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Clique-width with an inactive label

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Abstract

An inactive label in a clique-width expression cannot be used to create edges, and vertices that are labelled inactive have already received their incident edges. We study properties of clique-width expressions with inactive labels. The major results are: a characterisation of the distance-hereditary graphs as a syntactic clique-width class, a characterisation of the linear clique-width of disconnected graphs, and the complete set of disconnected minimal graphs of linear clique-width at least 4.

1 Introduction

Graph theory knows many graph representations. Graph representations are often used to succinctly represent graphs and to provide easy access to structural properties of graphs. Graph representations are of theoretical and practical importance. An example are clique-width expressions [9]. A clique-width expression defines a construction procedure that iterates this routine: take the disjoint union of two already constructed induced subgraphs and add the missing edges between the subgraphs. The special feature of this construction procedure is that edges are not added between vertices but between groups of vertices. Groups are identified by labels, and vertices with the same label belong to the same group. The number of different labels necessary during the construction is measured, and the smallest possible value is the clique-width of the graph. Clique-width has strong algorithmic applications, since many generally hard problems are efficiently solvable on graphs of bounded clique-width [10, 26].

There are two basic questions in the study of clique-width: about the clique-width of a given graph, and about the graphs of a given clique-width bound. Both questions are about a *syntactic property* of clique-width expressions, namely a bound on the number of used labels during the construction procedure. The number of used labels is a syntactic property, since this number is explicitly expressed in the clique-width expressions. We can so say that the clique-width parameter is a syntactic property of clique-width expressions. The two basic questions for clique-width can be seen as specific questions in the following two research directions on clique-width: given a graph, determine syntactic properties of its clique-width expressions, and given syntactic properties of clique-width expressions, determine the represented graphs. This paper is purely dedicated to a study in the latter research direction.

Both described research directions have been investigated. We briefly summarise results about syntactic properties of clique-width expressions. Some combinatorial results: forbidden

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induced subgraphs and special types of iterative constructions define graph classes of bounded clique-width [11, 13, 6, 4, 5, 28], closure properties define graph classes of unbounded clique-width [21]. Some algorithmic results: the clique-width and linear clique-width decision problems are NP-hard [12], polynomial-time algorithms for computing the clique-width or linear clique-width of some graph classes of unbounded clique-width or linear clique-width exist [13, 19, 23, 2].

This paper is purely dedicated to the study of graphs that have clique-width expressions of specific syntactic properties. The simplest such property is a bound on the number of used labels, which is a bound on the clique-width. Some results are known. Algorithmic results: graphs of clique-width at most 2 and 3 and of linear clique-width at most 2 and 3 are polynomial-time recognisable [11, 7, 14, 17]. Combinatorial results: the graphs of clique-width at most 2 are exactly the cographs [11], equivalently the P_4-free graphs [8], the graphs of linear clique-width at most 2 are the graphs that do not contain P_4 and $2K_2$ and co-$2P_3$ as an induced subgraph [14], some forbidden induced subgraphs for graphs of linear clique-width at most 3 are known [17, 18], and single further forbidden induced subgraphs for larger bounds on the clique-width and linear clique-width are known [13, 24]. So far, the graphs of clique-width and of linear clique-width at most 2 are the only graphs with clique-width expressions of specific syntactic properties that are completely known.

The syntactic property of clique-width expressions that we study in this paper is the existence of an *inactive label*: we are interested in the graphs that have clique-width expressions with an inactive label. During the evaluation of a clique-width expression, labels specify the groups the vertices belong to, and they implicitly specify the neighbourhoods of the vertices. A label is called *inactive* if its vertices will never again receive new edges. Vertices that are labelled with an inactive label already see their full neighbourhoods, and will not receive a different label. We study the graphs that have clique-width expressions with an inactive label. Among the major results of this paper, we will completely characterise the graphs of clique-width at most 3 and of linear clique-width at most 3 with an inactive label.

Inactive labels in clique-width expressions first appeared implicitly in the study of linear clique-width of disconnected graphs. It is known that each disconnected graph has a linear clique-width expression that constructs the connected components separately by assigning a special label to the vertices of already completed connected components [17]. This special label is an inactive label. A consequence of the specialties of linear clique-width expressions is that linear clique-width is not invariant with respect to the disjoint union of graphs; this means that the linear clique-width of a disconnected graph may be strictly larger than the linear clique-width of each of its connected components. A simple example is the disjoint union of complete graphs on at least two vertices each: complete graphs have linear clique-width at most 2, and their disjoint union has linear clique-width 3, since $2K_2$ is an induced subgraph. The linear clique-width of disconnected graphs cannot be determined from the linear clique-width of its connected components. Our first main result in this paper is a characterisation of the linear clique-width of disconnected graphs, and this characterisation is based on inactive labels. We even generalise this characterisation result and consider connected graphs with a small separator. This result is surprising and has no equivalent prior to this work.

In a second part of the paper, we study the graphs that have clique-width expressions with at most three labels including an inactive label. We will completely characterise these graphs. First, we consider general clique-width expressions, and we show that the graphs which have clique-width expressions with at most three labels and an inactive label are precisely the distance-hereditary graphs. It is known that distance-hereditary graphs are of clique-width at most 3 [13], and there are graphs of clique-width at most 3 that are not distance-hereditary, such as the chordless cycle on five vertices. It is therefore a surprise that the distance-hereditary graphs admit a characterisation by a purely syntactic property of clique-width expressions. After this characterisation, we restrict to linear clique-width expressions with at most three labels and an inactive label. We will precisely characterise also these graphs, by giving the set of the minimal forbidden induced subgraphs and by an explicit graph-structural description. Combining this result with the characterisation of the linear clique-width of disconnected graphs, we are able to give the complete set of disconnected minimal forbidden induced subgraphs for the graphs of linear clique-width at most 3. This is a big step toward a complete list of forbidden induced subgraphs of linear clique-width at most 3 and an understanding of graphs of bounded (linear) clique-width.

Organisation of the paper. We define clique-width and linear clique-width and inactive labels in Section 2, and we give some basic facts. We study the linear clique-width of graphs with small separators in Section 3, and in Sections 4 and 5, we characterise the graphs with clique-width expressions with at most three labels and an inactive label and with linear clique-width expressions with at most three labels and an inactive label. The Conclusions summarises our results and points out some further useful results that were established in the paper.

2 Definitions and notation, clique-width, and inactive labels

Basic graph terminology. The graphs in this paper are simple, finite, undirected. For a graph $G = (V, E)$, $V = V(G)$ is the *vertex set* and $E = E(G)$ is the *edge set* of G. Edges are denoted as uv, where $u \neq v$. If uv is an edge of G then u and v are *adjacent* in G and u and v are a *neighbour* of each other in G, and if uv is not an edge of G then u and v are *non-adjacent* in G. The edge uv of G is *incident* to u and v in G. The *(open) neighbourhood* of a vertex u of G, denoted as $N_G(u)$, is the set of the neighbours of u in G, and $N_G[u] =_{\mathrm{def}} N_G(u) \cup \{u\}$ is the *closed neighbourhood* of u in G. The *degree* of a vertex is the number of its neighbours in G. For x a vertex of G, if $N_G(x) = \emptyset$ then x is called *isolated*, and if $N_G[x] = V(G)$ then x is called *universal*. If G has no vertices then G is called *empty*; otherwise, G is *non-empty*. If the vertices of G are pairwise non-adjacent then G is called *edgeless*, and if the vertices of G are pairwise adjacent then G is called *complete*.

Let G and H be graphs. We say that H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $X \subseteq V(G)$. The set X *induces* the subgraph H of G, also denoted as $G[X]$, if $V(H) = X$ and for every vertex pair u, v of H, $uv \in E(H)$ if and only if $uv \in E(G)$. We write $G \setminus X$ instead of $G[V(G) \setminus X]$, and for $X = \{x\}$, we write $G{-}x$ instead of $G \setminus \{x\}$. We can say that $G{-}x$ is the graph obtained from G by deleting vertex x and its incident edges. For u, v a pair of possibly non-adjacent vertices of G, where $u \neq v$, $G{+}uv$ is the graph on vertex set $V(G)$ and with edge set $E(G) \cup \{uv\}$, i.e., the graph obtained from G by adding the edge uv. We say that H is an *induced subgraph* of G if there is $Y \subseteq V(G)$ such that $H = G[Y]$. We say that G *contains* H *as an induced subgraph* if H can be *embedded into* G, which means that there is a total, injective mapping $\varphi : V(H) \to V(G)$ such that for every vertex pair u, v of H, $uv \in E(H)$ if and only if

$\varphi(u)\varphi(v) \in E(G)$.

Let G be a graph. Let a, b be a vertex pair of G, and let r be an integer with $r \geq 0$. An *a, b-path of length r of G* is a sequence $(x_0, \ldots, x_r)$ of pairwise different vertices of G such that $x_0 = a$ and $x_r = b$ and $x_{i-1}x_i \in E(G)$ for every $1 \leq i \leq r$. The a, b-path $(x_0, \ldots, x_r)$ is called *chordless* if $x_i x_j \notin E(G)$ for every index pair i, j with $0 \leq i < i+1 < j \leq r$. Graph G is *connected* if for every vertex pair u, v of G, G has a u, v-path; otherwise, G is *disconnected*. The maximal connected induced subgraphs of a graph are called *connected components*. For G and H vertex-disjoint graphs, the *disjoint union* of G and H, denoted as $G \oplus H$, is the graph on vertex set $V(G) \cup V(H)$ and with edge set $E(G) \cup E(H)$. Note that the disjoint union of non-empty graphs is a disconnected graph.

In the course of the paper, we will encounter *marked* and *labelled* graphs. The given definitions and notation are extended to these graphs in a natural fashion.

Clique-width and linear clique-width. Let k be an integer with $k \geq 1$. The families of the k-expressions and the linear k-expressions, denoted as respectively $\mathcal{E}(k)$ and $\mathcal{E}_{\text{lin}}(k)$, are inductively defined:

- *atomic expression*
  $() \in \mathcal{E}(k)$ and $() \in \mathcal{E}_{\text{lin}}(k)$
- *one-step extension*
  - for $\delta \in \mathcal{E}(k)$ and $s, o \in \{1, \ldots, k\}$ with $s \neq o$:
    $\eta_{s,o}(\delta), \rho_{s \to o}(\delta) \in \mathcal{E}(k)$, and if $\delta \in \mathcal{E}_{\text{lin}}(k)$ then $\eta_{s,o}(\delta), \rho_{s \to o}(\delta) \in \mathcal{E}_{\text{lin}}(k)$
  - for $\beta, \delta \in \mathcal{E}(k)$ and $o \in \{1, \ldots, k\}$ and u a vertex name:
    $\beta \oplus \delta, \delta \oplus o(u) \in \mathcal{E}(k)$, and if $\delta \in \mathcal{E}_{\text{lin}}(k)$ then $\delta \oplus o(u) \in \mathcal{E}_{\text{lin}}(k)$.

It is as easy as important to observe: $\mathcal{E}_{\text{lin}}(k) \subseteq \mathcal{E}(k)$. A *$k$-labelled graph* is an ordered pair $\Gamma = (G, \ell)$ where G is a graph and $\ell : V(G) \to \{1, \ldots, k\}$ is a mapping that assigns a label to each vertex of G; the *vertices* and *edges* of Γ are the vertices and edges of G, and $G = [\Gamma] = [(G, \ell)]$. Let $\alpha \in \mathcal{E}(k)$. The k-labelled graph $\text{val}(\alpha)$, that is *represented* by α, is inductively defined:

- if $\alpha = ()$
  $\text{val}(\alpha)$ is the k-labelled graph with empty vertex set
- if $\alpha = \eta_{s,o}(\delta)$
  $\text{val}(\alpha)$ is obtained from $\text{val}(\delta)$ by adding all missing edges between the vertices with label s and with label o
- if $\alpha = \rho_{s \to o}(\delta)$
  $\text{val}(\alpha)$ is obtained from $\text{val}(\delta)$ by assigning label o to all vertices with label s
- if $\alpha = \beta \oplus \delta$
  $\text{val}(\alpha)$ is the disjoint union of $\text{val}(\beta)$ and $\text{val}(\delta)$
- if $\alpha = \delta \oplus o(u)$
  $\text{val}(\alpha)$ is obtained from $\text{val}(\delta)$ by adding a new vertex with name u and label o; the new vertex u has no neighbours in $\text{val}(\alpha)$.

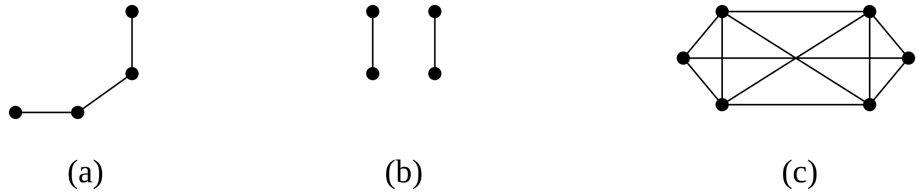

Figure 1: The minimal forbidden induced subgraphs for the graphs of clique-width at most 2 and of linear clique-width at most 2: (a) P_4 and (b) $2K_2$ and (c) co-$2P_3$.

The graph *represented* by α is $[\mathrm{val}(\alpha)]$, that is the graph G with $(G, \ell) = \mathrm{val}(\alpha)$. For G a graph and $\alpha \in \mathcal{E}(k)$, we say that α is a k-expression for G if $G = [\mathrm{val}(\alpha)]$, and we say that α is a linear k-expression for G if $G = [\mathrm{val}(\alpha)]$ and $\alpha \in \mathcal{E}_{\mathrm{lin}}(k)$. The *clique-width* of a graph G, denoted as $\mathrm{cwd}(G)$, is the smallest integer k such that G has a k-expression, and the *linear clique-width* of a graph G, denoted as $\mathrm{lcwd}(G)$, is the smallest integer k such that G has a linear k-expression.

The graphs of small clique-width and linear clique-width are completely known. It is easy to observe from the definition of $\mathcal{E}(k)$ and $\mathcal{E}_{\mathrm{lin}}(k)$ that the edgeless graphs are exactly the graphs of clique-width at most 1, and they are exactly the graphs of linear clique-width at most 1. The graphs of clique-width at most 2 and of linear clique-width at most 2 are more complex. They can be characterised by (forbidden) induced subgraphs. Because of future applications in the paper, we state the characterisation results as lower-bound results. The graphs in the statements are depicted in Figure 1.

Theorem 2.1 ([11, 14])**.** *Let G be a graph.*

1) $\mathrm{cwd}(G) \geq 3$ *if and only if G contains P_4 as an induced subgraph.*

2) $\mathrm{lcwd}(G) \geq 3$ *if and only if G contains P_4 or $2K_2$ or* co-$2P_3$ *as an induced subgraph.*

Let G be a graph. A *vertex ordering* for G is a linear arrangement $\sigma = \langle u_1, \ldots, u_n \rangle$ of the vertices of G. For a vertex pair x, y of G, we write $x \prec_\sigma y$ if $x = u_i$ and $y = u_j$ and $i < j$. We associate linear k-expressions with vertex orderings, that we define inductively. Let $k \geq 1$, and let $\alpha \in \mathcal{E}_{\mathrm{lin}}(k)$:

- the vertex ordering associated with $\alpha = ()$ is $\langle\rangle$, and
  the vertex ordering associated with $\alpha = \eta_{s,o}(\delta)$ and $\alpha = \rho_{s\to o}(\delta)$ is the vertex ordering associated with δ
- for $\alpha = \delta \oplus o(u)$ and $\langle u_1, \ldots, u_n \rangle$ the vertex ordering associated with δ
  $\langle u_1, \ldots, u_n, u \rangle$ is the vertex ordering associated with α.

Vertex orderings are a useful tool in the study of linear k-expressions. Another useful tool are subexpressions, that are defined in two steps. Let $k \geq 1$, and let $\alpha, \gamma \in \mathcal{E}(k)$. We say that γ is a *reduction* of α if γ is constructed through one-step extensions analogous to α but may skip one-step extensions of the form $\delta \oplus o(u)$. Informally, γ is a reduction of α if γ can be obtained from α by deleting vertices. Reductions are particularly interesting when considering induced subgraphs. We say that γ is a *subexpression* of α if one of the following applies:

- γ is a reduction of α
- there are $\beta, \delta \in \mathcal{E}(k)$ and γ is a subexpression of δ and either $\alpha = \eta_{s,o}(\delta)$ or $\alpha = \rho_{s \to o}(\delta)$ or $\alpha = \delta \oplus o(u)$ or $\alpha = \beta \oplus \delta$ or $\alpha = \delta \oplus \beta$.

Informally, we can say that γ is a subexpression of α if γ can be obtained from α by un-doing late one-step extensions and then deleting further vertices. Viewing α as an operation tree, γ corresponds to a subtree. Let G be a graph, let $\alpha \in \mathcal{E}(k)$ be a k-expression for G, and let γ be a subexpression of α. Let u, v be an adjacent vertex pair of G, and assume that u is a vertex of $\mathrm{val}(\gamma)$. We say that v is a *non-visible* neighbour of u in $\mathrm{val}(\gamma)$ if uv is not an edge of $\mathrm{val}(\gamma)$. Note that v may or may not be a vertex of $\mathrm{val}(\gamma)$. Let H be a subgraph of G, that will mostly be an induced subgraph of G. We say that γ is a *full subexpression* for H if $V(\mathrm{val}(\gamma)) = V(H)$. Full subexpressions will be particularly useful when proving properties about vertex labels for lower-bound results.

Sample application: Vertex orderings, subexpressions and non-visible neighbours are useful for obtaining lower bounds on the (linear) clique-width. We discuss a local property of linear 3-expressions. Let G be a graph on at least two vertices and without isolated vertices, and assume $\mathrm{lcwd}(G) \leq 3$. Let $\alpha \in \mathcal{E}_{\mathrm{lin}}(3)$ be a 3-expression for G with associated vertex ordering $\langle u_1, \ldots, u_n \rangle$. We assume $u_{n-1}u_n \notin E(G)$. We consider a specific subexpression of α. Let $\gamma = \delta \oplus o(u_{n-1})$ be a subexpression of α that is a full subexpression for $G - u_n$, and let $\Gamma =_{\mathrm{def}} \mathrm{val}(\gamma)$. Let L_1, L_2, L_3 be the sets of the vertices of Γ with label 1, 2, 3, respectively. By symmetry, we may assume $u_{n-1} \in L_3$. Since u_{n-1} and u_n are not isolated vertices of G and they are non-adjacent, the following is the case:

- $N_G(u_{n-1}) = L_1$ or $N_G(u_{n-1}) = L_2$ or $N_G(u_{n-1}) = L_1 \cup L_2$
- $N_G(u_n) = L_1$ or $N_G(u_n) = L_2$ or $N_G(u_n) = L_1 \cup L_2$;
  recall here that $L_3 \cap N_G(u_n) \neq \emptyset$ means $L_3 \subseteq N_G(u_n)$, and $u_{n-1} \in N_G(u_n)$ in particular, a contradiction.

It follows: either $N_G(u_{n-1}) = N_G(u_n)$ or $N_G(u_{n-1}) \cap N_G(u_n) = \emptyset$ or $N_G(u_{n-1}) \subset N_G(u_n)$ or $N_G(u_n) \subset N_G(u_{n-1})$. We will draw an easy but useful consequence at the end of the section. $\square$

Inactive labels. Let $k \geq 1$, and let $\alpha \in \mathcal{E}(k)$. Let l be a label, where $l \in \{1, \ldots, k\}$. We say that α satisfies the *inactivity condition* for label l if one of the following applies:

- $\alpha = ()$
- $\alpha = \delta \oplus o(u)$ and δ satisfies the inactivity condition for label l, or
  $\alpha = \beta \oplus \delta$ and β and δ satisfy the inactivity condition for label l
- $\alpha = \eta_{s,o}(\delta)$ and $s \neq l$ and $o \neq l$ and δ satisfies the inactivity condition for label l, or
  $\alpha = \rho_{s \to o}(\delta)$ and $s \neq l$ and δ satisfies the inactivity condition for label l.

We say that α is a k-expression *with label l as an inactive label* if α satisfies the inactivity condition for label l. Let $\mathcal{E}^{\mathrm{inac}}(k)$ and $\mathcal{E}^{\mathrm{inac}}_{\mathrm{lin}}(k)$ denote the families of respectively k-expressions and linear k-expressions with label 1 as an inactive label. The following inclusions are clear:

$\mathcal{E}^{\text{inac}}_{\text{lin}}(k) \subseteq \mathcal{E}^{\text{inac}}(k)$ and $\mathcal{E}^{\text{inac}}_{\text{lin}}(k) \subseteq \mathcal{E}_{\text{lin}}(k)$ and $\mathcal{E}^{\text{inac}}(k) \subseteq \mathcal{E}(k)$. It is also not difficult to see, and we will give the necessary formal arguments in Section 3.2: if a graph G has a k-expression with some label as an inactive label then there is $\alpha \in \mathcal{E}^{\text{inac}}(k)$ that is a k-expression for G; analogously for linear k-expressions. It is therefore no restriction to define $\mathcal{E}^{\text{inac}}(k)$ and $\mathcal{E}^{\text{inac}}_{\text{lin}}(k)$ for k-expressions with label 1 as an inactive label. For a graph G, $\text{cwd}_{\text{inac}}(G)$ denotes the smallest integer k such that G has a k-expression with an inactive label, and, analgously, $\text{lcwd}_{\text{inac}}(G)$ denotes the smallest integer k' such that G has a linear k'-expression with an inactive label. Consequently, for k and k' integers with $k \geq 1$ and $k' \geq 1$,

- $\text{cwd}_{\text{inac}}(G) \leq k$ if and only if $\mathcal{E}^{\text{inac}}(k)$ has a k-expression for G
- $\text{lcwd}_{\text{inac}}(G) \leq k'$ if and only if $\mathcal{E}^{\text{inac}}_{\text{lin}}(k')$ has a k'-expression for G.

The following properties are straightforward to observe.

Lemma 2.2. *Let G be a graph.*

1) $\text{cwd}(G) \leq \text{cwd}_{\text{inac}}(G) \leq \text{cwd}(G) + 1$, *and*
$\text{lcwd}(G) \leq \text{lcwd}_{\text{inac}}(G) \leq \text{lcwd}(G) + 1$

2) $\text{cwd}_{\text{inac}}(G) \leq \text{lcwd}_{\text{inac}}(G)$

3) $\text{cwd}_{\text{inac}}(G) \leq 2$ *if and only if* $\text{lcwd}(G) = 1$.

Proof. The first and second statement are a direct consequence of the above observed k-expression family inclusions. For the third statement, it suffices to observe that a 2-expression with an inactive label cannot contain an η-operation, so that each 2-expression with label 1 or label 2 as an inactive label defines an edgeless graph, and the edgeless graphs are exactly the graphs of linear clique-width 1. ∎

The third statement of the lemma implies that the smallest non-trivial value of k about k-expressions with an inactive label is 3.

The following two lemmas are particularly useful when proving lower bounds.

Lemma 2.3. *Let G be a graph and let H be an induced subgraph of G. Then,*

1) $\text{cwd}_{\text{inac}}(H) \leq \text{cwd}_{\text{inac}}(G)$

2) $\text{lcwd}_{\text{inac}}(H) \leq \text{lcwd}_{\text{inac}}(G)$.

