness set of size 5 for H: if $B \cap \{w_1, \ldots, w_l\} \neq \emptyset$ then we choose $y = w_i$ for $w_i \in B$ of smallest index, and if $B \cap \{w_1, \ldots, w_l\} = \emptyset$ then we choose $y = v'_j$ for $v'_j \in B$ of largest index. Thus, $t \geq 5$. □ ■

7.2 Characterisation and computation

We show that open k-models, short-end k-models and k-models with small separators exactly capture the full bubble model graphs of clique-width at most $k + 1$. We also show that open k-models and short-end k-models exactly capture the full bubble model graphs of linear clique-width at most $k + 1$. We show these results by proving that a full bubble model graph is an induced subgraph of a graph with a specified bubble model or contains one of the special graphs of large (linear) clique-width as an induced subgraph.

Let $\mathcal{B} = \langle b_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$ and $\mathcal{B}' = \langle b'_{i,j} \rangle_{1 \leq j \leq s', 1 \leq i \leq r'_j}$ be full bubble models. We say that $\mathcal{B}'$ is *embeddable* into $\mathcal{B}$ if there is an integer p with $0 \leq p \leq s - s'$ such that $r'_j \leq r_{j+p}$ for every $1 \leq j \leq s'$. We can understand p as a column-index offset, and we can understand the defined embeddability notion is a special induced subgraph notion, since $\mathcal{B}$ is a full bubble model.

The goal is to embed bubble models into one of the three special classes of full bubble models, and the main property is the depth of columns. Let k be an integer with $k \geq 3$. Let $\mathcal{B} = \langle b_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$ be a full bubble model. Let p, q be column indices with $1 \leq p \leq q \leq s$. We call $[p, q]$, or $\mathcal{B}[p, q]$, a *deep pseudo-rectangle* if $r_p, \ldots, r_q > k$, and either $p = 1$ or $r_{p-1} \leq k$, and either $q = s$ or $r_{q+1} \leq k$. The *size* of $[p, q] = \mathcal{B}[p, q]$ is $q - p + 1$. Observe that deep pseudo-rectangles are the deep analogue of shallow pseudo-rectangles that were used in the proof of Lemma 4.6.

Lemma 7.3. *Let k be an integer with $k \geq 3$. Let G be a connected full bubble model graph and let $\mathcal{B}$ be a full bubble model for G. If $\mathcal{B}$ is not embeddable into an open k-model or a short-end k-model then G contains Z_k or S_k or $M_{k,1,l}$ for some $l \geq 0$ as an induced subgraph.*

Proof. Let $\mathcal{B} = \langle b_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$. If $\mathcal{B}$ has no deep pseudo-rectangles then $r_j \leq k$ for every $1 \leq j \leq s$, and $\mathcal{B}$ is clearly embeddable into an open k-model. As the other case, assume that $\mathcal{B}$ has deep pseudo-rectangles. If every deep pseudo-rectangle of $\mathcal{B}$ has size at most $k-2$ then $\mathcal{B}$ is embeddable into an open k-model. Otherwise, $\mathcal{B}$ has a deep pseudo-rectangle of size at least $k-1$. For the following arguments, recall that the connectedness of G and the result of Lemma 3.2 implies $r_j \geq 2$ for every $1 \leq j \leq s-1$. Assume that $[p,q]$ is a deep pseudo-rectangle of $\mathcal{B}$ of size at least $k-1$:

- *assume that $p \geq 2$ and $q \leq s-3$*
  G contains S_k as an induced subgraph, induced by
  $$\{b_{1,p-1}, b_{2,p-1}\} \cup \bigcup_{p \leq j \leq p+k-2} \{b_{1,j}, \ldots, b_{k+1,j}\} \cup \{b_{1,p+k-1}, b_{2,p+k-1}, b_{1,p+k}, b_{2,p+k}\};$$
  recall here that $p+k \leq q+2 \leq s-1$, and thus, $r_{p+k} \geq 2$, and the listed vertices indeed exist