Proof. Let $k \geq 1$, let $\alpha \in \mathcal{E}^{\text{inac}}(k)$, and assume that α is a k-expression for G. Then, α has a subexpression γ that is a reduction of α and a k-expression for H. Since α satisfies the inactivity condition for label 1, so γ satisfies the inactivity condition for label 1, and $\gamma \in \mathcal{E}^{\text{inac}}(k)$. The two inequalities directly follow. ∎

Lemma 2.4. *Let G be a graph on at least two vertices and without isolated vertices, and assume* $\text{lcwd}_{\text{inac}}(G) \leq 3$. *Let $\alpha \in \mathcal{E}^{\text{inac}}_{\text{lin}}(3)$ be a 3-expression for G, and let $\langle u_1, \ldots, u_n \rangle$ be the vertex ordering associated with α. If $u_{n-1}u_n \notin E(G)$ then $N_G(u_{n-1}) = N_G(u_n)$.*

Proof. Assume $u_{n-1} \notin N_G(u_n)$. Let $\gamma = \delta \oplus o(u_{n-1})$ a subexpression of α that is a full subexpression for $G - u_n$, and let $\Gamma =_{\text{def}} \text{val}(\gamma)$. Let L_1, L_2, L_3 be the sets of the vertices of Γ with label 1, 2, 3, respectively. Since G has no isolated vertices and all neighbours of u_{n-1} in G are non-visible in Γ, u_{n-1} has a non-visible neighbour in Γ, so that $u_{n-1} \notin L_1$. By symmetry, we assume $u_{n-1} \in L_3$. As we have shown in the sample application above, $N_G(u_{n-1}) \subseteq L_1 \cup L_2$ and $N_G(u_n) \subseteq L_1 \cup L_2$, and since α and γ have label 1 as an inactive label, $N_G(u_{n-1}) = L_2 = N_G(u_n)$ follows. ■

3 Linear clique-width and graphs with (very) small separators

We consider graphs that are disconnected or that have a cut edge or a cut vertex. The three cases are unified by demanding a vertex whose deletion results in a disconnected graph. We show that such graphs have large induced subgraphs with linear clique-width expressions with an inactive label. We call this main technical result the "decomposition lemma". The decomposition lemma "reduces" the linear clique-width of induced subgraphs. The decomposition lemma is applied in order to characterise the linear clique-width of disconnected graphs and of graphs with a cut edge, and we apply the decomposition lemma to graphs with a cut vertex. In a first part, we prove the linear clique-width characterisation results, and in a second part, we prove the decomposition lemma and discuss further consequences.

3.1 Characterisation results

We want to characterise the linear clique-width of graphs with small separators through the linear clique-width of large induced subgraphs. Two implications are to be shown. To begin, we show the easy implication of the characterisation results, that constructs linear clique-width expressions for larger graphs from linear clique-width expressions for smaller graphs. Let G and H be vertex-disjoint graphs and let a and b be vertices of respectively G and H. By $G[a \bowtie b]H$, we denote the graph that is obtained from the disjoint union of $G - a$ and $H - b$ by adding a new vertex w with neighbourhood $N_G(a) \cup N_H(b)$. We can say that $G[a \bowtie b]H$ is the graph obtained from *joining* a and b.

Lemma 3.1. *Let G and H be vertex-disjoint graphs. Let k be an integer with $k \geq 3$, and assume* $\text{lcwd}(G) \leq k$ *and* $\text{lcwd}_{\text{inac}}(H) \leq k$*. Then, there is a vertex pair a, b with $a \in V(G)$ and $b \in V(H)$ such that the following is the case:*

1) $\text{lcwd}(G \oplus H) \leq k$

2) $\text{lcwd}((G \oplus H) + ab) \leq k$

3) $\text{lcwd}(G[a \bowtie b]H) \leq k$.
If G is not a complete graph then a can be chosen as a vertex that is not universal in G.

If $\text{lcwd}_{\text{inac}}(G) \leq k$ *then the three statements analogously hold for* $\text{lcwd}_{\text{inac}}$*, except for the third statement in the following situation: $k = 3$ and* $\text{lcwd}(G) = 2$ *and G has a universal vertex.*

Proof. The three statements can be proved very similarly. Let $\beta \in \mathcal{E}_{\text{lin}}(k)$ be a k-expression for G and let $\delta \in \mathcal{E}_{\text{lin}}^{\text{inac}}(k)$ be a k-expression for H. Let $\sigma = \langle u_1, \dots, u_n \rangle$ and $\tau = \langle v_1, \dots, v_m \rangle$ be the vertex orderings associated with respectively β and δ. Let $b =_{\text{def}} v_1$, and let $\delta = \delta'(\delta'' \oplus o(v_1))$. Note that $\delta'(() \oplus o(v_1))$ is a k-expression for H. We will use b and δ' in the proofs of the second and the third statement.

To prove the first statement, it suffices to observe that

$$\alpha^1 \;=_{\text{def}}\; \delta'\Big(\rho_{k\to 1}\Big(\cdots\Big(\rho_{2\to 1}(\beta)\Big)\cdots\Big) \oplus o(v_1)\Big)$$

is a linear k-expression for $G \oplus H$. It is important to recall that δ has label 1 as an inactive label. If $\beta \in \mathcal{E}_{\text{lin}}^{\text{inac}}(k)$ then β and δ have label 1 as an inactive label, and α^1 has label 1 as an inactive label, so that $\alpha^1 \in \mathcal{E}_{\text{lin}}^{\text{inac}}(k)$ in this case.

We prove the second statement. Let $a =_{\text{def}} u_n$. Since $k \geq 3$, we can modify β into a linear k-expression β' such that in $\text{val}(\beta')$: a has label 3 and all other vertices have label 1. This modification can be achieved, for instance, by changing the labels of the non-neighbours of a to label 1 and the labels of the neighbours of a to label 2 before inserting a, and a is inserted with label 3. Note that this modification of β into β' is possible also in case of $\beta \in \mathcal{E}_{\text{lin}}^{\text{inac}}(k)$, particularly since no neighbour of a can have label 1 before the insertion of a.

Let

$$\alpha^2 \;=_{\text{def}}\; \delta'\Big(\rho_{2\to o}\Big(\rho_{3\to 1}\Big(\eta_{2,3}\Big(\beta' \oplus 2(b)\Big)\Big)\Big)\Big)\,.$$

Note that $\rho_{2\to o}$ is only necessary (and valid) if $o \neq 2$. It is straightforward to verfiy that α^2 is a linear k-expression for $(G \oplus H)+ab$, and it is not difficult to see that α^2 satisfies the inactivity condition for label 1 if β, and thus β', satisfies the inactivity condition for label 1.

We prove the third statement. We choose vertex a and find a k-expression for G of special properties. If G is a complete graph then let $a =_{\text{def}} u_n$, and if G is not a complete graph then let $a =_{\text{def}} u_p$ for p largest with $1 \leq p \leq n$ such that u_p is not a universal vertex of G. If $a = u_n$ then let $\beta'' =_{\text{def}} \beta'$, where β' is the k-expression of the proof of the second statement.

As the other case, assume $a \neq u_n$, i.e., $p < n$. Then, $u_{p+1}, \dots, u_n$ are universal vertices of G. Let γ be a subexpression of β such that γ is a k-expression for $G \setminus \{u_{p+1}, \dots, u_n\}$. Note here that $\langle u_1, \dots, u_p \rangle$ is the vertex ordering associated with γ. Analogous to the definition of β', we obtain γ' from γ such that γ' is a linear k-expression for $G \setminus \{u_{p+1}, \dots, u_n\}$ and in $\text{val}(\gamma')$: u_p has label 3 and all other vertices have label 1. A linear k-expression for $G \setminus \{u_{p+2}, \dots, u_n\}$ is:

$$\rho_{2\to 1}\Big(\eta_{2,1}\Big(\eta_{2,3}\Big(\gamma' \oplus 2(u_{p+1})\Big)\Big)\Big)\,.$$

Iterating the construction, we obtain a linear k-expression β'' for G such that in $\text{val}(\beta'')$: a has label 3 and all other vertices have label 1.

Let

$$\alpha^3 \;=_{\text{def}}\; \delta'\Big(\rho_{3\to o}(\beta'')\Big)\,;$$

recall that $\rho_{3\to o}$ is valid and necessary only in case of $o \neq 3$. It is straightforward to verify that α^3 is indeed a linear k-expression for $G[a{\bowtie}b]H$.

It remains to consider k-expressions with an inactive label, more precisely, if $\beta \in \mathcal{E}_{\text{lin}}^{\text{inac}}(k)$. We need to show that β'' of the required properties exists. If $k \geq 4$ or if u_n is not a universal vertex of G then we can apply the above construction for β'' directly or with minor modifications, namely by using labels $2, 3, 4$ instead of $1, 2, 3$.

If $k = 3$ and u_n is a universal vertex of G then the construction of β'' as described is not possible. This is a special situation. Let $\beta_n = \beta_{n-1} \oplus o'(u_n)$ be a subexpression of β that is a full subexpression for G. Since u_n is adjacent to each other vertex of G, each vertex of G has a non-visible neighbour in $\mathrm{val}(\beta_n)$, and thus, no vertex of $\mathrm{val}(\beta_n)$ has label 1; moreover, the vertices of $\mathrm{val}(\beta_{n-1})$ have the same label and that is different from label o'. As a consequence, $[\mathrm{val}(\beta_{n-1})] = G{-}u_n$, and β_{n-1} is a linear 3-expression for $G{-}u_n$ and for each subexpression γ'_{n-1} of β_{n-1}, no vertex has label 1 in $\mathrm{val}(\gamma'_{n-1})$. Thus, $G{-}u_n$ has a linear 3-expression that does not use label 1, so that $G{-}u_n$ has a linear 2-expression, and $\mathrm{lcwd}(G{-}u_n) \leq 2$, and since u_n is a universal vertex of G, $\mathrm{lcwd}(G) = 2$. ∎

We established three easy results about how to join graphs without increasing the linear clique-width. The proof of Lemma 3.1 explores some constructions, and it also presents some easy arguments to reason about clique-width properties on the basis of given clique-width expressions. As an example, we refer to the special situation in the proof of the third statement. It is not difficult to see that more complex join operations are possible, that yield similar results. As an example, instead of connecting a and b through an edge in the second statement, a and b may be connected through a chordless path of arbitrary length. This raises the question of how "good" these join operations are. We are going to show that they are in fact best possible.

The approach to proving the asked optimality of the join operations of Lemma 3.1 is by giving equivalence statements. We are going to show that the assumptions of Lemma 3.1 are not only sufficient but also necessary. The main technical tool is the already mentioned decomposition lemma, that we state here in a simplified version and that we prove in the next subsection.

Lemma 3.2 (Decomposition lemma, simplified version)**.** *Let G be a graph and let w be a vertex of G. Assume that $G{-}w$ is disconnected. Let C be a connected component of $G{-}w$ on at least two vertices, and let $D =_{\mathrm{def}} G \setminus V(C)$. Assume that D is connected and has at least two vertices, and assume $V(C) \not\subseteq N_G(w)$.*

Let k be an integer with $k \geq 3$. Assume $\mathrm{lcwd}(G) \leq k$. Let $\alpha \in \mathcal{E}_{\mathrm{lin}}(k)$ be a k-expression for G and let $\langle u_1, \ldots, u_n \rangle$ be the vertex ordering associated with α. Assume that u_1 is a vertex of C. Then, $\mathrm{lcwd}_{\mathrm{inac}}(D) \leq k$.

As a first and easy application of Lemma 3.2, we characterise the linear clique-width of disconnected graphs. Recall that this is not a trivial result, since the linear clique-width of disconnected graphs is not a function in the linear clique-width of the connected components. A simple example is $2K_2$ of Figure 1, that has linear clique-width 3 and each proper induced subgraph has linear clique-width at most 2.

Proposition 3.3. *Let G be a graph, and let k be an integer with $k \geq 1$.*

1) $\mathrm{lcwd}_{\mathrm{inac}}(G) \leq k$ *if and only if* $\mathrm{lcwd}_{\mathrm{inac}}(C) \leq k$ *for every connected component C of G.*

2) $\mathrm{lcwd}(G) \leq k$ *if and only if* $\mathrm{lcwd}(C) \leq k$ *for every connected component C of G and there is at most one connected component C' of G such that* $\mathrm{lcwd}_{\mathrm{inac}}(C') > k$.

Proof. If G is connected then the two statements of the proposition are obviously correct. It is equally easy to see that the two statements are correct if G is edgeless and if $k = 1$. We consider the case when $k = 2$. If $\mathrm{lcwd}_{\mathrm{inac}}(G) \leq 2$ then G is edgeless due to Lemma 2.2, and the first statement clearly holds. If $\mathrm{lcwd}(G) \leq 2$ then at most one connected component of G has at least two vertices, since G does not contain $2K_2$ as an induced subgraph due to Theorem 2.1, so that G is the disjoint union of a connected graph of linear clique-width at most 2 and an edgeless graph, and the second statement follows. In the following, we assume $k \geq 3$. Let $C_1, \ldots, C_r$ be the connected components of G, where $r \geq 2$.

We prove the first statement. If $\mathrm{lcwd}_{\mathrm{inac}}(G) \leq k$ then $\mathrm{lcwd}_{\mathrm{inac}}(C_i) \leq k$ for every $1 \leq i \leq r$ due to Lemma 2.3. For the converse, assume $\mathrm{lcwd}_{\mathrm{inac}}(C_i) \leq k$ for every $1 \leq i \leq r$. Let $G_1 =_{\mathrm{def}} C_1$, and let $G_i =_{\mathrm{def}} G_{i-1} \oplus C_i$ for $2 \leq i \leq r$. An iterative application of the first statement of Lemma 3.1 to G_{i-1} and C_i yields $\mathrm{lcwd}_{\mathrm{inac}}(G_i) \leq k$ for every $1 \leq i \leq r$, and since $G_r = G$, we conclude $\mathrm{lcwd}_{\mathrm{inac}}(G) \leq k$.

We prove the second statement. We may assume $\mathrm{lcwd}(C_1) \leq k$ and $\mathrm{lcwd}_{\mathrm{inac}}(C_i) \leq k$ for every $2 \leq i \leq r$. Thus, $\mathrm{lcwd}_{\mathrm{inac}}(C_2 \oplus \cdots \oplus C_r) \leq k$ due to the first statement of the proposition, and $\mathrm{lcwd}(G) \leq k$ by applying the first statement of Lemma 3.1 to C_1 and $C_2 \oplus \cdots \oplus C_r$.

For the converse, assume $\mathrm{lcwd}(G) \leq k$. Then, $\mathrm{lcwd}(C_i) \leq k$ for every $1 \leq i \leq k$. Let $\beta \in \mathcal{E}_{\mathrm{lin}}(k)$ be a k-expression for G with associated vertex ordering $\sigma = \langle u_1, \ldots, u_n \rangle$. We assume that u_1 is a vertex of C_1. We are going to show $\mathrm{lcwd}_{\mathrm{inac}}(C_i) \leq k$ for every $2 \leq i \leq r$. Let $2 \leq p \leq r$. Let $H_p =_{\mathrm{def}} C_1 \oplus C_p$. Observe that H_p is an induced subgraph of G, and thus, $\mathrm{lcwd}(H_p) \leq \mathrm{lcwd}(G)$. Let β_p be a subexpression of β that is a k-expression for H_p, and let σ_p be the vertex ordering associated with β_p. Recall that β_p is a linear k-expression for H_p, and σ_p is the restriction of σ to the vertices of H_p. It is a direct consequence that $u_1 = x$ or $u_1 \prec_{\sigma_p} x$ for every vertex x of H_p.

We are going to apply Lemma 3.2. If C_p has at most two vertices then $\mathrm{lcwd}_{\mathrm{inac}}(C_p) \leq 3 \leq k$. So, assume that C_p has at least three vertices. Let w be a vertex of C_p such that $C_p{-}w$ is connected; observe that w does exist. Then, $H_p{-}w$ has exactly two connected components, namely C_1 and $C_p{-}w$, and $C_p = H_p \setminus V(C_1)$ and C_p has at least two vertices and $V(C_1) \not\subseteq N_{H_p}(w)$. The assumptions of Lemma 3.2 are satisfied for H_p and w and β_p, and we obtain the desired result of $\mathrm{lcwd}_{\mathrm{inac}}(C_p) \leq k$. ■

The result of Proposition 3.3 characterises the linear clique-width of disconnected graphs. We give two applications of this characterisation, applications to connected graphs. Our first application is to connected graphs with a cut edge, i.e., to connected graphs with an edge whose removal yields a disconnected graph.

Proposition 3.4. *Let G and H be vertex-disjoint connected graphs, and let k be an integer with $k \geq 3$. Then,* $\mathrm{lcwd}(G \oplus H) \leq k$ *if and only if there is a vertex pair a, b with $a \in V(G)$ and $b \in V(H)$ such that* $\mathrm{lcwd}((G \oplus H){+}ab) \leq k$. *Analogously for* $\mathrm{lcwd}_{\mathrm{inac}}$.

Proof. Assume $\mathrm{lcwd}(G \oplus H) \leq k$. Due to the second statement of Proposition 3.3 and without loss of generality, $\mathrm{lcwd}(G) \leq k$ and $\mathrm{lcwd}_{\mathrm{inac}}(H) \leq k$. Thus, the assumptions of Lemma 3.1 are satisfied: there is a vertex pair a, b with $a \in V(G)$ and $b \in V(H)$ such that $\mathrm{lcwd}((G \oplus H){+}ab) \leq k$. Analogously, if $\mathrm{lcwd}_{\mathrm{inac}}(G \oplus H) \leq k$ then $\mathrm{lcwd}_{\mathrm{inac}}(G) \leq k$ and $\mathrm{lcwd}_{\mathrm{inac}}(H) \leq k$ due to Lemma 2.3, and $\mathrm{lcwd}_{\mathrm{inac}}((G \oplus H){+}ab) \leq k$ for some $a \in V(G)$ and $b \in V(G)$ due to Lemma 3.1.

We prove the converse. Let a, b be a vertex pair with $a \in V(G)$ and $b \in V(H)$, and let $F =_{\text{def}} (G \oplus H) + ab$. Observe that G and H are induced subgraphs of F. First, if $\text{lcwd}_{\text{inac}}(F) \leq k$ then $\text{lcwd}_{\text{inac}}(G) \leq k$ and $\text{lcwd}_{\text{inac}}(H) \leq k$ due to Lemma 2.3, and $\text{lcwd}_{\text{inac}}(G \oplus H) \leq k$ due to the first statement of Proposition 3.3.

Next, assume $\text{lcwd}(F) \leq k$. Then, $\text{lcwd}(G) \leq k$ and $\text{lcwd}(H) \leq k$. If G or H has at most three vertices then $\text{lcwd}_{\text{inac}}(G) \leq 3$ or $\text{lcwd}_{\text{inac}}(H) \leq 3$, and $\text{lcwd}(G \oplus H) \leq k$ due to the second statement of Proposition 3.3. We henceforth assume that G and H have at least four vertices each. Let $\alpha \in \mathcal{E}_{\text{lin}}(k)$ be a k-expression for F with associated vertex ordering $\langle u_1, \ldots, u_n \rangle$. We may assume $u_1 \in V(G)$. We verify the assumptions of Lemma 3.2: $F{-}b$ is disconnected and G is a connected component of $F{-}b$ and $H = F \setminus V(G)$ and H is connected and $V(G) \not\subseteq N_F(b)$, particularly since G has at least four vertices and $N_F(b) \cap V(G) = \{a\}$. Then, $\text{lcwd}_{\text{inac}}(H) \leq k$, and $\text{lcwd}(G \oplus H) \leq k$ due to the second statement of Proposition 3.3. ∎

Our second application of the characterisation in Proposition 3.3 is to connected graphs with a cut vertex, i.e., to connected graphs with a vertex whose removal yields a disconnected graph. To state a result of full strength, technical assumptions are necessary, that we want to avoid here, and that is why the given statement is weaker than actually possible. Let G be a graph and let $S \subseteq V(G)$. An *S-component* of G is an induced subgraph C of G such that $S \subseteq V(C)$ and $C \setminus S$ is a connected component of $G \setminus S$. We can say that an S-component of G is obtained from a connected component of $G \setminus S$ by adding the vertices in S.

Proposition 3.5. *Let k be an integer with $k \geq 3$. Let G be a connected graph and let w be a vertex of G. Assume that $G{-}w$ is disconnected and no $\{w\}$-component of G has w as a universal vertex. If $\text{lcwd}(G) \leq k$ then there is at most one $\{w\}$-component C of G such that $\text{lcwd}_{\text{inac}}(C) > k$.*

Proof. Assume $\text{lcwd}(G) \leq k$. Let $\alpha \in \mathcal{E}_{\text{lin}}(k)$ be a k-expression for G with associated vertex ordering $\sigma = \langle u_1, \ldots, u_n \rangle$ such that $w \neq u_1$ and $N_G(u_1) \neq \{w\}$. It is important to note that this assumption is indeed possible and such a k-expression α does exist, since there are also linear k-expressions for G with associated vertex orderings $\langle u_2, u_1, u_3, \ldots, u_n \rangle$ and $\langle u_3, u_2, u_1, u_4, \ldots, u_n \rangle$.

Let C be the $\{w\}$-component of G containing u_1; since $u_1 \neq w$, C is unique. Let D be an arbitrary $\{w\}$-component of G, where $D \neq C$. If D has at most three vertices then $\text{lcwd}_{\text{inac}}(D) \leq 3 \leq k$. Otherwise, D has at least four vertices. Let $G_D =_{\text{def}} G[V(C) \cup V(D)]$, let α_D be a subexpression of α that is a k-expression for G_D, and let σ_D be the vertex ordering associated with α_D. Recall that α_D is a linear k-expression for G_D and $u_1 = x$ or $u_1 \prec_{\sigma_D} x$ for every vertex x of G_D. We verify the assumptions of Lemma 3.2: $G_D{-}w$ is disconnected and $C{-}w$ is a connected component of $G_D{-}w$ on at least two vertices and $D = G_D \setminus V(C{-}w)$ and D is connected and has at least two vertices and $D{-}w$ is connected. Then, $\text{lcwd}_{\text{inac}}(D) \leq k$. ∎

The converse of Proposition 3.5 requires some technical assumptions, and it is proved analogous to the proof of Proposition 3.4 by applying the third statement of Lemma 3.1. The two results, Propositions 3.4 and 3.5, can be seen as additional characterisations of the linear clique-width of disconnected graphs. The usefullness of the two results is better understood when considering their contrapositions: they provide lower bounds on the linear clique-width of connected graphs with a cut vertex through the linear clique-width of special induced subgraphs. For example: if G and H are vertex-disjoint connected graphs with $\text{lcwd}_{\text{inac}}(G) > k$

and $\mathrm{lcwd}_{\mathrm{inac}}(H) > k$, where $k \geq 3$, then $\mathrm{lcwd}(G \oplus H) > k$ and $\mathrm{lcwd}((G \oplus H)+ab) > k$ for every vertex pair a, b with $a \in V(G)$ and $b \in V(H)$. We will discuss more concrete consequences in the Conclusions section at the end of the paper.

3.2 Proof of the decomposition lemma

We are going to prove the decomposition lemma, Lemma 3.2 of the preceding subsection. The lemma shows that a graph with a cut vertex has a large component of smaller linear clique-width. We split the proof into several parts, that we prove as individual results. We do this for two reasons: to better structure the proof, and to provide a guideline for the proof of similar results.

The proof of the decomposition lemma employs two invariants for clique-width expressions. Let k be an integer with $k \geq 1$. Let a be a label with $a \in \{1, \ldots, k\}$, and let $\delta \in \mathcal{E}(k)$. We say that label a is *used* in δ if label a occurs as s or o in the one-step extensions $\eta_{s,o}$ or $\rho_{s \to o}$ or $\delta \oplus o(v)$ in δ. By $\mathrm{used}(\delta)$, we denote the set of the labels from $\{1, \ldots, k\}$ that are used in δ. We also need to express the existence of an inactive label, that we formalise through $\mathrm{inac}(\delta)$: $\mathrm{inac}(\delta) \in \{0, 1, \ldots, k\}$, and if $\mathrm{inac}(\delta) = 0$ then δ has no used label as an inactive label, and if $\mathrm{inac}(\delta) = b$ for $b \in \{1, \ldots, k\}$ then $b \in \mathrm{used}(\delta)$ and δ has label b as an inactive label. It is important to note that an inactive label shall be a used label, and δ may have more than one inactive labels.

We consider the three types of one-step extensions separately. We begin with the easiest case.

Lemma 3.6. *Let k be an integer with $k \geq 3$. Let $\delta \in \mathcal{E}(k)$, and let $\alpha =_{\mathrm{def}} \rho_{s \to o}(\delta)$, where $s, o \in \{1, \ldots, k\}$ and $s \neq o$. Then, there is $\gamma \in \mathcal{E}(k)$ with $\mathrm{val}(\gamma) = \mathrm{val}(\alpha)$ such that the two conditions are satisfied:*

1) $\mathrm{used}(\gamma) = \mathrm{used}(\delta)$*, or*
$s \in \mathrm{used}(\delta)$ *and* $o \notin \mathrm{used}(\delta)$ *and* $\mathrm{used}(\gamma) = (\mathrm{used}(\delta) \setminus \{s\}) \cup \{o\}$

2) $\mathrm{inac}(\gamma) = \mathrm{inac}(\delta)$*, or*
$\mathrm{inac}(\delta) = s$ *and* $\mathrm{inac}(\gamma) = o$*,*

and if $\delta \in \mathcal{E}_{\mathrm{lin}}(k)$ then $\gamma \in \mathcal{E}_{\mathrm{lin}}(k)$.

Proof. Let $\Delta =_{\mathrm{def}} \mathrm{val}(\delta)$ and $\Gamma =_{\mathrm{def}} \mathrm{val}(\alpha)$. For $i \in \{1, \ldots, k\}$, let L_i and L'_i be the sets of the vertices of respectively Δ and Γ with label i. Observe the easy observation: $L_i = L'_i$ for $i \in \{1, \ldots, k\} \setminus \{s, o\}$, and $L'_s = \emptyset$ and $L'_o = L_s \cup L_o$. If $L_s = \emptyset$ then $\Gamma = \Delta$, and we can choose $\gamma =_{\mathrm{def}} \delta$, and the two conditions are clearly satisfied. Otherwise, $L_s \neq \emptyset$, and therefore, $s \in \mathrm{used}(\delta)$. If $\mathrm{inac}(\delta) = o$ then $o \in \mathrm{used}(\delta)$, and we can choose $\gamma =_{\mathrm{def}} \alpha$ and $\mathrm{inac}(\gamma) = o$, and the two conditions are satisfied. Otherwise, $\mathrm{inac}(\delta) \neq o$.

Obtain δ' from δ by replacing each occurrence of label s by label o and each occurrence of label o by label s. In other words, we exchange labels s and o in δ against each other. It is important to observe $[\Delta] = [\mathrm{val}(\delta')]$, and, more importantly, $\Gamma = \mathrm{val}(\rho_{s \to o}(\delta'))$. Assume $L_o \neq \emptyset$. Then, $o \in \mathrm{used}(\delta)$, and we let $\gamma =_{\mathrm{def}} \rho_{s \to o}(\delta')$. Observe $\mathrm{used}(\gamma) = \mathrm{used}(\delta)$, and γ satisfies the first condition. If $\mathrm{inac}(\delta) \neq s$ then we can choose $\mathrm{inac}(\gamma)$ such that $\mathrm{inac}(\gamma) = \mathrm{inac}(\delta)$, and if inac($\delta$) $= s$ then we can choose inac(γ) such that inac(γ) $= o$, and γ also satisfies the second condition.

Assume $L_o = \emptyset$. Then, $\Gamma = \mathrm{val}(\delta')$ already. Let $\gamma =_{\mathrm{def}} \delta'$. If $o \in \mathrm{used}(\delta)$ then $\mathrm{used}(\gamma) = \mathrm{used}(\delta') = \mathrm{used}(\delta)$, and if $o \notin \mathrm{used}(\delta)$ then $\mathrm{used}(\gamma) = (\mathrm{used}(\delta) \setminus \{s\}) \cup \{o\}$, and the first condition is satisfied. For the second condition, if $\mathrm{inac}(\delta) \neq s$ then we can choose inac(γ) such that $\mathrm{inac}(\gamma) = \mathrm{inac}(\delta)$, and if $\mathrm{inac}(\delta) = s$ then we can choose inac(γ) such that $\mathrm{inac}(\gamma) = o$. ∎

For the other two one-step extensions, we need to restrict the structure of the considered graph. Throughout the remaining subsection, let the following definitions and assumptions be valid: G is a graph without isolated vertices, and G has a vertex w such that $G-w$ is disconnected. Observe that G may already be disconnected. Let C be a connected component of $G-w$, and let $D =_{\mathrm{def}} G \setminus V(C)$. Observe that w is a vertex of D. Assume that D is connected and has at least two vertices. This means that G has at most two connected components. We also assume that C has at least two vertices. Observe that these are assumptions of Lemma 3.2.

Lemma 3.7. *Let k be an integer with $k \geq 3$, and let $s, o \in \{1, \ldots, k\}$ and $s \neq o$. Let Γ be a k-labelled graph such that $[\Gamma]$ is a subgraph of G. Let L_s and L_o be the sets of the vertices of Γ with label s and o, respectively. Assume that the vertices in L_s and L_o are adjacent in G.*

Let $\delta \in \mathcal{E}(k)$ be with $\Gamma \setminus V(C) = \mathrm{val}(\delta)$, and let $\alpha =_{\mathrm{def}} \eta_{s,o}(\delta)$. Then, one of the two cases applies:

1) $\mathrm{val}(\alpha) = \mathrm{val}(\delta)$, *or* $L_s \cup L_o \subseteq V(D)$

2) $\mathrm{val}(\alpha) \neq \mathrm{val}(\delta)$, *and if* $L_s \cap V(C) \neq \emptyset$ *then* $L_o = \{w\}$ *and* $L_s \subseteq N_G(w)$ *and the vertices in* L_s *have no non-visible neighbour in* Γ *other than* w.

Proof. We assume $\mathrm{val}(\alpha) \neq \mathrm{val}(\delta)$. This means that there is an adjacent vertex pair u, v of D with $u \in L_s$ and $v \in L_o$ and u is a non-visible neighbour of v in Γ.

We assume $L_s \cap V(C) \neq \emptyset$; let $z \in L_s \cap V(C)$. According to the assumptions of the lemma, $xy \in E(G)$ for every $x \in L_s$ and $y \in L_o$, in particular, $L_o \subseteq N_G(z) \cap N_G(u)$. The structure of G implies $L_o \cap V(D) = \{w\}$. It follows $L_s \subseteq N_G(w)$, and $w \notin L_s$. If $L_o \cap V(C) \neq \emptyset$ then u has a neighbour in $V(C)$, which means $u = w$ according to the structure of G, but contradicts $w \notin L_s$. So, $L_o \cap V(C) = \emptyset$, and $L_o = \{w\}$.

Suppose there is a vertex pair x, y of G with $x \in L_s$ and $y \neq w$ such that x is a non-visible neighbour of y in Γ. Clearly, $L_s \subseteq N_G(y)$ must hold, and since $L_s \cap V(C) \neq \emptyset$ and $L_s \cap V(D) \neq \emptyset$, $y \neq w$ yields a contradiction. ∎

We prove the decomposition lemma by modifying a linear clique-width expression for G, mainly by omitting one-step extensions that are not necessary for D. Let k be an integer with $k \geq 3$. Let $\alpha \in \mathcal{E}_{\mathrm{lin}}(k)$ be a k-expression for G. Let $\alpha^0, \ldots, \alpha^r$ be the subexpressions of α such that $\alpha^0 = ()$ and $\alpha^r = \alpha$ and α^i is obtained from α^{i-1} by a one-step extension, for $0 < i \leq r$. For $0 \leq i \leq r$, let $\Gamma_i =_{\mathrm{def}} \mathrm{val}(\alpha^i)$ and $\Delta_i =_{\mathrm{def}} \Gamma_i \setminus V(C)$.

The linear k-expression for D that we are going to obtain from α will satisfy to invariants: about the number of used labels and about an inactive label. We need the following definitions. For $0 \leq i \leq r$, let a_i be the number of the labels occurring on the vertices of Δ_i. The following is clear: $a_0 = 0$, and for $0 < i \le r$,

$$a_{i-1} \left\{ \begin{array}{c} \le \\ = \\ \ge \end{array} \right\} a_i \quad \text{if} \left\{ \begin{array}{l} \alpha^i = \alpha^{i-1} \oplus o(v) \text{ and } v \in V(D) \\ \alpha^i = \eta_{s,o}(\alpha^{i-1}), \text{ or } \alpha^i = \alpha^{i-1} \oplus o(v) \text{ and } v \in V(C) \\ \alpha^i = \rho_{s\to o}(\alpha^{i-1})\,. \end{array} \right.$$

For $0 \le i \le r$, let $m_i =_{\text{def}} \max\{a_0, \dots, a_i\}$. Clearly, $m_0 \le \cdots \le m_r$, and if $m_{i-1} < m_i$ then $a_0, \dots, a_{i-1} < a_i$. These are the parameters that we employ for the used labels.

We need to determine an indicator for an inactive label. We say that (x, y, j) is a *witness triple* if $0 \le j \le r$ and $x \in V(C)$ and $y \in V(D)$ and x and y are vertices of Γ_j with the same label; let $b_{(x,y,j)}$ be the label of x and y in Γ_j. Observe that (x, y, i) is a witness triple for every i with $j \le i \le r$ then. We say that a witness triple (x, y, j) satisfies the *inactivity condition* if $x \in N_G(w)$ implies $w \in V(\Gamma_j)$ and if for every i with $j < i \le r$ and $\alpha^i = \alpha^{i-1} \oplus o(v)$ and $v \in V(D)$, (x, v, i) is not a witness triple, i.e., if $o \ne b_{(x,y,i)}$. We choose two indices p and q. If there is no witness triple that satisfies the inactivity condition then let $p =_{\text{def}} q =_{\text{def}} r$. Otherwise, let p with $0 \le p \le r$ be smallest such that (x, y, p) is a witness triple that satisfies the inactivity condition. We fix an arbitrary such witness triple, and we let $d_C =_{\text{def}} x$ and $d_D =_{\text{def}} y$. And let q with $p \le q \le r$ be largest such that $m_p = m_q$. If $q < r$ then for $p \le i \le r$, let $b_i =_{\text{def}} b_{(d_C,d_D,i)}$.

Finally, we need to identify a special phase for the special vertex w. Let t' with $0 < t' \le r$ be such that $\alpha^{t'} = \alpha^{t'-1} \oplus o(w)$, where $o \in \{1, \dots, k\}$. If G is disconnected then let $t =_{\text{def}} t'$, and if G is connected then let t be the largest index with $t' \le t \le r$ such that no other vertex of Γ_t has the same label as w. Observe that t is well-defined in the latter case, and it suffices to show that no second vertex of $\Gamma_{t'}$ has label o: if $\Gamma_{t'}$ has a second vertex with label o, say u, then $u \prec_\sigma w$, and so, $N_G(w) \subseteq N_G(u)$, so that $G{-}w$ cannot be disconnected, a contradiction. For $0 \le i \le r$, if G is disconnected then let $F_i =_{\text{def}} \emptyset$, and if G is connected then let

$$F_i \;=_{\text{def}} \begin{cases} \emptyset & , \text{ if } 0 \le i < t' \\ \big\{xw \in E(G) : x \in V(\Delta_i)\big\} & , \text{ if } t' \le i \le t \\ F_t & , \text{ if } t < i \le r\,. \end{cases}$$

We will see that we need to take special care of these edges.

We are ready to prove the main technical lemma. The statement of the lemma appears almost straightforward and expectable, and the proof shows the technical difficulties.

Lemma 3.8. *For $0 \le i \le r$, there is $\delta^i \in \mathcal{E}_{\text{lin}}(k)$ such that $\text{val}(\delta^i) = \Delta_i{+}F_i$ and the following two conditions are satisfied:*

1) δ^i uses at most m_i labels

2) if $q < i$ then δ^i has label b_i as an inactive label.

Proof. We prove the claim by induction on i. The induction base, i.e., the case when $i = 0$, is easy: $\Delta_0{+}F_0$ is empty, and $\delta^0 =_{\text{def}} ()$ is a linear k-expression for $\Delta_0{+}F_0$ that uses at most 0 labels, and since $a_0 = 0$ and $m_0 = 0$ and $q \ge p \ge 2$, the two conditions of the lemma are satisfied.

For the induction step, let $i > 0$. Assume that δ^{i-1} is a linear k-expression for $\Delta_{i-1}+F_{i-1}$ that satisfies the two conditions of the lemma. We show that a k-expression of the desired properties exists also for Δ_i+F_i, and we distinguish between the three possible one-step extensions of α^{i-1} to α^i.

As the first case, assume $\alpha^i = \alpha^{i-1} \oplus o(v)$. If $v \in V(C)$ then $\Delta_i+F_i = \Delta_{i-1}+F_{i-1}$, and $\delta^i =_{\text{def}} \delta^{i-1}$ is a linear k-expression for Δ_i+F_i that uses at most m_{i-1} labels. Since $a_i = a_{i-1}$ and $m_i = m_{i-1}$, the first condition of the lemma is satisfied, and since $q < i$ implies $q < i-1$, also the second condition is satisfied. Note here that $q < i$ also means $b_i = b_{i-1}$.

The main situation to consider is $v \in V(D)$. Let $\gamma =_{\text{def}} \delta^{i-1} \oplus o(v)$. Clearly, γ is a linear k-expression for Δ_i+F_{i-1}. If $F_i = F_{i-1}$ then γ is a k-expression for Δ_i+F_i. Otherwise, $F_i \neq F_{i-1}$, which is possible only if G is connected. According to the definition of F_i, this means $t' \leq i \leq t$ and $v \in N_D[w]$. Let L_o be the set of the vertices of Γ_{i-1} with label o. Observe that $L_o \cup \{v\}$ is the set of the vertices of Γ_i with label o. We distinguish between two cases. First, assume $v = w$, and this means $i = t'$. Then, $L_o = \emptyset$, as we proved about the choice of t. We consider $N_D(w) \cap V(\Delta_{i-1})$: let $c_1, \ldots, c_h$ be the labels of the vertices of Δ_{i-1} in $N_D(w)$. It is not difficult to see that $\beta =_{\text{def}} \eta_{c_1,o}(\cdots \eta_{c_h,o}(\gamma) \cdots)$ is a linear k-expression for Δ_i+F_i. Since $F_i \neq F_{i-1}$, $N_D(w) \cap V(\Delta_{i-1})$ is non-empty.

Second, assume $v \in N_D(w)$. Since w is a non-visible neighbour of v in Γ_i, $L_o \cup \{v\} \subseteq N_G(w)$ follows. This particularly means $w \notin L_o \cup \{v\}$. According to the definition of t', w is a vertex of Γ_i and, because of $w \neq v$, also of Γ_{i-1}. Let l be the label of w in Γ_{i-1}, and let L_l be the set of the vertices of Γ_{i-1} with label l. According to the definition of t, $L_l = \{w\}$. With the given arguments, it directly follows that $\eta_{l,o}(\gamma)$ is a linear k-expression for Δ_i+F_i.

We consider the two conditions of the lemma, that may or may not be satisfied by γ or β or $\eta_{l,o}(\gamma)$ already. We define δ^i appropriately to satisfy the two conditions, where δ^i will be fixed after the consideration of the two conditions.

1) *First condition.* If $o \in \text{used}(\delta^{i-1})$ then $\text{used}(\gamma) = \text{used}(\delta^{i-1})$, and γ uses at most m_{i-1} labels. Otherwise, $o \notin \text{used}(\delta^{i-1})$. This particularly means that no vertex of Δ_{i-1} has label o, and thus, $a_i = a_{i-1} + 1$. If $m_{i-1} < m_i$ then γ uses at most m_i labels.

   Otherwise, $m_{i-1} = m_i$, which means $a_{i-1} < a_i \leq m_i = m_{i-1}$. Let $s \in \text{used}(\delta^{i-1})$ be a label such that no vertex of Δ_{i-1} has label s; such a label must exist. We obtain δ' from δ^{i-1} by replacing every occurrence of label s by label o. The following is easy but crucial: δ' uses at most m_{i-1} labels and $s \notin \text{used}(\delta')$ and $o \in \text{used}(\delta')$ and δ' is a linear k-expression for $\Delta_{i-1}+F_{i-1}$. If $F_i = F_{i-1}$ then $\gamma' =_{\text{def}} \delta' \oplus o(v)$ is a linear k-expression for Δ_i+F_i, and if $F_i \neq F_{i-1}$ then $\beta' =_{\text{def}} \eta_{c_1,o}(\cdots \eta_{c_h,o}(\delta' \oplus o(v)) \cdots)$ or $\gamma' =_{\text{def}} \eta_{l,o}(\delta' \oplus o(v))$ is a linear k-expression for Δ_i+F_i, and β' and γ' use at most m_i labels, and they satisfy the first condition.

2) *Second condition.* We assume $q < i$, and we distinguish between two cases. As the first and easy case, assume $q < i-1$. Then, δ^{i-1} has label b_{i-1} as an inactive label. Note $b_{i-1} = b_i$. It suffices to prove, depending on the case about the first condition above: $o \neq b_i$ and $c_1, \ldots, c_h \neq b_i$ and $l \neq b_i$ and $s \neq b_i$ (when they are chosen). Since d_D of Δ_{i-1} has label b_{i-1}, $s \neq b_i$ directly follows. Since w has label l and $L_l = \{w\}$ and d_C and d_D of Γ_{i-1} have label b_{i-1}, also $l \neq b_i$ directly follows. And since (d_C, d_D, p) satisfies the

inactivity condition, $o \neq b_i$ is a definitional consequence. So, we can choose $\delta^i =_{\text{def}} \gamma'$, and δ^i satisfies the second condition. In the case of $v = w$, we easily observe $w \notin V(\Gamma_{i-1})$, so that the inactivity condition implies $d_C \notin N_G(w)$, and thus, $c_1, \ldots, c_h \neq b_{i-1}$ directly follows, and β' also satisfies the second condition.

As the second case, assume $q = i - 1$. Recall: $m_q < m_{q+1}$ and $m_{q+1} = a_{q+1} = a_i$ and $o \notin \text{used}(\delta^{i-1})$. Obtain δ'' from δ^{i-1} by replacing each occurrence of label b_{i-1} by label o. Note $b_{i-1} \notin \text{used}(\delta'')$. Also note that $\rho_{o\to b_{i-1}}(\delta'')$ is a linear k-expression for $\Delta_{i-1}+F_{i-1}$ that has label b_{i-1} as an inactive label, and that uses $m_{i-1} + 1$ labels however. If $F_i = F_{i-1}$ then let $\delta^i =_{\text{def}} \rho_{o\to b_{i-1}}(\delta'') \oplus o(v)$, and if $F_i \neq F_{i-1}$ and $v = w$ then let $\delta^i =_{\text{def}} \eta_{c_1,o}(\cdots \eta_{c_h,o}(\rho_{o\to b_{i-1}}(\delta'') \oplus o(v)) \cdots)$, and if $F_i \neq F_{i-1}$ and $v \neq w$ then let $\delta^i =_{\text{def}} \eta_{l,o}(\rho_{o\to b_{i-1}}(\delta'') \oplus o(v))$. Observe that δ^i uses at most m_i labels, and since $c_1, \ldots, c_h, l \neq b_i$, δ^i has label b_{i-1} as an inactive label, so that the two conditions of the lemma are satisfied. It remains to see that δ^i is a linear k-expression for Δ_i+F_i, which is straightforward.

We conclude that δ^i is a linear k-expression for Δ_i+F_i that satisfies the two conditions of the lemma, and the first case of the proof is complete.

As the second case, assume $\alpha^i = \eta_{s,o}(\alpha^{i-1})$. Note here that $F_i = F_{i-1}$ is the case, and if $p < i$ then $b_i = b_{i-1}$, and if $q < i$ then $q < i - 1$. If $\Delta_i+F_i = \Delta_{i-1}+F_{i-1}$ then δ^{i-1} is a linear k-expression for Δ_i+F_i that satisfies the two conditions. Otherwise, $\Delta_i+F_i \neq \Delta_{i-1}+F_{i-1}$.

Let L_s and L_o be the sets of the vertices of Γ_{i-1} with label s and o, respectively. Since $\Delta_i+F_i \neq \Delta_{i-1}+F_{i-1}$, it is clear that $L_s \cap V(D)$ and $L_o \cap V(D)$ are non-empty, and $s, o \in \text{used}(\delta^{i-1})$ in particular. If $L_s \cup L_o \subseteq V(D)$ and $p < i$ then $s \neq b_{i-1}$ and $o \neq b_{i-1}$, since d_C of Γ_{i-1} has label b_{i-1}, and $\delta^i =_{\text{def}} \eta_{s,o}(\delta^{i-1})$ is a linear k-expression for Δ_i+F_i that satisfies the two conditions. Otherwise, $L_s \cup L_o \not\subseteq V(D)$.

By a symmetry argument, we may assume $L_s \cap V(C) \neq \emptyset$. We show that this yields a contradiction. Observe that we can apply Lemma 3.7, and the second case of Lemma 3.7 applies: $L_o = \{w\}$ and $L_s \subseteq N_G(w)$ and the vertices in L_s have no non-visible neighbour in $\Delta_{i-1}+F_{i-1}$ other than w. Observe that G is connected, since $N_G(w)$ contains vertices of C and D, and $L_o = \{w\}$ implies $t' < i < t$. It simply follows that the vertices in $L_s \cap V(D)$ are adjacent to w in $\Delta_{i-1}+F_{i-1}$ already, so that $\Delta_i+F_i = \Delta_{i-1}+F_{i-1}$, the contradiction.

As the third case, assume $\alpha^i = \rho_{s\to o}(\alpha^{i-1})$. Note here that $F_i = F_{i-1}$ is the case. We apply Lemma 3.6 to δ^{i-1} for $\Delta_{i-1}+F_{i-1}$ and obtain a linear k-expression δ^i for Δ_i+F_i that satisfies the two conditions of Lemma 3.6.