- *assume that $[p,q]$ has size at least k and one of the following applies:*
  (1) $[p,q]$ has size at least $k+1$, or (2) $q \leq s-2$, or (3) $q = s-1$ and $r_s \geq 2$
  G contains Z_k as an induced subgraph, induced by
  $$\bigcup_{p \leq j \leq p+k-1} \{b_{1,j}, \ldots, b_{k+1,j}\} \cup \{b_{1,p+k}, b_{2,p+k}\}.$$

So, if G contains Z_k or S_k as an induced subgraph then the claim of the lemma trivially holds. We henceforth assume that G does not contain Z_k and S_k as an induced subgraph. Then, the following is the case for each deep pseudo-rectangle $[p,q]$ of $\mathcal{B}$:

- *if $[p,q]$ is of size $k-1$:*
  (1) $p = 1$ or (2) $p \geq 2$ and $q \geq s-2$ and if $q = s-2$ then $r_s = 1$

- *if $[p,q]$ is of size at least k:*
  $[p,q]$ has size exactly k and $q \geq s-1$ and if $q = s-1$ then $r_s = 1$.

We distinguish between the two situations about whether $\mathcal{B}$ has a deep pseudo-rectangle of size at least k.

As the first situation, assume that $\mathcal{B}$ has a deep pseudo-rectangle $[p,q]$ of size k. Note that $q \geq s-1$, and thus, $[p,q]$ is the unique deep pseudo-rectangle of $\mathcal{B}$ of size k. If $\mathcal{B}$ has no deep pseudo-rectangle of size $k-1$ then $\mathcal{B}$ is embeddable into an open k-model. Otherwise, $\mathcal{B}$ has a deep pseudo-rectangle $[p', q']$ of size $k - 1$. Note that $q' \leq p - 2$ must hold, since $p' \geq q + 2$ is not possible. Then, G contains $M_{k,1,l}$ as an induced subgraph, that is induced by

$$\bigcup_{p' \leq j \leq q'} \{b_{1,j}, \ldots, b_{k+1,j}\} \setminus \{b_{1,p'}\} \cup \bigcup_{q' < j < p} \{b_{1,j}, b_{2,j}\} \cup \bigcup_{p \leq j < q} \{b_{1,j}, \ldots, b_{k+1,j}\} \cup \{b_{1,q}\} .$$

It is to note that F_k of $M_{k,1,l}$ also requires $b_{1,q'+1}$ and $b_{2,q'+1}$, and the value of l is $2(p - q' - 2)$.

As the second situation, assume that $\mathcal{B}$ has no deep pseudo-rectangle of size k. Let p be smallest possible with $1 \leq p \leq s$ such that $[p, q]$ is a deep pseudo-rectangle of $\mathcal{B}$ of size $k - 1$. Recall from the beginning that $\mathcal{B}$ has a deep pseudo-rectangle of size $k - 1$, so that p indeed exists. According to the above: (1) $p = 1$ or (2) $q = s - 2$ and $r_s = 1$ or (3) $q \geq s - 1$. If $q \geq s - 2$ then all other deep pseudo-rectangles of $\mathcal{B}$ have size at most $k - 2$. It follows for $q \geq s - 2$ that $\mathcal{B}$ is embeddable into an open k-model. As the other case, assume $p = 1$. If all other deep pseudo-rectangles of $\mathcal{B}$ have size at most $k - 2$ then $\mathcal{B}$ is embeddable into a short-end k-model. Otherwise, there is a smallest p' with $q + 2 \leq p' \leq s$ such that $[p', q']$ is a deep pseudo-rectangle of size $k - 1$. If $q' < s$ then G contains $M_{k,1,l}$ as an induced subgraph, analogous to the construction of the preceding paragraph. If $q' = s$ then all deep pseudo-rectangles different from $[p, q]$ and $[p', q']$ have size at most $k - 2$, and $\mathcal{B}$ is embeddable into a short-end k-model. ■