We verify the two conditions of the lemma. Since $a_{i-1} \geq a_i$ and $m_{i-1} = m_i$, δ^i uses at most m_{i-1} labels according to the first condition of Lemma 3.6, and δ^i satisfies the first condition of the lemma. For the second condition of the lemma, we may assume $q < i$, which also means $q < i - 1$. According to the induction hypothesis, δ^{i-1} has label b_{i-1} as an inactive label, which means we may assume $\text{inac}(\delta^{i-1}) = b_{i-1}$. According to the second condition of Lemma 3.6: if $b_{i-1} \neq s$ then $\text{inac}(\delta^i) = \text{inac}(\delta^{i-1})$, and if $b_{i-1} = s$ then $b_i = o$ and $\text{inac}(\delta^i) = o$, so that $\text{inac}(\delta^i) = b_i$. We conclude that δ^i satisfies the second condition of the lemma. ■

Since $\Delta_r+F_r = \Delta_r = \Gamma_r \setminus V(C)$ and $[\Gamma_r \setminus V(C)] = G \setminus V(C) = D$, Lemma 3.8 shows that D has a linear k-expression that uses at most m_r labels and that has an inactive label, if $q < r$.

The two important parameters here are m_r and q. These two parameters are usually not known, and their definitions strongly depend on the given linear k-expression α. As a first application of Lemma 3.8 and in order to approach the proof of the decomposition lemma, we show how to bound these parameters combinatorially.

Lemma 3.9. *Let $\langle u_1, \ldots, u_n \rangle$ be the vertex ordering associated with α, and assume that u_1 is a vertex of C. Assume $V(C) \not\subseteq N_G(w)$. Then, $m_r < k$ or $q < r$.*

Proof. We begin with an auxiliary observation about the inactivity condition. Let $0 \leq i \leq r$ and assume $\alpha^i = \alpha^{i-1} \oplus o(v)$ with $v \in V(D)$. Assume that (x, v, i) is a witness triple. Recall that this means $N_G(v) \subseteq N_G(x)$, and since G has no isolated vertices, the structure of G implies $N_G(v) = \{w\}$, and x has no non-visible neighbour from $V(C)$ in Γ_i, and $x \in N_G(w)$. We will apply this result below.

Assume $m_r = k$. Since $m_0 = 0$, there is $0 < i \leq r$ with $m_{i-1} < k$ and $m_i = k$. This means $a_{i-1} < k$ and $a_i = k$ in particular, and $\alpha^i = \alpha^{i-1} \oplus o(v)$ and $v \in V(D)$. We show that $q < r$ must hold, by finding a witness triple that satisfies the inactivity condition appropriately. It is important to observe that $a_i \geq 1$ implies that Γ_i and Γ_{i-1} contain a vertex of C, namely u_1 according to the assumptions of the lemma.

For every vertex c from $V(\Gamma_i) \cap V(C)$, let $\hat{c}$ be a vertex from $V(\Gamma_i) \cap V(D)$ such that $(c, \hat{c}, i)$ is a witness triple; recall that $\hat{c}$ exists, because of $a_i = k$. If c has a non-visible neighbour from $V(C)$ in Γ_i then $\hat{c} = w$ due to the structure of G. We distinguish between two cases.

As the first case, assume that Γ_i has a vertex x from $V(C)$ with a non-visible neighbour from $V(C)$. Then, $\hat{x} = w$ and (x, w, i) is a witness triple. If (x, v, i) is also a witness triple then x, v, w have the same label in Γ_i, namely label o, so that $x, w \notin N_G(v)$, and this contradicts $N_G(v) = \{w\}$ of the auxiliary observation, so that (x, v, i) is not a witness triple and $v \neq w$. Since x and w are vertices of Γ_{i-1}, $(x, w, i-1)$ is also a witness triple. If $(x, w, i-1)$ does not satisfy the inactivity condition then there is $i - 1 < i' \leq r$ with $\alpha^{i'} = \alpha^{i'-1} \oplus o'(v')$ and $v' \in V(D)$ and $o' = b_{(x,w,i')}$, so that (x, v', i') is a witness triple, and the auxiliary observation yields $N_G(v') = \{w\}$, which is a contradiction, since w has label o' in $\Gamma_{i'}$. Thus, $(x, w, i-1)$ satisfies the inactivity condition, and $p \leq q < i \leq r$.

As the second case, assume that Γ_i has no vertex from $V(C)$ with a non-visible neighbour from $V(C)$. Since C is connected, this directly means $V(C) \subseteq V(\Gamma_i)$. Let $x \in V(C) \setminus N_G(w)$, and $(x, \hat{x}, i)$ is a witness triple. If $\hat{x} = v$ then the auxiliary observation observes $x \in N_G(w)$, a contradiction. So, $\hat{x} \neq v$, and $(x, \hat{x}, i-1)$ is a witness triple, and $(x, \hat{x}, i-1)$ satisfies the inactivity condition due to the auxiliary observation and $x \notin N_G(w)$. We conclude $q < i$.

As a conclusion, we conclude that $m_r = k$ implies $q < r$. ■

What if $V(C) \subseteq N_G(w)$? Our technique is not applicable in this case. We leave it as an open problem whether and how Lemma 3.9 can be extended to include this case.

We are finally ready to prove the decomposition lemma of the preceding subsection. Recall that our assumptions about G and the choice of w and C and D are in accordance with Lemma 3.2.

Proof of Lemma 3.2. If $m_r < k$ then there is $\delta \in \mathcal{E}_{\text{lin}}(k)$ for D that uses at most $k - 1$ labels according to Lemma 3.8. In this case, it follows that there is $\delta' \in \mathcal{E}_{\text{lin}}(k-1)$ for D, and $\text{lcwd}(D) \leq k - 1$, and thus, $\text{lcwd}_{\text{inac}}(D) \leq k$ due to Lemma 2.2.

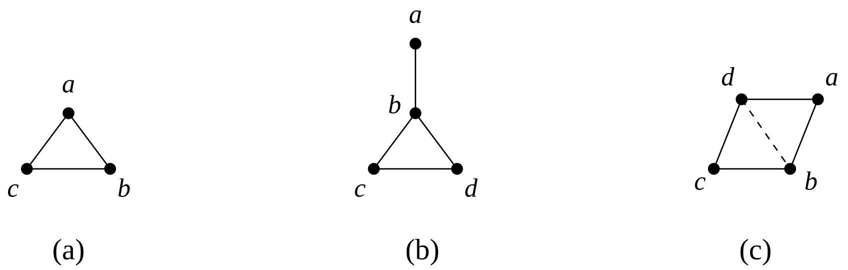

Figure 2: Lemma 4.1 establishes properties of 3-expressions for these three graphs. The dashed edge of graph (c) may or may not be an edge of the graph.

If $m_r = k$ then $q < r$ due to Lemma 3.9, and there is $\delta \in \mathcal{E}_{\text{lin}}(k)$ for D that has label b_r as an inactive label due to Lemma 3.8. So, there is $\delta' \in \mathcal{E}_{\text{lin}}^{\text{inac}}(k)$ for D, and $\text{lcwd}_{\text{inac}}(D) \leq k$. ■

As a final remark, and hinting at another direction of future research, note the case when u_n is a vertex of C. The auxiliary observation in the proof of Lemma 3.9 can be employed to show $m_r < k$. This directly leads to graph composition operations generating graphs of arbitrary linear clique-width.

4 Graphs of clique-width at most 3 with an inactive label

In the preceding section, we studied graphs of bounded linear clique-width and determined large induced subgraphs with linear clique-width expressions with an inactive label. In this and the next section, we ask about the graphs that have clique-width expressions with an inactive label. We consider 3-expressions with an inactive label, and will precisely and comprehensively characterise the represented graphs.

In this section, we consider the 3-expressions with an inactive label, and we will show that the represented graphs are exactly the distance-hereditary graphs. The result is proved in two steps, by proving necessity and sufficiency. Distance-hereditary graphs have a structural characterisation through forbidden induced subgraphs. We first show that these graphs do not have 3-expressions with an inactive label. This will prove the necessity part of the characterisation result. For the sufficiency part, we can employ and adapt a result from the literature. The definition of distance-hereditary graphs and useful properties are given later in this section.

We prove lower bounds. The graphs that we are interested in are: chordless cycle of length at least 5, house, gem, domino. The latter three graphs are depicted in Figure 3. We show that these graphs do not have 3-expressions with an inactive label. We will obtain this result by applying results for special induced subgraphs and their 3-expressions. These special induced subgraphs and their 3-expressions are considered in the next two lemmas. The graphs considered in the first lemma are depicted in Figure 2.

Lemma 4.1. *Let G be a graph and let $\alpha \in \mathcal{E}^{\text{inac}}(3)$ be a 3-expression for G. Let γ be a subexpression of α that is a full subexpression for G, and let $\Gamma =_{\text{def}} \text{val}(\gamma)$.*

1) If $V(G) = \{a, b, c\}$ and $E(G) = \{ab, bc, ca\}$ then two vertices of Γ have the same label.

2) If $V(G) = \{a, b, c, d\}$ and $E(G) = \{ab, bc, cd, db\}$ then a has label 1 in Γ or c and d have label 1 in Γ or a, c, d have the same label in Γ.

3) If $V(G) = \{a, b, c, d\}$ and $E(G) = \{ab, bc, cd, da\}$ or $E(G) = \{ab, bc, cd, da, bd\}$ then a and c have the same label in Γ or b and d have the same label in Γ.

Proof. If $\gamma = \eta_{s,o}(\delta)$ and the claims are true for $\mathrm{val}(\delta)$ then the claims are also true for Γ. Similarly, if $\gamma = \rho_{s\to o}(\delta)$ and the claims are true for $\mathrm{val}(\delta)$ then the claims are true also for Γ; recall here that $s \neq 1$ must hold. By repeatedly applying these easy arguments, it suffices to assume about γ that $\gamma = \beta \oplus \delta$ and $\mathrm{val}(\beta)$ and $\mathrm{val}(\delta)$ are not empty; if $\gamma = \beta \oplus o(u)$ then we assume $\gamma = \beta \oplus \delta$ with $\delta = () \oplus o(u)$ in order to satisfy the formal requirements.

We prove the first statement; consider graph (a) of Figure 2. By symmetry, we may assume that a and b are vertices of $\mathrm{val}(\beta)$ and c is the vertex of $\mathrm{val}(\delta)$. Then, a and b are non-visible neighbours of c in Γ, so that a, b, c have label 2 or 3 in Γ and the label of c is different from the labels of a and b. It follows that a and b must have the same label in Γ.

We prove the second statement; consider graph (b) of Figure 2. Assume that a does not have label 1 in Γ, and we may assume that a has label 2 in Γ. Since $[\Gamma]$ is disconnected, there is a pair of vertices of Γ that are non-visible neighbours of each other, so that they have label 2 and 3 in Γ; let x be that vertex with label 3. Then, all vertices with label 2 in Γ are neighbours of x in G, in particular, $a \in N_G(x)$, and thus, $x = b$. It also follows that c and d must have label 1 or 2 in Γ, since c and d are non-adjacent to a in G. Let γ' be a subexpression of γ that is a full subexpression for $G[\{b, c, d\}]$. Note that γ' is a subexpression of α. We can apply the first statement of the lemma to γ' and conclude that c and d have the same label in γ', and thus, c and d have label 1 or have label 2 in Γ.

We prove the third statement; consider graph (c) of Figure 2. We may assume that a is a vertex of $\mathrm{val}(\beta)$ and b is a vertex of $\mathrm{val}(\delta)$ and a is a non-visible neighbour of b in Γ. By symmetry, a has label 2 and b has label 3 in Γ. Since a and c are non-adjacent in G, c does not have label 3 in Γ. Assume that c does not have label 2 in Γ: c has label 1 in Γ. Since label 1 is an inactive label, bc and cd are edges already of Γ, so that c is a vertex of $\mathrm{val}(\delta)$, and thus, d is a vertex of $\mathrm{val}(\delta)$. Hence, d is a non-visible neighbour of a in Γ, and d must have label 3 in Γ, i.e., b and d have the same label in Γ. ■

Lemma 4.2. *Let G be a graph on n vertices, where $n \geq 4$. Assume that G has a path $P = (v_1, \ldots, v_n)$. Let $\alpha \in \mathcal{E}(3)$ be a 3-expression for G. Let γ be a subexpression of α that is a full subexpression for G, and let $\Gamma =_{\mathrm{def}} \mathrm{val}(\gamma)$.*

1) Assume that P is a chordless path of G. If $n \geq 6$ then v_1 and v_2 have the same label in Γ or v_{n-1} and v_n have the same label in Γ.

2) Assume that P is a chordless path of G, and assume $\alpha \in \mathcal{E}^{\mathrm{inac}}(3)$. Then, v_1 or v_n has label 1 in Γ.

3) Let $n = 5$. Assume that v_2v_5 is an edge of G and P is a chordles path of $G - v_2v_5$, and assume $\alpha \in \mathcal{E}^{\mathrm{inac}}(3)$. Then, v_1 or v_5 has label 1 in Γ, or v_1 and v_3 and v_5 have the same label in Γ.

Proof. Analogous to the introductory considerations in the proof of Lemma 4.1, it suffices to consider $\gamma = \beta \oplus \delta$ with $\text{val}(\beta)$ and $\text{val}(\delta)$ being non-empty, and we may assume that v_1 is a vertex of $\text{val}(\beta)$. Let p be the index with $1 \leq p < n$ such that $v_1, \ldots, v_p$ are vertices of $\text{val}(\beta)$ and v_{p+1} is a vertex of $\text{val}(\delta)$. Since v_{p+1} is a non-visible neighbour of v_p in Γ, we may assume that v_p has label 2 and v_{p+1} has label 3 in Γ. Let D be the set of the vertices of Γ with label 2 or 3. Clearly, the vertices of Γ that are not in D have label 1.

We prove the first and second statement. Since P is a chordless path of G, Γ cannot have two vertices with label 2 and two vertices with label 3, and since no vertex of G has degree at least 3, Γ does not have three vertices with label 2 or three vertices with label 3. It follows: $|D| \leq 3$. If $|D| = 2$ then $D = \{v_p, v_{p+1}\}$. If $|D| = 3$ then Γ has a second vertex of label 2, that must be v_{p+2}, or Γ has a second vertex of label 3, that must be v_{p-1}. Thus, $D = \{v_{p-1}, v_p, v_{p+1}\}$ or $D = \{v_p, v_{p+1}, v_{p+2}\}$. For the first statement: if $D \subseteq \{v_1, \ldots, v_4\}$ then v_{n-1} and v_n have the same label in Γ and if $D \not\subseteq \{v_1, \ldots, v_4\}$ then $D \subseteq \{v_3, \ldots, v_n\}$ and v_1 and v_2 have the same label in Γ. For the second statement: $v_1 \notin D$ or $v_n \notin D$.

We prove the third statement. We assume that v_1 and v_5 do not have label 1 in Γ. Thus, $v_1, v_5 \in D$, so that $\{v_1, v_p, v_{p+1}, v_5\} \subseteq D$, and v_1 is adjacent to v_p or v_{p+1}, so that $p \leq 2$ must hold. If $p = 2$ then v_1 and v_5 are non-adjacent to v_{p+1} in G, so that v_1, v_{p+1}, v_5 have the same label in Γ. Otherwise, $p = 1$, and v_1 and v_5 have label 2 and v_2 has label 3 in Γ. Since (v_1, v_2, v_3, v_4) is a chordless path of G, we can apply the second statement of the lemma and observe that v_4 has label 1 in Γ. Similarly, we can apply the third statement of Lemma 4.1 to $G[\{v_2, v_3, v_4, v_5\}]$: since v_2 and v_4 do not have the same label in Γ, v_3 and v_5 have the same label in Γ, and thus, v_1, v_3, v_5 have the same label in Γ. ■

As the first result, we characterise the clique-width of chordless cycles. Let n be an integer with $n \geq 3$. A *chordless cycle of length n*, denoted as C_n, is a graph G on n vertices that has a vertex ordering $\langle v_1, \ldots, v_n \rangle$ such that $(v_1, \ldots, v_n)$ is a chordless v_1, v_n-path of $G - v_1 v_n$ and $v_1 v_n \in E(G)$. The clique-width result for chordless cycles may already be fully known. We give a proof for completeness reasons and in order to provide an example for a lower-bound proof.

Proposition 4.3. *Let n be an integer with $n \geq 5$. The following is the case:*

1) $\text{cwd}(C_3) = \text{cwd}(C_4) = 2$ *and* $\text{cwd}(C_5) = \text{cwd}(C_6) = 3$

2) $\text{cwd}(C_n) = 4$ *for $n \geq 7$, and* $\text{cwd}_{\text{inac}}(C_n) = 4$.

Proof. Let $\langle v_1, \ldots, v_n \rangle$ be a vertex ordering for C_n such that $(v_1, \ldots, v_n)$ is a chordless v_1, v_n-path of $C_n - v_1 v_n$. We consider the first statement. The lower bounds for C_3 and C_4 are easy. The lower bounds for C_5 and C_6 and the upper bounds for C_3 and C_4 are due to Theorem 2.1. A 3-expression for C_5 can be obtained on the structure of this reduced expression: $((v_1 \oplus v_2) \oplus (v_3 \oplus v_4)) \oplus v_5$, and a 3-expression for C_6 can be obtained on the structure of this reduced expression: $(((v_1 \oplus v_2) \oplus (v_4 \oplus v_5)) \oplus v_3) \oplus v_6$.

We consider the second statement. The upper bounds are easy and straightforward exercises. We prove the lower bounds. Suppose that C_n has a 3-expression α. In case of $\text{cwd}(C_n) \geq 4$ for $n \geq 7$, we will construct a contradiction, and in case of $\text{cwd}_{\text{inac}}(C_5) \geq 4$ and $\text{cwd}_{\text{inac}}(C_6) \geq 4$, we will construct a contradiction when assuming $\alpha \in \mathcal{E}^{\text{inac}}(3)$. Let $\gamma = \beta \oplus \delta$ be a subexpression of α that is a full subexpression for C_n and such that $\text{val}(\beta)$ and $\text{val}(\delta)$ are non-empty; let

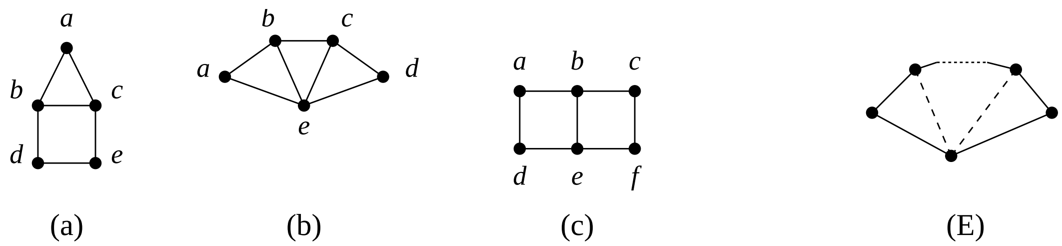

Figure 3: The figure depicts these special graphs: (a) house, (b) gem, (c) domino, and graph family (E) represents the family of graphs obtained from chordless cycles of length at least 5 by adding edges that are incident to a common vertex. The possible added edges are shown as dashed edges.

$\Gamma =_{\text{def}} \text{val}(\gamma)$. Without loss of generality and by easy symmetry arguments, we may assume that $\text{val}(\beta)$ has at least two vertices and there are i, i' with $1 < i < i' \leq n$ such that v_1 and $v_{i'}$ are vertices of $\text{val}(\beta)$ and v_i is a vertex of $\text{val}(\delta)$. Let p be the index with $1 \leq p \leq n-2$ such that $v_1, \ldots, v_p$ are vertices of $\text{val}(\beta)$ and v_{p+1} is a vertex of $\text{val}(\delta)$. We may assume that v_p has label 2 and v_{p+1} has label 3 in Γ.

According to our assumptions, v_p and v_{p+2} are the neighbours of v_{p+1} in C_n. Assume that v_{p+2} is a vertex of $\text{val}(\beta)$. Then, v_p and v_{p+2} are non-visible neighbours of v_{p+1} in Γ, and v_p and v_{p+2} have label 2 and v_{p+1} has label 3 and all other vertices have label 1 in Γ. Observe that $(v_p, v_{p-1}, \ldots, v_1, v_n, v_{n-1}, \ldots, v_{p+2})$ is a chordless v_p, v_{p+2}-path of $C_n - v_{p+1}$ on $n-1$ vertices. Let γ' be a subexpression of γ that is a full subexpression for $C_n - v_{p+1}$. If $n \geq 7$ then we can apply the first statement of Lemma 4.2, which yields a contradiction to the observed labels, and if $n \leq 6$ and $\alpha \in \mathcal{E}^{\text{inac}}(3)$ then the second statement of Lemma 4.2 yields a contradiction, since the labels of v_p and v_{p+2} must not be inactive.

As the other case, assume that v_{p+2} is a vertex of $\text{val}(\delta)$. Let q be the index with $p+2 \leq q < n$ such that $v_{p+1}, \ldots, v_q$ are vertices of $\text{val}(\delta)$ and v_{q+1} is a vertex of $\text{val}(\beta)$. It is easy to observe that $v_p, v_{p+1}, v_q, v_{q+1}$ must have pairwise different labels in Γ, a contradiction. ∎

As the second result, we characterise the clique-width of graphs that are obtained from chordless cycles by adding edges, that are incident to a common vertex. We start from chordless cycles of length at least 5. If the obtained graph does not have a chordless cycle of length at least 5 as an induced subgraph then it has one of the three left-side graphs of Figure 3 as an induced subgraph: house or gem or domino. We characterise the clique-width of these graphs.

Proposition 4.4. *Let G be a* house *or a* gem *or a* domino*. Then,* $\text{cwd}(G) = \text{lcwd}(G) = 3$ *and* $\text{cwd}_{\text{inac}}(G) = \text{lcwd}_{\text{inac}}(G) = 4$.

Proof. For the considered graphs and used vertex names, we refer to Figure 3. We prove the desired bounds by mixing general arguments for the three graphs and separate arguments for the individual graphs. We abbreviate the names of the three graphs as follows: (H) and (G) and (D) for house, gem, domino, respectively. We first show the upper bounds. It is not difficult to verify that house and gem and domino have linear 3-expressions, that are associated with these vertex orderings: (H) $\langle a, b, e, d, c \rangle$, and (G) $\langle a, b, c, d, e \rangle$, and (D) $\langle d, e, b, c, a, f \rangle$. So, $\text{cwd}(G) \leq \text{lcwd}(G) \leq 3$. The upper bound of $\text{cwd}_{\text{inac}}(G) \leq \text{lcwd}_{\text{inac}}(G) \leq 4$ follows from

Lemma 2.2. The lower bound of $\mathrm{cwd}(G) \geq 3$ follows from Theorem 2.1, since G has P_4 as an induced subgraph. It remains to show $\mathrm{cwd}_{\mathrm{inac}}(G) \geq 4$.

Suppose for a contradiction that $\mathrm{cwd}_{\mathrm{inac}}(G) \leq 3$. Let $\alpha \in \mathcal{E}^{\mathrm{inac}}(3)$ be a 3-expression for G. Let $\gamma = \beta \oplus \delta$ be a subexpression of α that is a full subexpression for G and such that $\mathrm{val}(\beta)$ and $\mathrm{val}(\delta)$ are non-empty; let $\Gamma =_{\mathrm{def}} \mathrm{val}(\gamma)$. It will be important to recall that vertices with label 1 in Γ do not have non-visible neighbours. We always assume that a is a vertex of $\mathrm{val}(\beta)$. We distinguish between the three cases.

(H) Assume that a has label 1 in Γ. Then, a, b, c are vertices of $\mathrm{val}(\beta)$, and we may assume by symmetry that e is a vertex of $\mathrm{val}(\delta)$. This also means that c and d and e do not have label 1 in Γ, in particular, a and c do not have the same label in Γ. We obtain a contradiction to the second statement of Lemma 4.1 when applied to $G{-}e$.

Assume that a does not have label 1 in Γ. We apply the second statement of Lemma 4.2 to $G{-}c$ and $G{-}b$: since a does not have label 1 in Γ, d and e must have label 1 in Γ. So, b, c, d, e are the vertices of $\mathrm{val}(\delta)$, and a is the unique vertex of $\mathrm{val}(\beta)$, and b and c are non-visible neighbours of a in Γ. Then, b and e do not have the same label and c and d do not have the same label in Γ, and we obtain a contradiction to the third statement of Lemma 4.1.

(G) We apply the second statement of Lemma 4.2 to $G{-}e$: a or d has label 1 in Γ, and we assume that a has label 1 in Γ. Then, a, b, e are vertices of $\mathrm{val}(\beta)$, and c or d is a vertex of $\mathrm{val}(\delta)$. Since c, d, e are pairwise adjacent in G, c, d, e do not have label 1 in Γ.

We apply the second statement of Lemma 4.1 to $G{-}c$: since d does not have label 1 and a and d do not have the same label, a and b must have label 1 in Γ. Observe that c must be a vertex of $\mathrm{val}(\beta)$ then, so that d is the unique vertex of $\mathrm{val}(\delta)$ and c and e are non-visible neighbours of d in Γ, that do not have label 1. We apply the third statement of Lemma 4.1 to $G{-}d$: a and c have the same label or b and e have the same label in Γ, a contradiction.

(D) We apply the second statement of Lemma 4.2 to $G{-}b$ and $G{-}e$: a or c has label 1 in Γ and d or f has label 1 in Γ. If a, c, e have the same label in Γ then a, c, e have label 1 in Γ, and $\mathrm{val}(\delta)$ is empty. Analogously, f, d, b cannot have the same label in Γ, since this would be label 1. We apply the third statement of Lemma 4.2 to $G{-}a$, $G{-}c$, $G{-}d$ and $G{-}f$: b or d, and b or f, and a or e, and c or e has label 1 in Γ. Then, by combining the two steps, Γ has four vertices with label 1, which yields a contradition.

We conclude a contradiction in each of the cases. We therefore conclude $\mathrm{cwd}_{\mathrm{inac}}(G) > 3$. ∎

Let $\mathcal{D}$ be the graph family containing the chordless cycles of length at least 5 and house and gem and domino. Recall from a previous discussion that a graph G has one of the graphs in $\mathcal{D}$ as an induced subgraph if and only if G has an induced subgraph that is obtained from a chordless cycle of length at least 5 by adding edges that are incident to a common vertex. We collect these graphs in *graph family (E)*, and this graph family (E) is depicted in the right-side figure of Figure 3.

Corollary 4.5. *Let G be a graph. If G contains one of the graphs in $\mathcal{D}$ as an induced subgraph then* $\mathrm{cwd}_{\mathrm{inac}}(G) \geq 4$.

Proof. Due to Propositions 4.3 and 4.4, $\mathrm{cwd}_{\mathrm{inac}}(H) = 4$ for every graph H in $\mathcal{D}$. The claimed lower bound follows from Lemma 2.3. ■

We are ready to complete the characterisation of the graphs with 3-expressions with an inactive label. Corollary 4.5 shows that such graphs must not contain a graph in $\mathcal{D}$ as an induced subgraph. The graphs without induced subgraphs from $\mathcal{D}$ are a well-known graph class, namely the distance-hereditary graphs. A graph G is called *distance-hereditary* if for every vertex pair a, b of G, the chordless a, b-paths of G are of the same length [20]. It is not difficult to see that trees are distance-hereditary graphs, and cographs are distance-hereditary. Cographs are the graphs that do not contain P_4 as an induced subgraph (see the first statement of Theorem 2.1). Distance-hereditary graphs admit several characterisations [20, 3, 16], some of which we state in the next theorem.

Theorem 4.6 ([3, 16]). *Let G be a graph. The following statements are equivalent:*

1) G is distance-hereditary

2) G does not contain a graph in $\mathcal{D}$ as an induced subgraph

3) G is a graph on a single vertex, or
G has a vertex pair u, v where $u \neq v$ such that $G - v$ is distance-hereditary and one of the three applies: $N_G(v) = \{u\}$ or $N_G(v) = N_G(u)$ or $N_G[v] = N_G[u]$.

Golumbic and Rotics showed that the clique-width of distance-hereditary graphs is at most 3 [13]. The upper-bound proof is based on an algorithm to construct a 3-expression, by applying the deconstruction characterisation of distance-hereditary graphs in the third statement of Theorem 4.6. A careful analysis of the constructed 3-expression shows that the expression has label 1 as an inactive label. We can therefore directly conclude the following result.

Theorem 4.7 ([13]). *Let G be a distance-hereditary graph. Then,* $\mathrm{cwd}_{\mathrm{inac}}(G) \leq 3$.

We can give our characterisation result of the graphs with 3-expressions with an inactive label.

Proposition 4.8. *Let G be a graph. Then,* $\mathrm{cwd}_{\mathrm{inac}}(G) \leq 3$ *if and only if G is a distance-hereditary graph.*

Proof. If G is distance-hereditary then $\mathrm{cwd}_{\mathrm{inac}}(G) \leq 3$ due to Theorem 4.7. If $\mathrm{cwd}_{\mathrm{inac}}(G) \leq 3$ then G does not contain a graph in $\mathcal{D}$ as an induced subgraph due to Corollary 4.5, and G is distance-hereditary due to Theorem 4.6. ■

The structure of graphs of clique-width at most 3 is unknown. The currently best known result is the polynomial-time recognition algorithm for graphs of clique-width at most 3 [7]. An easy description of the structure of these graphs is not likely, since these graphs can contain C_5 and C_6 and house and gem and domino as induced subgraphs (Propositions 4.3 and 4.4).

5 Graphs of linear clique-width at most 3 with an inactive label

We know the graphs of clique-width at most 3 with an inactive label: the distance-hereditary graphs. Now, we consider the restriction to linear 3-expressions. As is easy to see and proved in Lemma 2.2, the graphs of linear clique-width at most 3 with an inactive label are distance-hereditary graphs, and we will see that they form a proper subclass of the distance-hereditary graphs. We will precisely characterise this subclass in two ways: by constructively describing their structure and by a set of forbidden induced subgraphs. Our proof approach relies on the result of Proposition 4.8.

This section is partitioned into four subsections, that partition and structure the proof of the main result.

5.1 Base graphs and pendant vertex extension

We define a class of graphs, that generalise the graphs of linear clique-width at most 2 (see the second statement of Theorem 2.1). We prove structural properties of these graphs and we consider the extension of these graphs by adding a pendant vertex. In particular, we determine induced subgraphs that are obtained from adding a pendant vertex.

Definition 5.1. *Marked two-chain graphs and their building sequences are defined inductively.*

1) *Let G be an edgeless graph on vertex set M. Then, $(G; M, M)$ is a* marked two-chain graph*, and $\langle M \rangle$ is a* building sequence *for $(G; M, M)$.*

2) *Let $(G; M, A)$ be a marked two-chain graph with building sequence $\langle B_1, \ldots, B_t \rangle$. Let B be a set of vertices that are not vertices of G. Let K be a set of edges as follows: $K = \emptyset$ or $K = \{ab : a \in A \text{ and } b \in B\}$. Let H be obtained from G by adding the vertices in B and the edges in K. Then, $(H; M, A)$ and $(H; M, (A \cup B))$ are* marked two-chain graphs*, and $\langle B_1, \ldots, B_t, B \rangle$ is a building sequence for $(H; M, A)$ and $(H; M, (A \cup B))$.*

A graph G is a two-chain graph *if there are $M \subseteq V(G)$ and $A \subseteq V(G)$ such that $(G; M, A)$ is a marked two-chain graph, and a* building sequence *for G is a building sequence for a marked two-chain graph $(G; M, A)$ where $M, A \subseteq V(G)$.*

An illustration of the iterative construction process of marked two-chain graphs is given in Figure 4. It is visible in the figure example and is an easy consequence of the definition of marked two-chain graphs that the vertices in $V(G) \setminus A$ admit a vertex ordering by neighbourhood inclusion, and this inclusion chain corresponds to their order in the building sequence, except for vertices with the same neighbourhoods and for vertices without any neighbours. Vertices that are added in the same step are vertices with the same neighbours. Such vertices are twins. Let G be a graph, and let u, v be a vertex pair of G. We say that u and v are *false twins* if $N_G(u) = N_G(v)$, and we say that u and v are *true twins* if $N_G[u] = N_G[v]$. False twins are non-adjacent and true twins are adjacent. We say that u, v is a *twin pair* if u and v are false twins or true twins.

We begin by proving some structural properties of marked two-chain graphs, that will be most valuable.

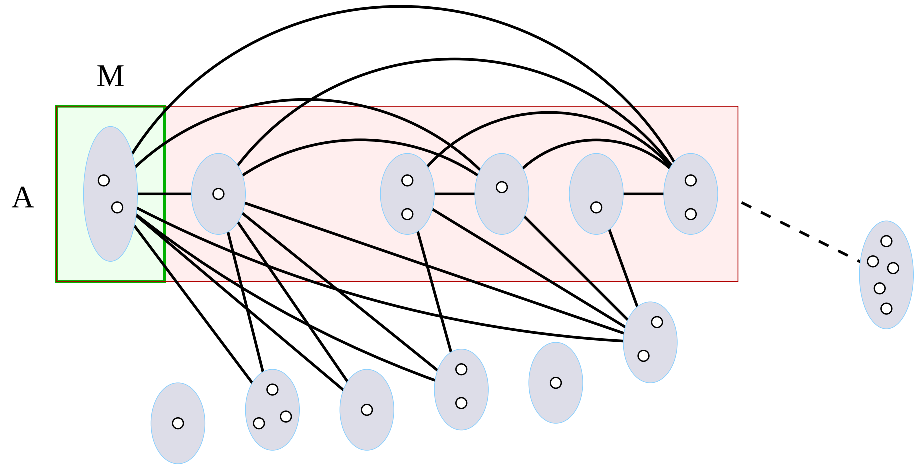

Figure 4: The figure illustrates the construction of marked two-chain graphs. The initial set M and the neighbourhood set A are highlighted. In each step, a set of new vertices is added and the vertices are made adjacent to all vertices of the current set A or to no vertex, and then, they are added to the neighbourhood set A or not. The building sequence that we can associate with the depicted marked two-chain graph is $\langle 2, 1, 1, 3, 1, 2, 2, 1, 1, 1, 2, 2, 5\rangle$, where we simply give the cardinalities instead of the sets of vertices. Note that the depicted marked two-chain graph has several building sequences.

Lemma 5.2. *Let $\Sigma = (G; M, A)$ be a marked two-chain graph where M and $V(G) \setminus M$ are non-empty. Then, one of the two cases applies:*

1) G has an isolated vertex, or $\langle M, (V(G) \setminus M)\rangle$ is a building sequence for Σ

2) G has a vertex pair x, y with $x, y \notin M$ such that either x, y is a twin pair of G or $N_G(x) \cap M = \emptyset$ and $M \cup \{x\} \subseteq N_G(y)$.

Proof. Let $\langle B_1, \ldots, B_t\rangle$ be a building sequence for Σ where $B_1 = M$. We assume that the first case of the lemma does not apply. Then, each vertex of G has a neighbour and at least two of the sets $B_2, \ldots, B_t$ are non-empty. If there is $2 \leq i \leq t$ such that $|B_i| \geq 2$ then the vertices in B_i are pairwise false twins of G and G has a twin pair that satisfies the condition of the second case of the lemma. We henceforth assume $|B_i| \leq 1$ for every $2 \leq i \leq t$. Note here that B_i may indeed be empty.

Assume that G has a non-adjacent vertex pair x, z with $x \notin M$ and $z \in M$. Let p with $2 \leq p \leq t$ be such that $x \in B_p$. Observe $N_G(x) \cap M = \emptyset$, and the definition of building sequences implies $(B_1 \cup \cdots \cup B_{p-1}) \cap N_G(x) = \emptyset$, so that $N_G(x) \subseteq B_{p+1} \cup \cdots \cup B_t$. Let $y \in N_G(x)$, that exists, since x is not an isolated vertex. According to Definition 5.1, $B_p \subseteq N_G(y)$, and thus, $A \cap (B_1 \cup \cdots \cup B_p) \subseteq N_G(y)$, so that $M \cup \{x\} \subseteq B_1 \cup B_p \subseteq N_G(y)$ in particular, and the second case of the lemma applies.

Assume that each vertex in $B_2 \cup \cdots \cup B_t$ is adjacent to each vertex in M. Let p and q with $1 < p < q \leq t$ be largest possible such that B_p and B_q are non-empty; let $B_p = \{x\}$ and $B_q = \{y\}$. Since x and y are not isolated vertices of G and $(B_{p+1} \cup \cdots \cup B_{q-1}) \cup (B_{q+1} \cup \cdots \cup B_t) = \emptyset$ by the choice of p and q:

$$\begin{array}{rcl} & N_G(y) & = & (A \cap (B_1 \cup \cdots \cup B_{p-1})) \cup (A \cap B_p) \\ A \cap (B_1 \cup \cdots \cup B_{p-1}) \;\subseteq & N_G(x) & \subseteq & (A \cap (B_1 \cup \cdots \cup B_{p-1})) \cup B_q \,. \end{array}$$

If $x \in A$ then $B_p \subseteq A$ and $N_G[x] = N_G[y]$, and if $x \notin A$ then $A \cap B_p = \emptyset$ and $N_G(x) = N_G(y)$. In both cases, x, y is a twin pair of G that satisfies the condition of the second case of the lemma. ∎

Let G be a two-chain graph, and let (A, C) be a partition of $V(G)$. We say that (A, C) is a *building partition* for G if $(G; M, A)$ is a marked two-chain graph for some $M \subseteq V(G)$. Analogously, we can associate a building partition with every building sequence for G.

Lemma 5.3. *Let G be a two-chain graph with building sequence $\langle B_1, \ldots, B_t\rangle$. Then, G has a building sequence $\langle D_1, \ldots, D_s\rangle$ such that $D_1, \ldots, D_s$ are maximal sets of pairwise false twins of G and $B_1 \subseteq D_s$.*

Proof. We prove the claim by induction on t, and we may assume that G is non-empty. If G is edgeless then $\langle B_1 \cup \cdots \cup B_t\rangle$ is a building sequence for G of the desired properties. Note that G is edgeless if $t = 1$ or if $t = 2$ and B_1 or B_2 is empty. We henceforth assume that G is not edgeless. If $t = 2$ then B_1 and B_2 are non-empty and G has an adjacent vertex pair x, y with $x \in B_1$ and $y \in B_2$, and thus, $N_G(x) = B_2$ and $N_G(y) = B_1$, and $\langle B_2, B_1\rangle$ is a building sequence for G. Since the vertices in B_1 are adjacent to the vertices in B_2, B_1 and B_2 are maximal sets of pairwise false twins, and $\langle B_2, B_1\rangle$ has the desired properties. We henceforth assume $t \geq 3$. If $B_2 = \emptyset$ or $B_t = \emptyset$ then $\langle B_1, B_3, \ldots, B_t\rangle$ or $\langle B_1, \ldots, B_{t-1}\rangle$ is a building sequence for G, and we can apply the induction hypothesis, and conclude. So, we henceforth assume $B_2 \neq \emptyset$ and $B_t \neq \emptyset$.

Let J be the set of the isolated vertices of G, and assume $J \neq \emptyset$. Note that J is a maximal set of pairwise false twins of G. If $B_1 \neq \emptyset$ and $B_1 \subseteq J$ then G is edgeless, a contradiction to the above assumptions. Let p with $1 \leq p \leq t$ be such that $B_i \cap J \neq \emptyset$. For every $1 \leq i \leq t$, let $B_i' =_{\text{def}} B_i \setminus J$. Since $B_1, \ldots, B_t$ are sets of pairwise false twins, $B_i' = B_i$ or $B_i' = \emptyset$, and $B_1' = B_1$ and $B_p' = \emptyset$, and $p \geq 2$. So, $\langle B_1', \ldots, B_{p-1}', B_{p+1}', \ldots, B_t'\rangle$ is a building sequence for $G \setminus J$. It is important to observe that $B_1' = B_1 = \emptyset$ is possible. We can apply the induction hypothesis and obtain a building sequence $\langle D_1, \ldots, D_s\rangle$ for $G \setminus J$ that has the desired properties. Since G is not edgeless, $G \setminus J$ is not edgeless, and therefore, $s \geq 2$. Then, $\langle D_1, J, D_2, \ldots, D_s\rangle$ is a building sequence for G, that has the desired properties. We henceforth assume that G has no isolated vertices.

Let (A, C) be the building partition for G associated with $\langle B_1, \ldots, B_t\rangle$. Assume $B_1 = \emptyset$. If $B_2 \subseteq C$ then the vertices in B_2 are isolated vertices of G, a contradiction, so that $B_2 \subseteq A$. Then, $\langle B_2, \ldots, B_t\rangle$ is a building sequence for G, and we can conclude by an application of the induction hypothesis. We henceforth assume $B_1 \neq \emptyset$. If $B_1 \cup B_2$ is a set of pairwise false twins or if $B_2 \cup B_3$ is a set of pairwise false twins then $\langle (B_1 \cup B_2), B_3, \ldots, B_t\rangle$ or $\langle B_1, (B_2 \cup B_3), B_4, \ldots, B_t\rangle$ is a building sequence for G, and we can conclude by applying the induction hypothesis. We henceforth assume that $B_1 \cup B_2$ and $B_2 \cup B_3$ are not sets of pairwise false twins. Since the vertices in B_t are not isolated vertices, it follows $N_G(x) = A \cap (B_1 \cup \cdots \cup B_{t-1}) = A \setminus B_t$ for every $x \in B_t$.

We distinguish between two cases about B_2. As the first case, assume $B_2 \subseteq C$. Then, $N_G(x) = B_1$ for every $x \in B_2$, which also means that B_1 is a maximal set of pairwise false twins of G. We show $B_3 \subseteq A$ and $t \geq 4$: if $B_3 \subseteq C$ or if $B_3 \subseteq A$ and $t = 3$ then $N_G(y) = B_1$ for every $y \in B_3$, and $B_2 \cup B_3$ is a set of pairwise false twins, a contradiction. It follows for every vertex u in $B_4 \cup \cdots \cup B_t$: $(B_1 \cup B_3) \cap N_G(u) = \emptyset$ or $B_1 \cup B_3 \subseteq N_G(u)$. An easy consequence: B_2 is a maximal set of pairwise false twins of G. Now, observe that $\langle B_1, B_3, \ldots, B_t \rangle$ is a building sequence for $G \setminus B_2$, and an application of the induction hypothesis yields a building sequence $\langle D_1, \ldots, D_s \rangle$ for $G \setminus B_2$ of the desired properties. If $D_s = B_1$ then $\langle D_1, \ldots, D_{s-1}, B_2, D_s \rangle$ is a building sequence for G of the desired properties, and if $B_1 \subset D_s$ then $\langle D_1, \ldots, D_{s-1}, (D_s \setminus B_1), B_2, B_1 \rangle$ is a building sequence for G of the desired properties.

As the second case, assume $B_2 \subseteq A$. Analogous to the preceding paragraph, for every vertex u in $B_3 \cup \cdots \cup B_t$: $(B_1 \cup B_2) \cap N_G(u) = \emptyset$ or $B_1 \cup B_2 \subseteq N_G(u)$. If the vertices in B_1 and B_2 are non-adjacent in G then $B_1 \cup B_2$ is a set of pairwise false twins, a contradiction to our assumptions, so that the vertices in B_1 and B_2 are adjacent, and B_1 and B_2 are maximal sets of pairwise false twins, particularly since B_1 and B_2 are non-empty. Observe that $\langle B_2, \ldots, B_t \rangle$ is a building sequence for $G \setminus B_1$, and an application of the induction hypothesis yields a building sequence $\langle D_1, \ldots, D_s \rangle$ for $G \setminus B_1$ of the desired properties. If $D_s = B_2$ then $\langle D_1, \ldots, D_s, B_1 \rangle$ is a building sequence for G of the desired properties, and if $B_2 \subset D_s$ then $\langle D_1, \ldots, D_{s-1}, (D_s \setminus B_2), B_2, B_1 \rangle$ is a building sequence for G of the desired properties. ∎

Let G be a two-chain graph, and let $\langle B_1, \ldots, B_t \rangle$ be a building sequence for G. We call $\langle B_1, \ldots, B_t \rangle$ a *normal building sequence* for G if $B_1, \ldots, B_t$ are maximal sets of pairwise false twins of G. This particularly means that $B_1, \ldots, B_t$ are non-empty, unless G is empty.

Corollary 5.4. *Let G be a two-chain graph that is not edgeless, and let x be a vertex of G. Then, G has a normal building sequence $\langle D_1, \ldots, D_s \rangle$ with building partition (A, C) such that $x \notin D_1$ and $x \in A$.*

Proof. Let $\langle B_1, \ldots, B_t \rangle$ be a building sequence for G. We apply Lemma 5.3: G has a normal building sequence $\langle D_1, \ldots, D_s \rangle$ with associated building partition (A, C). We may assume $D_s \subseteq A$. If $x \in A \cap (D_2 \cup \cdots \cup D_s)$ then we can already conclude. Otherwise, $x \in D_1$ or $x \in C$.

We apply Lemma 5.3 to $\langle D_1, \ldots, D_s \rangle$: G has a normal building sequence $\langle E_1, \ldots, E_s \rangle$ with $E_s = D_1$ and associated building partition (A', C'). Also here, we may assume $E_s \subseteq A'$. If $x \in A' \cap (E_2 \cup \cdots \cup E_s)$ then we can directly conclude, and if $x \in E_1$ then we conclude after another application of Lemma 5.3. Otherwise, $N_G(x) = \emptyset$, which we observe as follows: if $x \in D_1$ then $x \in A' \cap E_s$, and if $x \in C$ and $N_G(x) \neq \emptyset$ then $D_1 \subseteq N_G(x)$, and $x \in A'$.