Lemma 7.4. *Let k be an integer with $k \geq 3$. Let G be a connected full bubble model graph and let $\mathcal{B}$ be a full bubble model for G. If $\mathcal{B}$ is not embeddable into an open k-model or a short-end k-model or a k-model with small separators then G contains Z_k or S_k^+ or $M_{k,k',l}$ for some integers k' and l with $k > k' \geq 1$ and $k' + \lfloor \frac{k'}{2} \rfloor \geq k + 1$ and $l \geq 0$ as an induced subgraph.*

Proof. The proof is similar to the proof of Lemma 7.3, and we can therefore refer to the constructions there. Let $\mathcal{B} = \langle b_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$. If $\mathcal{B}$ has no deep pseudo-rectangle of size at least $k - 1$ then $\mathcal{B}$ is embeddable into an open k-model, and if G does not contain Z_k as an induced subgraph then for every deep pseudo-rectangle $[p, q]$ of $\mathcal{B}$ of size at least k: $[p, q]$ has size exactly k and (1) $q = s$ or (2) $q = s - 1$ and $r_s = 1$. Assume that $\mathcal{B}$ has a deep pseudo-rectangle $[p, q]$ of size $k - 1$, and assume $p \geq 2$ and (1) $q \leq s - 3$ or (2) $q = s - 2$ and $r_s \geq 2$. Recall that $r_{p-1} \geq 2$ and $r_{q+1} \geq 2$ and $r_{q+2} \geq 2$. If $r_{p-1} = k$ or if $r_{q+1} \geq 3$ then G contains S_k^+ as an induced subgraph, more precisely, G contains one of the two graphs represented by S_k^+ as an induced subgraph.

We henceforth assume that G does not contain Z_k and S_k^+ as an induced subgraph. Then, for every deep pseudo-rectangle $[p, q]$ of size $k - 1$, one of the following applies: (1) $p = 1$, or (2) $p \geq 2$ and $q \geq s - 1$, or (3) $p \geq 2$ and $q = s - 2$ and $r_s = 1$, or (4) $p \geq 2$ and $q \leq s - 2$ and $r_{p-1} \leq k - 1$ and $r_{q+1} = 2$. The following cases are easy.

- $[1, k - 1]$ *is not a deep pseudo-rectangle*
  $\mathcal{B}$ is embeddable into a k-model with small separators.
  In order to formally satisfy the definition of k-models with small separators, we would embed into a bubble model whose second rectangle is of depth 2.

- $[1, k - 1]$ *is the only deep pseudo-rectangle of size at least $k - 1$*
  $\mathcal{B}$ is embeddable into a short-end k-model.

- $[1, k-1]$ *and* $[s-k+2, s]$ *are the only deep pseudo-rectangles of size at least* $k-1$
  $\mathcal{B}$ is embeddable into a short-end k-model.

Finally, assume that $[1, k-1]$ is a deep pseudo-rectangle and there is a smallest p' with $k+1 \leq p' \leq s-k+1$ such that $[p', q']$ is a deep pseudo-rectangle of size at least $k-1$. Note that $[p', q']$ has size at least k or it has size exactly $k-1$ and $q' < s$. Let k' be the largest integer such that $r_j \geq k'+1$ for every $k \leq j < p'$. Recall that $k \geq k'+1 \geq 2$: by the definition of deep pseudo-rectangles, $r_k \leq k$, and therefore, $k'+1 \leq k$, and $k'+1 \geq 2$ follows from the connectedness of G and Lemma 3.2. Then, G contains $M_{k,k',l}$ as an induced subgraph, induced by

$$\bigcup_{1 \leq j \leq k-1} \{b_{1,j}, \ldots, b_{k+1,j}\} \setminus \{b_{1,1}, \ldots, b_{k',1}\} \;\cup \bigcup_{k \leq j < p'} \{b_{1,j}, \ldots, b_{k'+1,j}\}$$
$$\cup \bigcup_{p' \leq j \leq p'+k-2} \{b_{1,j}, \ldots, b_{k+1,j}\} \;\cup\; \{b_{1,p'+k-1}\}\,.$$