So, let J be the set of the isolated vertices of G, and let $\langle D'_1, \ldots, D'_{s-1} \rangle$ be a normal building sequence for $G \setminus J$. It is easy to see that $\langle D'_1, \ldots, D'_{s-1}, J \rangle$ is a normal building sequence for G of the desired properties. ∎

Lemma 5.5. *Let G be a connected two-chain graph. Let $\langle D_1, \ldots, D_s \rangle$ be a normal building sequence for G with building partition (A, C), and assume $s \geq 3$. Let $z \in D_1$ and $d \in D_{s-1}$ and $c \in D_s$. Then, $D_1 \cup D_{s-1} \subseteq A$, and $(C \cup D_2) \subseteq N_G(z)$ and $N_G(c) = A \setminus D_s$, and if $D_1 \subseteq N_G(d)$ then $\langle D_1, \ldots, D_{s-2}, D_s, D_{s-1} \rangle$ is a normal building sequence for G.*

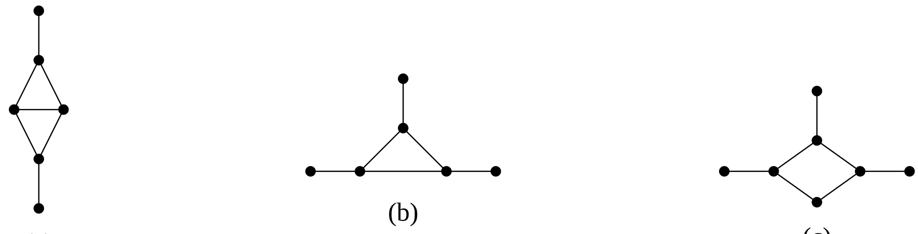

Figure 5: We use these names for the depicted graphs: (a) diamond net, (b) triangle net, (c) square net.

Proof. Since $s \geq 3$, G has at least three vertices, and since G is connected, every vertex of G has a neighbour. So, $N_G(c) = A \cap (D_1 \cup \cdots \cup D_{s-1}) = A \setminus D_s$. If $D_{s-1} \subseteq C$ then $N_G(d) = A \cap (D_1 \cup \cdots \cup D_{s-2}) = A \setminus D_s$, and $D_{s-1} \cup D_s$ is a set of pairwise false twins, a contradiction to the properties of normal building sequences, so that $D_{s-1} \subseteq A$. Note that $D_1 \subseteq A$ follows from Definition 5.1 directly.

We consider the vertices in $C \cup D_2$. Let $x \in C$, and let $x \in D_i$. Since $N_G(x) \neq \emptyset$, $N_G(x) = A \cap (D_1 \cup \cdots \cup D_{i-1})$, in particular, $D_1 \subseteq N_G(x)$, and thus, $D_i \subseteq N_G(z)$. Let $y \in D_2$. If $yz \notin E(G)$ then $D_2 \subseteq A$, and $D_1 \cup D_2$ is a set of pairwise false twins, a contradiction. Thus, $yz \in E(G)$, i.e., $D_2 \subseteq N_G(z)$.

Assume $D_1 \subseteq N_G(d)$. This means $N_G(d) = (A \cap (D_1 \cup \cdots \cup D_{s-2})) \cup D_s = (A \setminus D_{s-1}) \cup D_s$, and thus, $\langle D_1, \ldots, D_{s-2}, D_s, D_{s-1} \rangle$ is a building sequence for G with building partition $((A \cup D_s), (C \setminus D_s))$. ■

We are ready to prove the main results of this subsection. We want to extend two-chain graphs by a single vertex. We want to show that the extended graph is a two-chain graph or contains a specific graph as an induced subgraph. Let G be a graph, and let u be a vertex of G. Let v be a new vertex. We say that we obtain H from G by *adding v as a pendant vertex at u* if H is a graph and v is a vertex of H and $G = H - v$ and $N_H(v) = \{u\}$. In other words, H is a graph that is obtained from G by adding the new vertex v and making it adjacent to u. We consider graphs that are obtained from two-chain graphs by adding a pendant vertex. The specific graphs that we are going to identify are the three graphs of Figure 5.

We do not consider arbitrary two-chain graphs but two-chain graphs without twin pairs. Let G be a graph. We say that G is *twin-free* if G has no twin pairs, and we say that G is *weakly twin-free* if G has no true twin pairs and for every vertex pair x, y of G with $x \neq y$ and $N_G(x) = N_G(y)$, $|N_G(x)| = |N_G(y)| = 1$ must hold. Observe that weakly twin-free graphs are twin-free on the vertices of degree at least 2.

Lemma 5.6. *Let G be a connected two-chain graph on at least three vertices. Let u be a vertex of G and let v be a new vertex. Obtain H from G by adding v as a pendant vertex at u. Assume that H is weakly twin-free and $|N_G(u)| \geq 2$. Then, one of the two cases applies:*

1) *G has a normal building sequence $\langle D_1, \ldots, D_s \rangle$ such that $u \in D_s$*

2) *H contains one of the following graphs as an induced subgraph:* diamond net *or* triangle net *or* square net.

Proof. We assume that the first case of the lemma does not apply. Let $\langle D_1, \ldots, D_s \rangle$ be a normal building sequence for G with building partition (A, C), and we may assume $D_s \subseteq A$. We also assume $u \notin D_1 \cup D_s$. This particularly means $s \geq 3$. Let $z \in D_1$ and $d \in D_{s-1}$ and $c \in D_s$. Due to Lemma 5.5, $D_1 \cup D_{s-1} \subseteq A$ and $N_G(c) = A \setminus D_s$. Observe $D_s = \{c\}$, because: $D_1 \cup D_{s-1} \subseteq N_G(c)$, so that $|N_G(c)| \geq 2$, and $N_H(x) = N_G(x)$ for every $x \in D_s$, since $u \notin D_s$, so that $|D_s| \geq 2$ contradicts the weak twinfreeness of H. If $u \in D_{s-1}$ and $D_1 \subseteq N_G(u)$ then the first case of the lemma applies due to Lemma 5.5, a contradiction, and if $u \in D_{s-1}$ and $D_1 \not\subseteq N_G(u)$ then $N_G(u) = N_G(d) = D_s = \{c\}$, i.e., $|N_G(u)| = 1$, also a contradiction. Thus, $u \notin D_1 \cup D_{s-1} \cup D_s$. If $D_1 \subseteq N_G(d)$ then $N_G(d) = A \setminus D_{s-1}$ due to Lemma 5.5, and $|N_G(d)| \geq 2$ and either $|D_{s-1}| \geq 2$ or d and c are true twins. Both cases yield a contradiction. Thus, $D_1 \not\subseteq N_G(d)$ and $N_G(d) = D_s = \{c\}$. We apply these considerations to two building sequences for G.

Recall that G is connected and has at least three vertices, so that G is not edgeless. Due to Corollary 5.4, G has a normal building sequence $\langle E_1, \ldots, E_s \rangle$ with building partition (A, C) such that $u \in A \setminus E_1$. We apply Lemma 5.3 to $\langle E_1, \ldots, E_s \rangle$ and obtain a normal building sequence $\langle E'_1, \ldots, E'_s \rangle$ for G, where $E'_s = E_1$. According to our initial assumption: $u \notin E_1 \cup E_s \cup E'_1 \cup E'_s$, and the considerations of the preceding paragraph are applicable, in particular, $u \notin E_1 \cup E_{s-1} \cup E_s$ and $u \notin E'_1 \cup E'_{s-1} \cup E'_s$.

Let $z \in E_1$ and $a \in E'_{s-1}$ and $d \in E_{s-1}$ and $c \in E_s$. From the first paragraph, it follows: $N_G(d) = \{c\}$ and $N_G(a) = \{z\}$ and $zc \in E(G)$. Observe that z, a, d, c are pairwise different vertices, and they are different from u, and $uc \in E(G)$. If $uz \in E(G)$ then $\{z, u, c, a, v, d\}$ induces a triangle net in H. Otherwise, $uz \notin E(G)$. Note that $N_G(d) \subseteq \{c\} \subseteq N_G(u)$ is the case, and since $u \notin E_{s-1}$, there is $y \in N_G(u) \setminus N_G(d)$. Let $u \in E_p$; recall $2 \leq p \leq s - 2$. Thus, $y \in E_{p+1} \cup \cdots \cup E_s$, and $yz \in E(G)$ follows. Note also $y \neq d$ and $y \neq c$. If $yc \notin E(G)$ then $\{z, y, u, c, a, v, d\}$ induces a square net in H, and if $yc \in E(G)$ then $\{z, y, u, c, a, v\}$ induces a diamond net in H. We conclude that the second case of the lemma applies. ∎

Lemma 5.7. *Let G be a connected two-chain graph. Let u be a vertex of G, and assume that $N_G(u)$ is a set of pairwise false twins of G. Let $\langle D_1, \ldots, D_s \rangle$ be a normal building sequence for G with building partition (A, C), and assume $s \geq 3$. Then, one of the two cases applies:*

1) $u \in C \cap D_2$ *and* $N_G(u) = D_1$

2) $u \in A \cap D_{s-1}$ *and* $N_G(u) = D_s$.

Proof. Let $z \in D_1$ and $c \in D_s$. Due to Lemma 5.5, $D_1 \cup D_{s-1} \subseteq A$ and $(C \cup D_2) \subseteq N_G(z)$ and $N_G(c) = A \setminus D_s$. If $u \in D_1$ then $D_2 \cup D_s \subseteq N_G(u)$, and if $u \in D_s$ then $D_1 \cup D_{s-1} \subseteq A \setminus D_s = N_G(c) = N_G(u)$, a contradiction in both cases. Thus, $u \notin D_1 \cup D_s$. Let p with $1 < p < s$ be such that $u \in D_p$.

If $u \in C$ then $D_p \subseteq C$ and $N_G(u) = A \cap (D_1 \cup \cdots \cup D_{p-1})$, and $D_1 \subseteq A$ implies $N_G(u) = D_1$ and $D_2 \cup \cdots \cup D_{p-1} \subseteq C$. Since G is connected, $D_2 \cup \cdots \cup D_p$ is a set of pairwise false twins of G, which is only possible in case of $p = 2$, and the first case of the lemma applies.

If $u \in A$ then $D_p \subseteq A$, and $D_s \subseteq N_G(u)$ implies $N_G(u) = D_s$. Since G is connected and has no isolated vertices, $D_{p+1} \cup \cdots \cup D_{s-1} \subseteq A$ must follow. And since $N_G(z) \cap (D_{p+1} \cup \cdots \cup D_s) =$

$N_G(u) \cap (D_{p+1} \cup \cdots \cup D_s)$, it follows that $D_p \cup \cdots \cup D_{s-1}$ is a set of pairwise false twins of G, which is only possible in case of $p = s - 1$, and the second case of the lemma applies. ■

Let G be a graph. We call G a *star graph* if G is connected and has a vertex x such that each edge of G is incident to x. Clearly, $N_G[x] = V(G)$ and $N_G(y) = \{x\}$ for every vertex y of G with $y \neq x$.

Corollary 5.8. *Let G be a connected two-chain graph that is not a star graph. Let u be a vertex of G and let v be a new vertex. Obtain H from G by adding v as a pendant vertex at u. Assume that H is weakly twin-free and $|N_G(u)| \geq 2$. Also assume that H does not contain any of the following graphs as an induced subgraph:* diamond net *and* triangle net *and* square net.

Assume that G has a vertex pair a, b with $b \neq u$ such that $N_G(a) = \{b\}$. Then, H has a normal building sequence $\langle D_1, \ldots, D_s \rangle$ such that $D_1 = \{b\}$ and $a \in D_2$ and $v \in D_{s-1}$ and $D_s = \{u\}$.

Proof. Note that G has at least three vertices, and we can apply Lemma 5.6: G has a normal building sequence $\langle D_1, \ldots, D_s \rangle$ such that $u \in D_s$. If $s = 2$ and $|D_1| \geq 2$ and $|D_s| \geq 2$ then H is not weakly twin-free, and if $s = 2$ and $|D_1| = 1$ or $|D_s| = 1$ then G is a star graph. So, $s \geq 3$ must hold.

We determine the sets containing a and b. Since $N_G(a)$ is a set of pairwise false twins of G, we can apply Lemma 5.7: since $b \neq u$ and $u \in D_s$, $D_s \neq N_G(a)$ follows, so that $a \in D_2$ and $N_G(a) = \{b\} = D_1$.

Assume that v has no false twin in H. Then, $\{v\}$ is a maximal set of pairwise false twins of H, and $\langle D_1, D_2, \ldots, D_{s-1}, \{v\}, D_s \rangle$ or $\langle D_1, D_2, \ldots, D_{s-1}, (D_s \setminus \{u\}), \{v\}, \{u\} \rangle$ is a normal building sequence for H of the desired properties.

Assume that v has a false twin x in H. Then, $N_H(x) = N_H(v) = \{u\} = N_G(x)$, and $N_G(x)$ is a set of pairwise false twins of G, and we can apply Lemma 5.7: $x \notin D_2$ because of $u \neq b$ and $D_1 = \{b\}$, and thus, $x \in D_{s-1}$ and $N_G(x) = \{u\} = D_s$. Then, $\langle D_1, \ldots, D_{s-2}, (D_{s-1} \cup \{v\}), D_s \rangle$ is a normal building sequence for H of the desired properties. ■

5.2 Base graph combinations and pendant vertex extension

We combine two-chain graphs into sequences of two-chain graphs, thereby generalising two-chain graphs and graphs of linear clique-width at most 2. We prove a structural property of these graphs, that corresponds to Corollary 5.4 of the preceding subsection. We consider the extension of these graphs by adding a pendant vertex, mainly relying on the results of the preceding subsection.

Definition 5.9. *Marked sequence two-chain graphs and their building sequences are defined inductively.*

1) *Let $\Sigma = (G; M, A)$ be a marked two-chain graph. Then, $(G; A)$ is a* marked sequence two-chain graph, *and $\langle \Sigma \rangle$ is a* building sequence *for $(G; A)$.*

2) *Let $(G; A)$ be a marked sequence two-chain graph with building sequence $\langle \Sigma_1, \ldots, \Sigma_t \rangle$. Let $\Sigma = (H; M, B)$ be a marked two-chain graph. Let $K =_{\text{def}} \{ab : a \in A \text{ and } b \in M\}$. Let F*

be obtained from the disjoint union of G *and* H *by adding the edges in* K*. Then,* $(F;B)$ *is a* marked sequence two-chain graph*, and* $\langle \Sigma_1,\ldots,\Sigma_t,\Sigma\rangle$ *is a* building sequence *for* $(F;B)$*.*

A graph G *is a* sequence two-chain graph *if there is* $A \subseteq V(G)$ *such that* $(G;A)$ *is a marked sequence two-chain graph, and a* building sequence *for* G *is a building sequence for a marked sequence two-chain graph* $(G;A)$ *where* $A \subseteq V(G)$*.*

Observe that two-chain graphs are sequence two-chain graphs. As the main structural result of this subsection, we show that sequence two-chain graphs have building sequences with good properties. A graph G is called *complete bipartite* if G admits a partition (C,D) of $V(G)$ such that for every vertex pair u,v of G with $u \neq v$, $uv \in E(G)$ if and only if $u \in C$ and $v \in D$, or $v \in C$ and $u \in D$. Let $\Sigma = (H;M,A)$ be a marked two-chain graph. Set M of Σ is called the *initializer* of Σ. We call Σ *easy complete bipartite* if H is complete bipartite with vertex set partition $(M,(V(G)\setminus M))$. The first case of Lemma 5.2 does not apply to connected marked two-chain graphs that are not easy complete bipartite.

Lemma 5.10. *Let* G *be a connected sequence two-chain graph with building sequence* $\langle \Sigma_1,\ldots,\Sigma_t\rangle$*. Assume that* G *is not a complete bipartite graph. Then,* G *has a building sequence* $\langle \Delta_r,\ldots,\Delta_1\rangle$ *such that* $V(\Sigma_1) \subseteq V(\Delta_1)$ *and* Δ_r *is not edgeless and* Δ_1 *is not easy complete bipartite and* $\Delta_{r-1},\ldots,\Delta_1$ *have non-empty initializers.*

Proof. If $t = 1$ then Σ_1 is not easy complete bipartite, since otherwise, G would be complete bipartite, and $\langle \Sigma_1\rangle$ is a building sequence for G of the desired properties. We henceforth assume $t \geq 2$. For $1 \leq i \leq t$, let $\Sigma_i = (G_i;M_i,A_i)$, let $C_i =_{\mathrm{def}} V(G_i)\setminus A_i$, and let $A_0 =_{\mathrm{def}} M_{t+1} =_{\mathrm{def}} \emptyset$. According to Definition 5.9, for every $1 \leq p \leq t$ and $x \in V(G_p)$:

$$N_G(x) \cap \Big(V(G_{p+1}) \cup \cdots \cup V(G_t)\Big) \;=\; \begin{cases} \emptyset & ,\ \text{if } x \in C_p \\ M_{p+1} & ,\ \text{if } x \in A_p \end{cases}$$

$$\text{and}\quad N_G(x) \cap \Big(V(G_1) \cup \cdots \cup V(G_{p-1})\Big) \;=\; \begin{cases} \emptyset & ,\ \text{if } x \notin M_p \\ A_{p-1} & ,\ \text{if } x \in M_p\,. \end{cases}$$

Assume that G_1 is edgeless. Then, for every $x \in V(G_1)$, $N_G(x) \cap V(G_1) = \emptyset$, and thus, $N_G(x) = M_2$. Let $\Sigma_2' =_{\mathrm{def}} (G[V(G_1)\cup V(G_2)];M_2,A_2)$. Let $\langle B_1,\ldots,B_s\rangle$ be a building sequence for Σ_2 with $M_2 = B_1$. Then, $\langle B_1,V(G_1),B_2,\ldots,B_s\rangle$ is a building sequence for Σ_2', also in case of $V(G_1) = \emptyset$. Thus, Σ_2' is a marked two-chain graph, and $\langle \Sigma_2',\Sigma_3,\ldots,\Sigma_t\rangle$ is a building sequence for G. If $V(G_t) = \emptyset$ then $\langle \Sigma_1,\ldots,\Sigma_{t-1}\rangle$ is a building sequence for G. We henceforth assume that G_1 is not edgeless and G_t is non-empty.

Let $a \in V(G_1)$ and $b \in V(G_t)$. Since G is connected, G has an a,b-path, that contains a vertex x_p with $x_p \in V(G_p) \cup \cdots \cup V(G_t)$ such that $N_G(x_p) \cap \Big(V(G_1) \cup \cdots \cup V(G_{p-1})\Big) \neq \emptyset$ for every $2 \leq p \leq t$. Due to the above observations, $x_p \in M_p$ must hold, and thus, $M_2,\ldots,M_t$ are non-empty. And since G has no isolated vertices, we can also assume that M_1 is non-empty, as we argued in the proof of Lemma 5.3. For $1 \leq i \leq t$, let $z_i \in M_i$, and let

$$\begin{aligned} H_i \;&=_{\mathrm{def}}\; G\Big[V(G_i) \cup M_{i+1}\Big] \quad \text{and} \\ \Pi_i \;&=_{\mathrm{def}}\; \Big(H_i;\ M_{i+1},\ (N_{H_i}(z_i) \cup M_i)\Big)\,. \end{aligned}$$

Assume that $\Pi_1, \ldots, \Pi_t$ are marked two-chain graphs. Let $\Delta_1 =_{\text{def}} \Pi_1$ and $\Delta_i =_{\text{def}} \Pi_i \setminus M_i$ for $2 \leq i \leq t$, i.e., $\Delta_i = ((H_i \setminus M_i); M_{i+1}, N_{H_i}(z_i))$. Note that z_i is not a vertex of Δ_i. It follows that $\Delta_1, \ldots, \Delta_t$ are marked two-chain graphs. It is not difficult to see that $\langle \Delta_t, \ldots, \Delta_1 \rangle$ is a building sequence for G.

We verify the conditions of the lemma. Clearly, $V(\Sigma_1) \subseteq V(\Delta_1)$ is the case. The initializers of $\Delta_{t-1}, \ldots, \Delta_1$ are $M_t, \ldots, M_2$, that we already proved non-empty. If Δ_t is edgeless then we apply the simplification result from the beginning of the proof. If Δ_1 is easy complete bipartite then H_1 is complete bipartite with vertex set partition $(M_2, (V(H_1) \setminus M_2))$, where $V(H_1) \setminus M_2 = V(G_1)$, and G_1 is edgeless, a contradiction.

We prove that $\Pi_1, \ldots, \Pi_t$ are indeed marked two-chain graphs. Let $1 \leq p \leq t$. It suffices to show that H_p is a two-chain graph with an appropriate building sequence. Since G has no isolated vertices, $M_p \subseteq N_G(x)$ for every $x \in C_p$.

Let a, c, d be new vertices. Obtain F from H_p by adding a, c, d with these neighbourhoods: $N_F(a) = M_p$ and $N_F(c) = A_p \cup \{d\}$ and $N_F(d) = M_{p+1} \cup \{c\}$. Observe that F is connected: every vertex in $A_p \cup \{d\}$ is adjacent to c, and every vertex in $C_p \cup \{a\}$ is adjacent to z_p, and $z_p \in A_p$. Note here that (a, z_p, c, d) is a chordless path of F.

We show that F is a two-chain graph. Let $\langle E_1, \ldots, E_l \rangle$ be a building sequence for Σ_p with $E_1 = M_p$ and building partition (A_p, C_p). Then, $\langle E_1, \ldots, E_l, M_{p+1} \rangle$ is a building sequence for H_p with building partition $((A_p \cup M_{p+1}), C_p)$. And then, $\langle E_1, \{a\}, E_2, \ldots, E_l, \{d\}, (M_{p+1} \cup \{c\}) \rangle$ is a building sequence for F with building partition $((A_p \cup M_{p+1} \cup \{c, d\}), (C_p \cup \{a\}))$. Due to Corollary 5.4, F has a normal building sequence $\langle D_1, \ldots, D_s \rangle$ with building partition (A'', C'') such that $z_p \notin D_1$. We may assume $D_s \subseteq A''$. Since F is not edgeless and not complete bipartite, $s \geq 3$ must hold.

Observe that $N_F(a)$ and $N_F(d)$ are sets of pairwise false twins of F, and we can apply Lemma 5.7:

- since $z_p \in N_F(a)$ and $z_p \notin D_1$: $\quad N_F(a) = D_s$ and $a \in A'' \cap D_{s-1}$
- since $N_F(a) \neq N_F(d)$: $\quad N_F(d) \neq D_s$ and thus $N_F(d) = D_1$ and $d \in C'' \cap D_2$.

So, $D_s = M_p$ and $D_1 = M_{p+1} \cup \{c\}$. And due to Lemma 5.5 and our definitions, $N_F(z_p) = A'' \setminus D_s = A'' \setminus M_p$, and thus, $N_{H_p}(z_p) \cup M_p \cup \{a, c\} = A''$. We conclude that

$$\Big\langle M_{p+1},\ (D_2 \setminus \{d\}),\ D_3,\ \ldots,\ D_{s-2},\ (D_{s-1} \setminus \{a\}),\ M_p \Big\rangle$$

is a building sequence for H_p with building partition $((A'' \setminus \{c, a\}), (C'' \setminus \{d\}))$, and Π_p is a marked two-chain graph indeed.

It is noteworthy that the construction is also valid in case G_p is edgeless. Then, C_p is empty, and the vertices in $A_p \setminus M_p$ are false twins of d in F_p, and thus, $D_2 = (A_p \setminus M_p) \cup \{d\}$. ∎

Corollary 5.11. *Let G be a connected sequence two-chain graph that is not a two-chain graph, and let x be a vertex of G. Then, G has a building sequence $\langle \Delta_1, \ldots, \Delta_s \rangle$ with $s \geq 2$ such that $x \notin V(\Delta_1)$ and Δ_1 is not edgeless and Δ_s is not easy complete bipartite and $\Delta_2, \ldots, \Delta_s$ have non-empty initializers.*

Proof. Due to Lemma 5.10, G has a building sequence $\langle \Sigma_1, \ldots, \Sigma_s \rangle$ such that Σ_1 is not edgeles and Σ_s is not easy complete bipartite and $\Sigma_2, \ldots, \Sigma_s$ have non-empty initializers. If $s = 1$ then G is a two-chain graph, a contradiction to the assumptions. If $x \notin V(\Sigma_1)$ then we can already conclude. Otherwise, $x \in V(\Sigma_1)$.

We apply Lemma 5.10 to $\langle \Sigma_1, \ldots, \Sigma_s \rangle$ and obtain a building sequence $\langle \Delta_1, \ldots, \Delta_r \rangle$ for G with $V(\Sigma_1) \subseteq V(\Delta_r)$, and Δ_1 is not edgeless and Δ_r is not easy complete bipartite and $\Delta_2, \ldots, \Delta_r$ have non-empty initializers. If $r = 1$ then G is a two-chain graph, a contradiction, so that $r \geq 2$ must hold, and thus, $x \notin V(\Delta_1)$, and we can conclude. ∎

We are ready to prove our main result about the extension of sequence two-chain graphs by a pendant vertex. We partition the proof into two cases, where we consider two-chain graphs first and sequence two-chain graphs that are not two-chain graphs after.

Lemma 5.12. *Let G be a two-chain graph that is edgeless or connected. Let u be a vertex of G and let v be a new vertex. Obtain H from G by adding v as a pendant vertex at u. Assume that H is weakly twin-free. Then, H is a sequence two-chain graph or H contains one of the following graphs as an induced subgraph:* diamond net *or* triangle net *or* square net*.*

Proof. If G is edgeless then H clearly is a two-chain graph and thus a sequence two-chain graph. If G is a star graph then H is also a two-chain graph. So, we may assume that G is not a star graph and connected, and G has at least four vertices and each vertex of G has a neighbour. We assume that G does not contain any of the listed graphs as an induced subgraph.

Assume that G has a normal building sequence $\langle D_1, \ldots, D_s \rangle$ with building partition (A, C) such that $u \in D_s$. If $s \leq 2$ then G is edgeless or easy complete bipartite, and since H is weakly twin-free, $s = 2$ implies $|D_1| = 1$ or $|D_2| = 1$, and G is a star graph, a contradiction. So, $s \geq 3$ must hold. We can apply Lemma 5.5: $N_G(u) = A \setminus D_s$. Then, H is a two-chain graph with building sequence $\langle D_1, \ldots, D_{s-1}, (D_s \setminus \{u\}), \{v\}, \{u\} \rangle$. If $|N_G(u)| \geq 2$ then G has a normal building sequence of the requested properties due to Lemma 5.6.

As the other case, assume $|N_G(u)| \leq 1$. Since G is connected and has at least four vertices, $|N_G(u)| = 1$ follows. Let $N_G(u) = \{w\}$. Let $\langle D_1, \ldots, D_s \rangle$ be a normal building sequence for G with building partition (A, C) such that $w \notin D_1$, that exists due to Corollary 5.4. Note that $N_G(u)$ is a set of pairwise false twins of G, and $s \geq 3$ as shown in the preceding paragraph, so we can apply Lemmas 5.7 and 5.5: $N_G(u) = \{w\} = D_s$ and $N_G(w) = A \setminus D_s = A \setminus \{w\}$.

Let $\Sigma_1 =_{\text{def}} (H \setminus \{u, v, w\}; D_1, A \setminus \{u, w\})$ and $\Sigma_2 =_{\text{def}} (H[\{u, v, w\}]; \{w\}, \{w, v\})$. According to the preceding paragraph, Σ_1 is a marked two-chain graph, and Σ_2 is a marked two-chain graph clearly. It follows that H is a sequence two-chain graph with building sequence $\langle \Sigma_1, \Sigma_2 \rangle$. ∎

For the proof of the general result about the extension of sequence two-chain graphs by a pendant vertex, we need more graphs. These new graphs are depicted in Figure 6.

Lemma 5.13. *Let G be a connected sequence two-chain graph. Let u be a vertex of G and let v be a new vertex. Obtain H from G by adding v as a pendant vertex at u. Assume that H is twin-free. Then, H is a sequence two-chain graph or H contains one of the following graphs as an induced subgraph:* triangle-0 graph *or* triangle-1 graph *or* triangle-2 graph *or* diamond net *or* triangle net *or* square net *or* square-1 graph *or* square-2 graph*.*

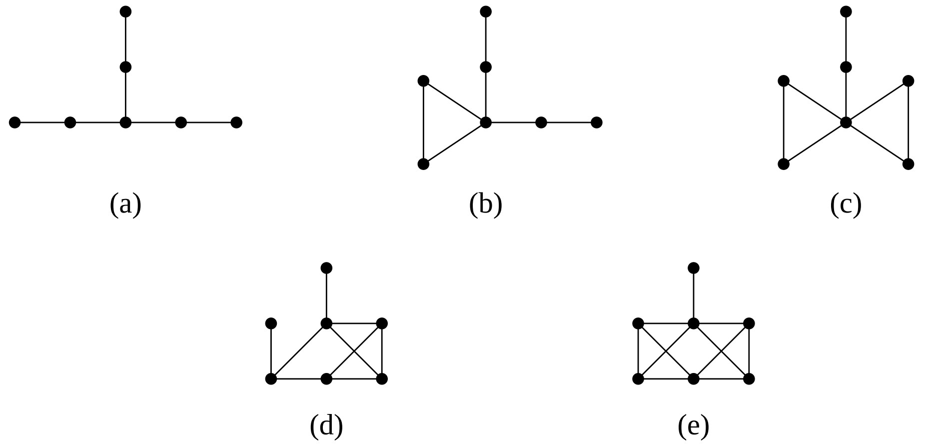

Figure 6: We use these names for the depicted graphs: (a) triangle-0 graph, (b) triangle-1 graph, (c) triangle-2 graph, (d) square-1 graph, (e) square-2 graph.

Proof. If G is a two-chain graph then we can apply Lemma 5.12, and conclude. We henceforth assume that G is not a two-chain graph. This particularly means that G has no isolated vertices. Due to Corollary 5.11, G has a building sequence $\langle \Sigma_1, \dots, \Sigma_t \rangle$ with $t \geq 2$ such that $u \notin V(\Sigma_1)$ and Σ_1 is not edgeless and Σ_t is not easy complete bipartite and $\Sigma_2, \dots, \Sigma_t$ have non-empty initializers.

For $1 \leq i \leq t$, let $\Sigma_i = (G_i; M_i, A_i)$ and let $C_i =_{\text{def}} V(G_i) \setminus A_i$. We choose useful vertices of G. For $2 \leq i \leq t$, let $z_i \in M_i$, that do exist, and let z_0, z_1 be an adjacent vertex pair of G_1 where $z_1 \in A_1$, and let z_{t+1}, z_{t+2} be an adjacent vertex pair of G_t where $z_{t+1}, z_{t+2} \notin M_t$ and $z_t z_{t+1} \in E(G_t)$. Since G_1 is not edgeless, z_0, z_1 do exist, and since G_t is not easy complete bipartite and connected, also z_{t+1}, z_{t+2} do exist. Observe that $(z_0, z_1, z_2, \dots, z_t, z_{t+1}, z_{t+2})$ is a z_0, z_{t+2}-path of G, and z_0 and z_2 may be adjacent and z_t and z_{t+2} may be adjacent in G.

We prove the claim of the lemma by considering different cases about u and its neighbourhood.

Case A: There is p with $2 \leq p \leq t$ such that $u \in M_p$.

Proof of the case. Assume $|M_p| = 1$, i.e., $M_p = \{u\}$. Let $\Sigma'_p =_{\text{def}} (H[V(G_p) \cup \{v\}]; \{u\}, A_p)$. Observe that Σ'_p is a marked two-chain graph. Then, $\langle \Sigma_1, \dots, \Sigma_{p-1}, \Sigma'_p, \Sigma_{p+1}, \dots, \Sigma_t \rangle$ is a building sequence for H, and H is a sequence two-chain graph.

Assume $|M_p| \geq 2$. Let $z \in M_p$ with $z \neq u$. Then, $\{z_{p-2}, z_{p-1}, z, u, v, z_{p+1}, z_{p+2}\}$ induces a square net or a square-1 graph or a square-2 graph in H, depending on $N_G(z) \cap \{z_{p-2}, z_{p+2}\}$. $\square$

We henceforth assume $u \notin M_2 \cup \dots \cup M_t$. Since H is twin-free, this particularly means $|M_2| = \dots = |M_t| = 1$, so that $M_i = \{z_i\}$ for $2 \leq i \leq t$.

Case B: There is $1 \leq p \leq t$ such that $N_G(u) = M_p$.

Proof of the case. If $p = 1$ and $N_G(u) = M_1$ then $u \in M_2$, a contradiction. So, $p \geq 2$.

Note that $N_G(u) = M_p = \{z_p\}$ is the case. Thus, u is of degree 1 in G, so that u is different from z_{p-1}, z_p, z_{p+1}, that are vertices of degree at least 2. Furthermore, since u is non-adjacent to z_{p-1} and z_{p+1}, u is also different from z_{p-2} and z_{p+2}. Thus, u is different from

$z_{p-2}, z_{p-1}, z_p, z_{p+1}, z_{p+2}$. Then, $\{z_{p-2}, z_{p-1}, z_p, u, v, z_{p+1}, z_{p+2}\}$ induces a triangle-0 graph or a triangle-1 graph or a triangle-2 graph in H, depending on the edges $z_{p-2}z_p$ and $z_p z_{p+2}$. $\square$

We henceforth assume $N_G(u) \neq M_i$ for $1 \leq i \leq t$. Recall that u is of degree at least 1. Let p with $2 \leq p \leq t$ be such that $u \in V(G_p)$.

Case C: $|N_G(u)| = 1$

Proof of the case. If $u \in C_p$ then $M_p \subseteq N_G(u)$, so that $M_p = N_G(u)$, a contradiction. Thus, $u \in A_p$. If $p < t$ then $M_{p+1} \subseteq N_G(u)$, so that $N_G(u) = M_{p+1}$, a contradiction. Thus, $p = t$.

We assume that H does not contain any of the listed graphs as an induced subgraph. Let $N_G(u) = \{w\}$. Due to our assumptions, $w \in V(G_t)$ and $w \neq z_t$. Let $G' =_{\text{def}} G[V(G_t) \cup \{z_{t-1}\}]$ and $H' =_{\text{def}} H[V(G_t) \cup \{z_{t-1}, v\}]$. Note that G' is a two-chain graph that is weakly twin-free, and $N_{G'}(z_{t-1}) = \{z_t\}$. By repeating the construction of the second case in the proof of Lemma 5.12, there are marked two-chain graphs Σ'_t, Σ'_{t+1} such that $\langle \Sigma'_t, \Sigma'_{t+1} \rangle$ is a building sequence for H' and M_t is the initializer of Σ'_t. Let $\Sigma'_t = (G'_t; M_t, A'_t)$, and let $\Sigma''_t =_{\text{def}} ((G'_t - z_{t-1}); M_t, (A'_t \setminus \{z_{t-1}\}))$. It directly follows that $\langle \Sigma_1, \ldots, \Sigma_{t-1}, \Sigma''_t, \Sigma'_{t+1} \rangle$ is a building sequence for H, and H is a sequence two-chain graph. $\square$

Case D: $|N_G(u)| \geq 2$

Proof of the case. Let

$$\begin{aligned} G'_p &=_{\text{def}} G\Big[V(G_p) \cup \{z_{p-1}, z_{p+1}, z_{p+2}\}\Big] \quad \text{and} \\ H'_p &=_{\text{def}} H\Big[V(G_p) \cup \{z_{p-1}, z_{p+1}, z_{p+2}\} \cup \{v\}\Big]. \end{aligned}$$

It is important to observe that G'_p is an induced subgraph of G and a two-chain graph, and H'_p is an induced subgraph of H and is obtained from G'_p by adding v as a pendant vertex at u. Furthermore, $N_G(u) = N_{G'_p}(u)$, since $u \notin M_p$ and, for $p < t$, $N_G(u) \cap V(G_{p+1}) \subseteq \{z_{p+1}\}$. Finally, H'_p is weakly twin-free, in particular, since $N_{H'_p}(x) = N_H(x)$ for every vertex x from $V(H'_p) \setminus \{z_{p-1}, z_p, z_{p+1}, z_{p+2}\}$.

Assume that H does not contain any of the listed graphs as an induced subgraph. Then, H'_p does not contain any of the listed graphs as an induced subgraph. Also observe $N_{G'_p}(z_{p-1}) = \{z_p\}$. Since G'_p is a connected two-chain graph that is not a star graph, we can apply Corollary 5.8: H'_p has a normal building sequence $\langle D_1, \ldots, D_s \rangle$ such that $D_1 = \{z_p\}$ and $z_{p-1} \in D_2$ and $v \in D_{s-1}$ and $D_s = \{u\}$. If $p = t$ then $\Sigma'_t =_{\text{def}} (H[V(G_t) \cup \{v\}]; M_t, N_H(u))$ is a marked two-chain graph and $\langle \Sigma_1, \ldots, \Sigma_{t-1}, \Sigma'_t \rangle$ is a building sequence for H. If $p < t$ then $N_{G'_p}(z_{p+2}) = \{z_{p+1}\}$ and $z_{p+1} \notin D_1$ and $z_{p+1} \notin D_s$, and we obtain a contradiction due to Lemma 5.7. $\square$

We have considered all possible cases about u and $N_G(u)$, and thus, we can conclude the lemma. $\blacksquare$

5.3 True- and false-twin extensions

In the preceding two subsections, we introduced the sequence two-chain graphs, studied structural properties and investigated the extension by a pendant vertex. The main result of these two subsections is Lemma 5.13. In this subsection, we consider two further single-vertex extensions: by adding a true twin or a false twin. Let G be a graph, and let u be a vertex of G. Let v be a new vertex. We say that we obtain H from G by *adding v as a true twin or a false twin of u* if H is a graph and v is a vertex of H and $G = H{-}v$ and $N_H[v] = N_H[u]$ or $N_H(v) = N_H(u)$, respectively.

We begin by considering false-twin extensions. This case is easy, and it is no surprise that the class of sequence two-chain graphs is closed under adding false twins.

Lemma 5.14. *Let G be a sequence two-chain graph. Let u be a vertex of G and let v be a new vertex. Obtain H from G by adding v as a false twin of u. Then, H is a sequence two-chain graph.*

Proof. Let $\langle \Sigma_1, \ldots, \Sigma_t \rangle$ be a building sequence for G, and let $u \in V(\Sigma_p)$. Let $\Sigma_p = (G_p; M_p, A_p)$ and let $C_p =_{\text{def}} V(G_p) \setminus A_p$.

Obtain H' from G_p by adding v as a false twin of u. We show that H' is a two-chain graph. Let $\langle B_1, \ldots, B_r \rangle$ be a building sequence for G_p with building partition (A_p, C_p) and where $B_1 = M_p$. Let $u \in B_q$. Then, $\langle B_1, \ldots, B_{q-1}, (B_q \cup \{v\}), B_{q+1}, \ldots, B_r \rangle$ is a building sequence for H' with building partition $((A_p \cup \{v\}), C_p)$ or $(A_p, (C_p \cup \{v\}))$, depending on whether $u \in A_p$ or $u \in C_p$. Let

$$M' =_{\text{def}} \begin{cases} M_p \cup \{v\} & \text{, if } u \in M_p \\ M_p & \text{, otherwise} \end{cases} \qquad \text{and} \qquad A' =_{\text{def}} \begin{cases} A_p \cup \{v\} & \text{, if } u \in A_p \\ A_p & \text{, otherwise,} \end{cases}$$

and let $\Sigma'_p =_{\text{def}} (H'; M', A')$. Clearly, Σ'_p is a marked two-chain graph.

Let the building sequence $\langle \Sigma_1, \ldots, \Sigma_{p-1}, \Sigma'_p, \Sigma_{p+1}, \ldots, \Sigma_t \rangle$ define the sequence two-chain graph F. We show $F = H$. This is clear: $F{-}v = H{-}v = G$. According to the definitions of H' and Σ'_p and M' and A':

$$\begin{array}{rrcccl} & N_F(v) \cap V(\Sigma'_p) & = & N_{H'}(v) = N_{H'}(u) & = & N_G(u) \cap V(\Sigma'_p) \\ \text{if } p > 1: & N_F(v) \cap V(\Sigma_{p-1}) & = & N_F(u) \cap V(\Sigma_{p-1}) & = & N_G(u) \cap V(\Sigma_{p-1}) \\ \text{if } p < t: & N_F(v) \cap V(\Sigma_{p+1}) & = & N_F(u) \cap V(\Sigma_{p+1}) & = & N_G(u) \cap V(\Sigma_{p+1})\,. \end{array}$$

We conclude $N_F(v) = N_F(u) = N_G(u)$, and $F = H$. ▪

The main part of this subsection is dedicated to the extension of sequence two-chain graphs by true twins. We separately consider two-chain graphs and sequence two-chain graphs. We need further graphs, that are depicted in Figure 7.

Lemma 5.15. *Let G be a connected two-chain graph. Let u be a vertex of G and let v be a new vertex. Obtain H from G by adding v as a true twin of u. Then, one of the three cases applies:*

1) *G has a normal building sequence $\langle D_1, \ldots, D_s \rangle$ such that either $u \in D_s$ or $u \in D_{s-1}$ and $D_s = N_G(u)$*

2) *H is a two-chain graph*

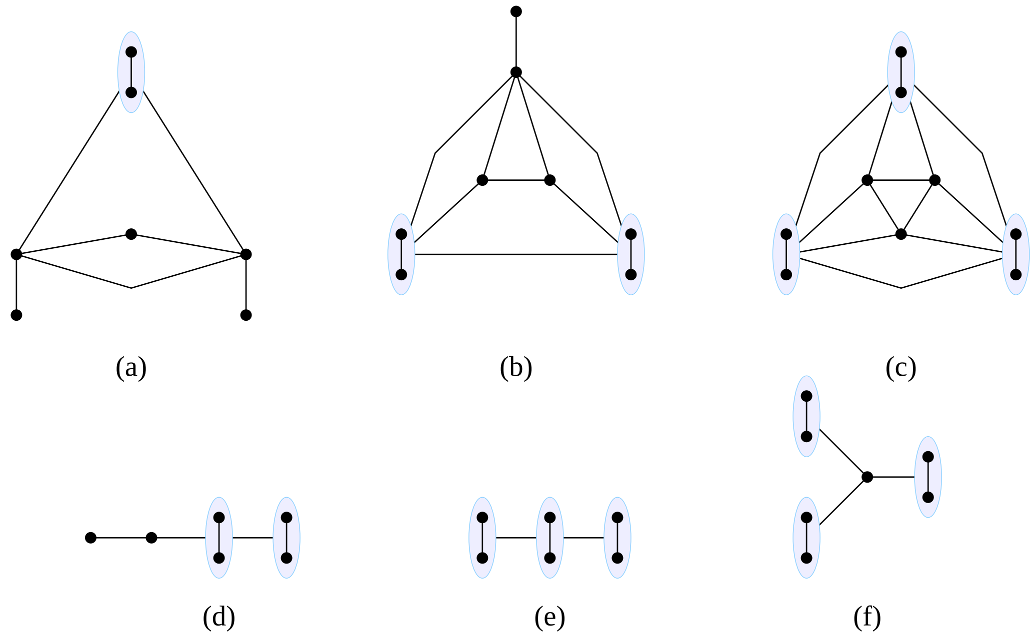

Figure 7: We use these names for the depicted graphs: (a) face-1 graph, (b) face-2 graph, (c) face-3 graph, (d) full house with antenna, (e) full domino, (f) triangle-3 graph. In all graphs, the grouped vertices represent true twin pairs.

3) *H contains one of the following graphs as an induced subgraph:* square-1 graph *or* square-2 graph *or* face-1 graph *or* face-2 graph *or* face-3 graph *or* full house with antenna *or* full domino.

Proof. If G is edgeless then $\langle V(G)\rangle$ is a normal building sequence for G, and the first case of the lemma applies. Otherwise, G is not edgeless. Due to Corollary 5.4, G has a normal building sequence $\langle D_1, \ldots, D_s\rangle$ with building partition (A, C) such that $u \in A \setminus D_1$. If $u \in D_s$ then the first case of the lemma applies. Otherwise, $u \notin D_s$, which particularly means $s \geq 3$. If $u \in D_{s-1}$ and $D_1 \subseteq N_G(u)$ then $\langle D_1, \ldots, D_{s-2}, D_s, D_{s-1}\rangle$ is a normal building sequence for G due to Lemma 5.5, and the first case of the lemma applies, and if $u \in D_{s-1}$ and $D_1 \not\subseteq N_G(u)$ then $N_G(u) = D_s$, and the first case of the lemma applies again. We henceforth assume $u \notin D_{s-1}$, i.e., $u \in D_2 \cup \cdots \cup D_{s-2}$. Let $u \in D_p$.

Let $z \in D_1$ and $d \in D_{s-1}$ and $c \in D_s$. Due to Lemma 5.5, $N_G(c) = A \setminus D_s$ and $D_{s-1} \subseteq A$ and $(C \cup D_2) \subseteq N_G(z)$, and $D_s \subseteq N_G(u)$ in particular. We distinguish between two cases about $N_G(u)$.

Case A: $z \notin N_G(u)$

Proof of the case. Note: $N_G(u) \subseteq D_{p+1} \cup \cdots \cup D_s$ and $D_s \subseteq N_G(u) \subseteq N_G(z)$. Let $x \in N_G(z) \setminus N_G(u)$, that exists, since u and z are not false twins of G. Note: $x \in D_2 \cup \cdots \cup D_{p-1}$, since $u \in A$.

Assume that $N_G(u)$ contains an adjacent vertex pair a, b. If $x \in C$ then $N_G(x) \cap \{a, b\} = \emptyset$ and if $x \in A$ then $\{a, b\} \subseteq N_G(x)$. In the former case, $\{x, z, a, b, u, v\}$ induces a full house with antenna in H, and in the latter case, $\{x, z, a, b, u, v\}$ induces a full domino in H.

Assume that the vertices in $N_G(u)$ are pairwise non-adjacent in G. Since $N_G(c) = A \setminus D_s$ and $D_s \subseteq N_G(u)$, $A \cap N_G(u) = D_s$ follows, and therefore, $N_G(d) = D_s$ and $N_G(u) = (C \cap (D_{p+1} \cup \cdots \cup D_{s-2})) \cup D_s$. Let $y \in N_G(u) \setminus N_G(d)$, that exists, since $N_G(d) \subseteq N_G(u)$ and u and d are not false twins of G. Recall that $y \in C \setminus D_s$ must hold, so that $yc \notin E(G)$. If $x \in C$ then $\{z, c, u, y, x, d, v\}$ induces a square-1 graph in H, and if $x \in A$ then $u, z, x \in N_G(y)$ and $xc \in E(G)$ and $\{z, c, u, y, x, d, v\}$ induces a square-2 graph in H. □

Case B: $z \in N_G(u)$

Proof of the case. Note: $A \cap (D_1 \cup \cdots \cup D_{p-1}) \subseteq N_G(u)$. If $D_p = \{u\}$ then

$$\Big\langle D_1,\ \ldots,\ D_{p-1},\ \{u\},\ \{v\},\ D_{p+1},\ \ldots,\ D_s \Big\rangle$$

is a building sequence for H with building partition $((A \cup \{v\}), C)$, and the second case of the lemma applies. Otherwise, $|D_p| \geq 2$. Let $u' \in D_p$ with $u' \neq u$. If $D_{p+1} \cup \cdots \cup D_{s-2} \subseteq A$ and $D_{p+1} \cup \cdots \cup D_{s-1} \subseteq N_G(u)$ then $\langle D_1, \ldots, D_{p-1}, D_{p+1}, \ldots, D_s, D_p \rangle$ is a normal building sequence for G, and the first case of the lemma applies. Similarly, if $D_2 \cup \cdots \cup D_{p-1} \subseteq A$ and $D_3 \cup \cdots \cup D_{p-1} \subseteq N_G(z)$ then $\langle D_p, D_1, \ldots, D_{p-1}, D_{p+1}, \ldots, D_s \rangle$ is a building sequence for G, and we can apply Lemma 5.3, and the first case of the lemma applies. We henceforth assume that these two situations do not occur.