If $k' + \lfloor \frac{k'}{2} \rfloor \leq k$ then $\mathcal{B}$ is embeddable into a k-model with small separators, and if $k' + \lfloor \frac{k'}{2} \rfloor \geq k+1$ then k' satisfies the requested conditions. ∎

We obtain the final characterisation results about the clique-width of full bubble model graphs. We give the two results separately.

Theorem 7.5. *Let k be an integer with $k \geq 3$. Let G be a connected full bubble model graph and let $\mathcal{B}$ be a full bubble model for G.*

1) $\operatorname{cwd}(G) \leq k+1$ *if and only if $\mathcal{B}$ is embeddable into an open k-model or a short-end k-model or a k-model with small separators.*

2) $\operatorname{lcwd}(G) \leq k+1$ *if and only if $\mathcal{B}$ is embeddable into an open k-model or a short-end k-model.*

Proof. The two results follow from Corollary 4.8, and Lemmas 7.3 and 7.4 and Propositions 7.1 and 7.2. ∎

Theorem 7.6. *Let G be a connected full bubble model graph.*

1) $\operatorname{cwd}(G) \leq 1$ *if and only if G does not contain Z_0 as an induced subgraph.*
$\operatorname{cwd}(G) \leq 2$ *if and only if G does not contain Z_1 as an induced subgraph.*
$\operatorname{cwd}(G) \leq 3$ *if and only if G does not contain Z_2 and S_2 as an induced subgraph.*

$\operatorname{lcwd}(G) \leq 3$ *if and only if G does not contain Z_2 and S_2 and $M_2^{\pm}$ as an induced subgraph.*

2) For $k \geq 3$:
$\operatorname{cwd}(G) \leq k+1$ *if and only if G does not contain Z_k and S_k^+ and $M_{k,k',l}$ where $k > k' \geq 1$ and $k' + \lfloor \frac{k'}{2} \rfloor \geq k+1$ and $l \geq 0$ as an induced subgraph.*

3) For $k \geq 3$:
$\operatorname{lcwd}(G) \leq k+1$ *if and only if G does not contain Z_k and S_k and $M_{k,1,l}$ for $l \geq 0$ as an induced subgraph.*

Figure 21: A full bubble model graph of linear clique-width at most 3: the graph to the left, and a full bubble model to the right. The shaded areas indicate cliques.

Proof. It is clear from the definition of clique-width that $\text{cwd}(G) \leq 1$ if and only if G has no edge, which is equivalent to G not containing Z_0 as an induced subgraph. It is analogously known that $\text{cwd}(G) \leq 2$ if and only if G does not contain P_4 as an induced subgraph [5], and Z_1 is isomorphic to P_4. The case of $\text{cwd}(G) \leq k+1$ and the case of $\text{lcwd}(G) \leq k+1$ for $k \geq 3$ is due to Propositions 7.1 and 7.2 and Lemmas 7.3 and 7.4 and Theorem 7.5.

We prove the remaining two cases, namely about $\text{cwd}(G) \leq 3$ and $\text{lcwd}(G) \leq 3$. Let $\mathcal{B} = \langle b_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$ be a full bubble model for G. If there is a column index p with $2 \leq p \leq s-2$ such that $r_{p-1} \geq 2$ and $r_p \geq 3$ and $r_{p+1} \geq 2$ and $r_{p+2} \geq 2$ then G contains S_2 as an induced subgraph. Analogously, if there is p with $1 \leq p \leq s-2$ such that $r_p \geq 3$ and $r_{p+1} \geq 3$ and $r_{p+2} \geq 2$ then G contains Z_2 as an induced subgraph. So, if G does not contain Z_2 and S_2 as an induced subgraph then $r_j \leq 2$ for every $2 \leq j \leq s-3$ and if $r_{s-2} \geq 3$ then $r_s = 1$. Observe that G must be almost like an induced path, informally spoken. By splitting G at $b_{1,3}$, it is not difficult to see that $\text{cwd}(G) \leq 3$ holds.

We consider the linear clique-width of G. If $r_1 = 2$ then G is an induced subgraph of a graph with a full bubble model as depicted in Figure 21, and $\text{lcwd}(G) \leq 3$. Otherwise, $r_1 \geq 3$. If $r_{s-2} \geq 3$ or if $r_{s-1} \geq 3$ and $r_s \geq 2$ then G contains $M_2^{\pm}$ as an induced subgraph, and $\text{lcwd}(G) \geq 4$ due to the third statement of Proposition 7.2. If $r_{s-2} = r_{s-1} = 2$ then G is an induced subgraph of a graph as represented in Figure 21, and $\text{lcwd}(G) \leq 3$. ■

We conclude with a consequence of the main characterisation results.

Theorem 7.7. *The clique-width and the linear clique-width of connected full bubble model graphs without true twins can be computed in linear time.*

Proof. The clique-width and linear clique-width of connected full bubble model graphs without true twins of clique-width at most 3 can be computed in linear time (Theorem 7.6 and its proof). Recall that such graphs of clique-width at most 2 are induced subgraphs of stars and such graphs of clique-width at most 3 are obtained from induced paths by adding vertices to the beginning and the end of the path.

We consider graphs of larger clique-width, and we will apply the characterisation of Theorem 7.5. Recall from Section 3 that full bubble model graphs can be recognised in linear time and a full bubble model for full bubble model graphs can be computed in linear time [15]. According to Theorem 7.5, it suffices to decide whether the computed full bubble model is embeddable into one of the two or three special models. The proofs of Lemmas 7.3 and 7.4 describe easy such algorithms, when k is given: it mainly suffices to determine the deep pseudo-rectangles. It is to note here that no particular bubble model for the input graph is required but any full bubble model will do. So, if a good "approximation" on the clique-width can be computed in $\mathcal{O}(n)$ time, we obtain the desired linear-time algorithm by trying the few possible values in the approximation range.

We compute a good approximation on the clique-width and linear clique-width of the input graph G. Let $\mathcal{B} = \langle b_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$ be the computed full bubble model for G. We determine the smallest integer t such that $\mathcal{B}$ has no deep pseudo-rectangle of size at least $t+1$, relative to t. That means more precisely: $\mathcal{B}$ has no $t+1$ consecutive columns of depth at least $t+1$. The choice of t implies that $\mathcal{B}$ is embeddable into an open $(t+2)$-model and $\mathcal{B}$ is not embeddable into a $(t-1)$-model. Thus, $t \leq \mathrm{cwd}(G) \leq \mathrm{lcwd}(G) \leq t+3$ due to Theorem 7.5.

We compute t by applying a sweep algorithm, that scans the column depths from right to left, i.e., the sequence $\langle r_1, \ldots, r_s \rangle$. We give an informal description of the algorithm. Assume the algorithm has already processed a right subsequence $\langle r_q, \ldots, r_s \rangle$: with the current value of t, t is the smallest integer such that $\langle r_q, \ldots, r_s \rangle$ has no $t+1$ consecutive numbers larger than t. The algorithm continues with r_{q-1} and decides upon the following cases:

- if $r_{q-1} \leq t$ then t satisfies the condition also for the right subsequence $\langle r_{q-1}, \ldots, r_s \rangle$
- if $r_{q-1} \geq t+1$ and $r_i \leq t$ for some index i with $q \leq i \leq q+t-1$ then also in this case, t satisfies the condition for the right subsequence $\langle r_{q-1}, \ldots, r_s \rangle$
- if $r_{q-1}, r_q, \ldots, r_{q+t-1} \geq t+1$ then $\mathcal{B}$ has $t+1$ consecutive columns of depth at least $t+1$, and we increase the value of t by 1, and the new value of t satisfies the condition for the right subsequence $\langle r_{q-1}, \ldots, r_s \rangle$.

To make this algorithm run in linear time, it suffices to decide the applying case in constant time, and it suffices to decide the existence of an index i with $q \leq i \leq q+t-1$ such that $r_i \leq t$. We keep and update a table storing for each column depth d with $t < d$ the smallest index j with $q \leq j \leq s$ such that $r_j = d$ and the smallest index j' with $r_{j'} \leq t$, if such indices exist. Using this table, the case distinction is constant-time decidable, and this table can be updated in constant time. Note that the update in case of incrementing the value of t means to take the minimum table entry for t and $t+1$. ■

We conclude this section with the announced consideration about true twins and linear clique-width. Recall from Section 3 that true twins may increase the linear clique-width of graphs. The situations when this is the case and when this is not the case are not easy to describe. An example of a comprehensive such study is about induced paths [13], that illustrates the complexity already for easy-structured graphs. We do not aim at studying such questions for full bubble model graphs. However, we do want to mention that our results already provide the description of some cases for which adding true twins is possible without increasing the linear clique-width. This is particularly the case for the lower-part vertices of deep rectangles. This can be seen by reconsidering the constructions for deep rectangles of Section 4.

8 Conclusions

We characterised the full bubble model graphs of bounded clique-width completely, by forbidden induced subgraphs and by an embedding notion into graphs. As a corollary, we obtained an easy linear-time algorithm for computing the clique-width of full bubble model graphs. We proved analogous results about the linear clique-width of connected full bubble model graphs without true twins. Full bubble model graphs are the first large graph class for which such results are known, since the previously known related results are for square grids [10] and path powers [14] only. We believe that our results provide a deeper understanding of the structure of graphs of bounded clique-width, and therefore of clique-width itself. We also believe that our results, the lower-bound proofs in particular, present interesting and useful techniques for proving lower clique-width bounds for other graphs.

A major contribution of our paper are the lower-bound results of Sections 5 and 6. They are summarised and completed in Propositions 7.1 and 7.2. The result of Theorem 7.6 shows that the lower bounds are optimal in the context of full bubble model graphs. However, our forbidden induced subgraphs may contain proper induced subgraphs of the same clique-width or linear clique-width, so that $Z_k, S_k, S_k^+, M_{k,k',l}$ may not be minimal forbidden induced subgraphs for graphs of clique-width or linear clique-width at most $k + 1$. We discuss this minimality question in an appendix note [19], where we show that each proper induced subgraph of Z_k and S_k and $M_{k,1,l}$ has linear clique-width at most $k+1$ and each proper induced subgraph of Z_k and S_k^+ has clique-width at most $k + 1$. So, Z_k and S_k^+, for example, are in fact minimal forbidden induced subgraphs for graphs of clique-width at most $k + 1$.

Finally, we want to repeat that the linear clique-width of connected full bubble model graphs without true twins can be larger than their clique-width. According to Theorem 7.5, the two parameters differ on graphs with k-models with small separators. Is it possible that clique-width and linear clique-width differ arbitrarily? In fact, this is not the case, since every k-model with small separators is embeddable into an open $(k+1)$-model. It follows that the linear clique-width of full bubble model graphs is equal to their clique-width up to a small additive constant. The construction of our clique-width expressions also shows that the optimal clique-width expressions for full bubble model graphs are easy: they are either linear expressions or their implicit tree structure has very small pathwidth. It turns out that each full bubble model graph has an optimal clique-width expression whose implicit tree structure has pathwidth at most 2. We keep these observations on this informal and intuitive level, but we hope that these observations can be helpful in the future and stimulate future research.

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