For a better understanding of the below conclusions, note this table of a schematic description of the considered construction sequence, where we assume $D_s \subseteq A$ for convenience.

	D_1	$D_2 \cup \cdots \cup D_{p-1}$	D_p	$D_{p+1} \cup \cdots \cup D_{s-2}$	D_{s-1}	D_s
A	$\{z\}$	e	$\{u, u'\}$	b	$\{d\}$	$\{c\}$
C		a		y		

Assume that vertex a exists. If vertex b can be chosen with $b \notin N_G(u)$ then $\{a, z\} \cup \{u', u, v\} \cup \{c, b\}$ induces a face-1 graph in H; note that $b = d$ is possible. If vertex b cannot be chosen with $b \notin N_G(u)$ then $D_{p+1} \cup \cdots \cup D_{s-1} \subseteq N_G(u)$, and vertex y must exist according to our above assumptions. In this case, $\{a, z\} \cup \{u', u, v\} \cup \{y, c, d\}$ induces a face-2 graph in H.

As the other case, assume that vertex a does not exist. Then, $D_2 \cup \cdots \cup D_{p-1} \subseteq A$, and our above assumptions imply that vertex e with $e \notin N_G(z)$ must exist. Let $x \in N_G(z) \setminus N_G(e)$, that exists, since $N_G(e) \subseteq N_G(z)$ and e and z are not false twins of G. Recall $x \in A \cap (D_2 \cup \cdots \cup D_{p-1})$. If vertex b with $b \notin N_G(u)$ exists then $\{e, z, x\} \cup \{u', u, v\} \cup \{c, b\}$ induces a face-2 graph in H. If vertex b cannot be chosen with $b \notin N_G(u)$ then vertex y exists, and $\{e, z, x\} \cup \{u', u, v\} \cup \{y, c, d\}$ induces a face-3 graph in H. □

To conclude: if the first or the second case of the lemma does not apply then H contains one of the listed graphs as an induced subgraph, and the third case of the lemma applies. ■

Lemma 5.16. *Let G be a connected sequence two-chain graph. Let u be a vertex of G and let v be a new vertex. Obtain H from G by adding v as a true twin of u. Then, H is a sequence two-chain graph or H contains one of the following graphs as an induced subgraph:* triangle-1 graph *or* triangle-2 graph *or* triangle-3 graph *or* diamond net *or* full house with antenna *or* full domino *or* square-1 graph *or* square-2 graph *or* face-1 graph *or* face-2 graph *or* face-3 graph.

Proof. We first assume that G is not a two-chain graph; the other case will be considered at the end of the proof. Due to Corollary 5.11, G has a building sequence $\langle \Sigma_1, \ldots, \Sigma_t \rangle$ with $t \geq 2$ such that $u \notin V(\Sigma_1)$ and Σ_1 is not edgeless and Σ_t is not easy complete bipartite and $\Sigma_2, \ldots, \Sigma_t$ have non-empty initializers. For $1 \leq i \leq t$, let $\Sigma_i = (G_i; M_i, A_i)$. As proved in the second paragraph of the proof of Lemma 5.13: G has a path $(z_0, z_1, \ldots, z_{t+2})$ where $z_i \in M_i$ for every $2 \leq i \leq t$ and $z_0 \in V(G_1)$ and $z_1 \in A_1$ and $z_{t+1}, z_{t+2} \in V(G_t) \setminus M_t$.

We distinguish between three cases.

Case A: There is p with $1 \leq p \leq t$ such that $N_G(u) = M_p$ or $u \in M_p$.

Proof of the case. Assume $N_G(u) = M_p$. Observe that $p = 1$ is not possible, since $N_G(u) = M_1$ and $u \notin V(G_1)$ implies $u \in M_2$ and $t = 2$ and $V(G_2) = M_2$, the latter contradicting the assumption about Σ_t being not easy complete bipartite. So, $p \geq 2$. Consider $\{z_{p-2}, z_{p-1}, z_p, z_{p+1}, z_{p+2}\}$: $N_G(u) \cap \{z_{p-2}, z_{p-1}, z_p, z_{p+1}, z_{p+2}\} = \{z_p\}$, and $\{z_{p-2}, z_{p-1}, z_p, z_{p+1}, z_{p+2}, u, v\}$ induces a triangle-1 graph or a triangle-2 graph or a triangle-3 graph in H.

Next, assume $u \in M_p$; clearly, $p \geq 2$. Observe: $u = z_p$ or u, z_p is a false twin pair of G. So, $\{z_{p-2}, z_{p-1}, u, v, z_{p+1}, z_{p+2}\}$ induces a diamond net or a full house with antenna or a full domino in H. □

We henceforth assume that Case A does not apply. Let p with $1 < p \leq t$ be such that u is a vertex of G_p. Note $u \notin M_p$ according to our assumption. Let $H_p =_{\text{def}} H[V(G_p) \cup \{v\}]$. Observe that H_p is obtained from G_p by adding v as a true twin of u. Let $A'_p =_{\text{def}} A_p$ if $u \notin A_p$, and let $A'_p =_{\text{def}} A_p \cup \{v\}$ if $u \in A_p$. Let $\Sigma'_p =_{\text{def}} (H_p; M_p, A'_p)$. If Σ'_p is a marked two-chain graph then $\langle \Sigma_1, \ldots, \Sigma_{p-1}, \Sigma'_p, \Sigma_{p+1}, \ldots, \Sigma_t \rangle$ is a building sequence for H, and H is a sequence two-chain graph. We apply this result in the following, and we distinguish between two cases about p.

Case B: $1 < p < t$

Proof of the case. Let $G'_p =_{\text{def}} G[V(G_p) \cup M_{p+1} \cup \{z_{p-1}, z_{p+2}\}]$. It is important to recall $N_{G'_p}(z_{p-1}) = M_p$ and $N_{G'_p}(z_{p+2}) = M_{p+1}$. Note also that G'_p is a connected two-chain graph, that is not edgeless or complete bipartite. Let $\langle D_1, \ldots, D_s \rangle$ be a normal building sequence for G'_p such that $z_{p+1} \notin D_1$, that exists due to Corollary 5.4. Note that $s \geq 3$ must hold. We can apply Lemma 5.7: $z_{p-1} \in D_2$ and $M_p = D_1$ and $z_{p+2} \in D_{s-1}$ and $M_{p+1} = D_s$. If $u \in D_1 \cup D_2 \cup D_{s-1} \cup D_s$ then Case A applies, a contradiction.

Let $H'_p =_{\text{def}} H[V(G_p) \cup M_{p+1} \cup \{z_{p-1}, z_{p+2}, v\}]$. Observe that H'_p is obtained from G'_p by adding v as a true twin of u. We apply Lemma 5.15: if the first case applies then Case A applies, as we have shown in the preceding paragraph, and if the third case applies then we can already conclude. So, assume that the second case applies, which means that H'_p is a two-chain graph. Due to Corollary 5.4, H'_p has a normal building sequence $\langle E_1, \ldots, E_r \rangle$ with building partition (A', C') such that $z_{p+1} \notin E_1$. Observe: $N_{H'_p}(z_{p-1}) = N_{G'_p}(z_{p-1}) = M_p$ and $N_{H'_p}(z_{p+2}) = N_{G'_p}(z_{p+2}) = M_{p+1}$, and due to Lemma 5.7: $E_r = M_{p+1}$ and $E_1 = M_p$. It directly follows that $(H'_p; M_p, A')$ is a marked two-chain graph. Recall: $N_{H'_p}(z_{p+1}) \setminus \{v\} = N_{G'_p}(z_{p+1}) = A_p \cup \{z_{p+2}\}$. Due to Lemma 5.5, $N_{H'_p}(z_{p+1}) = A' \setminus E_r$, and thus, $A' \setminus (E_r \cup \{z_{p+2}\}) = A'_p$. Hence, $\Sigma'_p = (H_p; M_p, A'_p)$ is a marked two-chain graph. □

Case C: $p = t$

Proof of the case. We adapt the proof of Case B. Let $G'_t =_{\text{def}} G[V(G_t) \cup \{z_{t-1}\}]$ and $H'_t =_{\text{def}} H[V(G_t) \cup \{z_{t-1}, v\}]$, and H'_t is obtained from G'_t by adding v as a true twin of u. Note also that G'_t is a connected two-chain graph, and $(z_{t-1}, z_t, z_{t+1}, z_{t+2})$ is a path of G'_t, and G'_t is not complete bipartite. We apply Lemma 5.15. If the third case applies then we can already conclude.

Assume that the second case applies and H'_t is a two-chain graph. Due to Corollary 5.4, H'_t has a normal building sequence $\langle D_1, \ldots, D_s \rangle$ with building partition (A', C'). If $z_{t-1} \in D_2$ then $z_{t-1} \in C'$ and $D_1 = M_t$ due to Lemma 5.7, so that $(H'_t; M_t, A')$ is a marked two-chain graph. Then, $\Sigma''_t =_{\text{def}} (H_t; M_t, A')$ is a marked two-chain graph and $\langle \Sigma_1, \ldots, \Sigma_{t-1}, \Sigma''_t \rangle$ is a building sequence for H. As a remark, we use Σ''_t instead of Σ'_t to avoid a here unnecessary argument about why also $(H_t; M_t, A'_t)$ is a marked two-chain graph. If $z_{t-1} \notin D_2$ then $z_{t-1} \in D_{s-1}$ and $D_s = M_t$ due to Lemma 5.7, and we apply Lemma 5.3 to $\langle D_1, \ldots, D_s \rangle$, obtain a normal building sequence $\langle E_1, \ldots, E_s \rangle$ for H'_t with $E_s = D_1$, and $z_{t-1} \in E_2$ follows due to Lemma 5.7.

Next, assume that the first case applies and G'_t has a normal building sequence $\langle D_1, \ldots, D_s \rangle$ with building partition (A', C') such that either $u \in D_s$ or $u \in D_{s-1}$ and $D_s = N_{G'_t}(u)$. If $z_{t-1} \in D_{s-1}$ then $M_t = N_{G'_t}(z_{t-1}) = D_s$ due Lemma 5.7, and either $u \in D_s$ where $D_s = M_t$ or $u \in D_{s-1}$ and $N_{G'_t}(u) = N_{G'_t}(z_{t-1}) = M_t$, and Case A applies, which contradicts our previous assumptions. So, $z_{t-1} \in C' \cap D_2$ and $N_{G'_t}(z_{t-1}) = M_t = D_1$ due to Lemma 5.7, and $(G_t; M_t, A')$ is a marked two-chain graph with building sequence $\langle D_1, (D_2 \setminus \{z_{t-1}), D_3, \ldots, D_s \rangle$. We show that H is a sequence two-chain graph, and we distinguish between the two cases about u. In both cases, it is important to recall $N_{G'_t}(x) = A' \setminus D_s$ for $x \in D_s$ due to Lemma 5.5, which is indeed applicable.

- $u \in D_s$

  Let $A'' =_{\text{def}} (A' \setminus D_s) \cup \{u, v\}$. Then, $(H_t; M_t, A'')$ is a marked two-chain graph with building sequence

  $$\Big\langle D_1,\ (D_2 \setminus \{z_{z-1}\}),\ D_3,\ \ldots,\ D_{s-1},\ (D_s \setminus \{u\}),\ \{u\},\ \{v\} \Big\rangle$$

  and building partition $(A'', ((C' \setminus \{z_{t-1}\}) \cup (D_s \setminus \{u\})))$, and H is a sequence two-chain graph.

- $u \in D_{s-1}$ and $D_s = N_{G'_t}(u) = N_G(u)$

  Let $A'' =_{\text{def}} A' \setminus (D_{s-1} \cup D_s)$ and $A''' =_{\text{def}} D_s \cup \{u, v\}$, and let

  $$\begin{aligned} \Sigma'_t &=_{\text{def}} \Big((G_t \setminus (D_{s-1} \cup D_s));\ M_t,\ A''\Big) \\ \Sigma'_{t+1} &=_{\text{def}} \Big(H[D_{s-1} \cup D_s \cup \{v\}];\ D_s,\ A'''\Big). \end{aligned}$$

  It is not difficult to see that Σ'_t and Σ'_{t+1} are marked two-chain graphs, and it follows that H is a sequence two-chain graph with building sequence $\langle \Sigma_1, \ldots, \Sigma_{t-1}, \Sigma'_t, \Sigma'_{t+1} \rangle$.

Thus, if H does not contain any of the listed graphs as an induced subgraph then H is a sequence two-chain graph. □

We have proved that the claim of the lemma is correct if G is not a two-chain graph. As the remaining case, assume that G is a two-chain graph. We can apply Lemma 5.15 to G and H

directly: if the second or the third case applies then we can already conclude. If the first case applies then we can conclude purely analogous to the last situation of above Case C, by giving a building sequence for H. ∎

5.4 Two characterisations

We are ready to prove the characterisation results. Most work was done in the preceding subsections. The main technical results of this subsection are lower-bound proofs.

Let $\mathcal{F}$ be the set of the following graphs: triangle-0 graph and triangle-1 graph and triangle-2 graph and triangle-3 graph and diamond net and triangle net and square net and square-1 graph and square-2 graph and face-1 graph and face-2 graph and face-3 graph and full house with antenna and full domino. These are exactly the graphs that are depicted in Figures 5 and 6 and 7. We summarise the results of the preceding subsections.

Proposition 5.17. *Let G be a connected distance-hereditary graph. Then, G is a sequence two-chain graph or G contains one of the graphs in $\mathcal{F}$ as an induced subgraph.*

Proof. We prove the claim by induction on the number of vertices of G, by applying the result of Theorem 4.6. If G is a graph on a single vertex then G is a two-chain graph and thus a sequence two-chain graph. Otherwise, G has at least two vertices. If G contains one of the graphs in $\mathcal{F}$ as an induced subgraph then we can already conclude. So, assume that G does not contain any of the graphs in $\mathcal{F}$ as an induced subgraph, and thus, there is no vertex x of G such that $G-x$ contains one of the graphs in $\mathcal{F}$ as an induced subgraph.

Since $G-x$ is distance-hereditary, the induction hypothesis implies that $G-x$ is a sequence two-chain graph for every $x \in V(G)$. Assume that G is not twin-free. Then, G has a twin pair u, v. Since $G-v$ is a sequence two-chain graph, if $N_G(u) = N_G(v)$, i.e., v is a false twin of u in G, then we apply Lemma 5.14 to $G-v$, and if $N_G[u] = N_G[v]$, i.e., v is a true twin of u in G, then we apply Lemma 5.16 to $G-v$. Otherwise, G is twin-free. Due to Theorem 4.6, G has a vertex pair u, v such that $N_G(v) = \{u\}$, i.e., v is a pendant vertex at u in G, and we apply Lemma 5.13 to $G-v$. In all cases, we conclude that G is a sequence two-chain graph.

For the application of Lemmas 5.16 and 5.13, it is important to observe that $G-v$ is connected. ∎

Observe that Proposition 5.17 does not show that sequence two-chain graphs are without induced subgraphs from $\mathcal{F}$. A main result of this subsection shows that this is nevertheless the case, and this will directly translate into a forbidden induced subgraph characterisation of the sequence two-chain graphs.

A second main result of this subsection is a characterisation of the linear clique-width of the sequence two-chain graphs. The following lemma proves an upper bound.

Lemma 5.18. *Let F be a sequence two-chain graph. Then,* $\mathrm{lcwd}_{\mathrm{inac}}(F) \leq 3$.

Proof. We show that F has a linear 3-expression with label 1 as an inactive label. We first consider marked two-chain graphs, and then, we consider marked sequence two-chain graphs.

Let $(G; M, A)$ be a non-empty marked two-chain graph with building sequence $\langle B_1, \ldots, B_t \rangle$ where $B_1 = M$, and let $M = \{m_1, \ldots, m_r\}$. We iteratively construct a 3-expression α such that

$A \cap (B_1 \cup \cdots \cup B_t)$ are the vertices with label 2 and the other vertices have label 1 in val(α). For a technical reason, α begins with some seemingly unnecessary one-step extensions. Let

$$\alpha_1 \ =_{\text{def}} \begin{cases} \rho_{2\to 1}() & , \text{ if } B_1 = \emptyset \\ \rho_{3\to 2}(\rho_{2\to 1}(\eta_{2,3}(() \oplus 3(m_1) \oplus \cdots \oplus 3(m_r)))) & , \text{ if } B_1 \neq \emptyset\,. \end{cases}$$

Clearly, $[\text{val}(\alpha_1)] = G[B_1]$ and the vertices in $A \cap B_1 = B_1$ have label 2 and all other vertices have label 1 in val(α_1).

Let $1 \leq i < t$. If $B_{i+1} = \emptyset$ then let $\alpha_{i+1} =_{\text{def}} \alpha_i$. Otherwise, $B_{i+1} \neq \emptyset$. Let $B_{i+1} = \{x_1, \ldots, x_q\}$, and let

$$\alpha'_{i+1} \ =_{\text{def}} \ \alpha_i \oplus 3(x_1) \oplus \cdots \oplus 3(x_q)\,.$$

Observe $V(\text{val}(\alpha'_{i+1})) = B_1 \cup \cdots \cup B_{i+1}$. We obtain α_{i+1} for $G[B_1 \cup \cdots \cup B_{i+1}]$ by distinguishing between four cases:

$$\alpha_{i+1} \ =_{\text{def}} \begin{cases} \rho_{3\to 1}(\alpha'_{i+1}) & , \text{ if } B_{i+1} \not\subseteq A \text{ and } N_G(x_1) \cap (B_1 \cup \cdots \cup B_i) = \emptyset \\ \rho_{3\to 2}(\alpha'_{i+1}) & , \text{ if } B_{i+1} \subseteq A \text{ and } N_G(x_1) \cap (B_1 \cup \cdots \cup B_i) = \emptyset \\ \rho_{3\to 1}(\eta_{2,3}(\alpha'_{i+1})) & , \text{ if } B_{i+1} \not\subseteq A \text{ and } N_G(x_1) \cap (B_1 \cup \cdots \cup B_i) \neq \emptyset \\ \rho_{3\to 2}(\eta_{2,3}(\alpha'_{i+1})) & , \text{ if } B_{i+1} \subseteq A \text{ and } N_G(x_1) \cap (B_1 \cup \cdots \cup B_i) \neq \emptyset\,. \end{cases}$$

Let $\alpha =_{\text{def}} \alpha_t$. It is easy to verify that α has label 1 as an inactive label, so that $\alpha \in \mathcal{E}^{\text{inac}}_{\text{lin}}(3)$, and it applies an easy inductive argument about α_i to verify that α is a 3-expression for G of the requested properties.

We prove the upper bound on the linear clique-width of F. Let $\langle \Sigma_1, \ldots, \Sigma_t \rangle$ be a building sequence for F, and for $1 \leq i \leq t$, let $\Sigma_i = (G_i; M_i, A_i)$. Let δ_i be the linear 3-expression for Σ_i that is constructed in the above construction. Recall that δ_i has label 1 as an inactive label and the vertices in A_i have label 2 and the other vertices have label 1 in val(δ_i), and δ_i has a very special beginning. If $t = 1$ then δ_1 already proves $\text{lcwd}_{\text{inac}}(F) \leq 3$.

Otherwise, $t \geq 2$. It is straightforward to verify that $\delta_t(\delta_{t-1}(\cdots \delta_1 \cdots))$ is a linear 3-expression for F, that has label 1 as an inactive label, and thus, $\text{lcwd}_{\text{inac}}(F) \leq 3$. ∎

We now prove lower bounds on the linear clique-width of the graphs in $\mathcal{F}$. The proofs apply similar ideas as the lower-bound proofs in Section 4.

Lemma 5.19. *Let G be a graph on n vertices and let $\alpha \in \mathcal{E}^{\text{inac}}_{\text{lin}}(3)$ be a 3-expression for G with associated vertex ordering $\sigma = \langle x_1, \ldots, x_n \rangle$. Let γ be a subexpression of α that is a full subexpression for G. Let $\Gamma =_{\text{def}}$ val(γ).*

1) *If $V(G) = \{a, b, c\}$ and $E(G) = \{ab, bc, ca\}$ then x_1 and x_2 have the same label in Γ.*

2) *If $V(G) = \{a, b, c, d\}$ and $E(G) = \{ab, cd\}$ and $x_4 \in \{a, b\}$ then c and d have label 1 in Γ.*

3) *If $V(G) = \{a, b, c, d\}$ and $E(G) = \{ab, bc, cd\}$ and $x_4 \in \{a, b\}$ then d has label 1 in Γ, and $d, c \prec_\sigma b, a$, and if c does not have label 1 in Γ then a and c have the same label in Γ.*

4) *If $V(G) = \{a, b, c, d, e\}$ and $E(G) = \{ab, cb, db, eb, de\}$ then a and c have the same label in Γ or d has label 1 in Γ.*

Proof. For all statements, we consider $\gamma = \delta \oplus o(x_n)$, and the claim directly follows.

We prove the first statement. Since x_1 and x_2 are non-visible neighbours of x_3 in Γ, x_3 has label 2 and x_1 and x_2 have label 3 in Γ or x_3 has label 3 and x_1 and x_2 have label 2 in Γ.

We prove the second statement. By symmetry, we can assume $x_4 = a$. Since b is a non-visible neighbour of a in Γ and since c and d are non-adjacent to a and b in G, a and b have label 2 and 3 in Γ and c and d cannot have label 2 or 3 in Γ, so that c and d have label 1 in Γ.

We prove the third statement. Observe that a and b are non-visible neighbours of each other in Γ and d is non-adjacent to a and b in G, so that a, b, d have pairwise different labels in Γ, and d has label 1 in Γ. Assume that c does not have label 1 in Γ. Then, c has the same label as a or b in Γ, and since a and c are non-adjacent in G, a and c must have the same label in Γ. If $x_4 = a$ then $x_3 = b$ due to Lemma 2.4, and if $x_4 = b$ then $d, c \prec_\sigma a$ follows from the fact that a does not have label 1 in Γ.

We prove the fourth statement. If $x_5 = b$ then a, c, d, e have the same label in Γ. If $x_5 = d$ or $x_5 = e$ then d and e are non-visible neighbours of each other, and a and c have label 1 in Γ, since a and c are non-adjacent to d and e in G. Assume $x_5 \in \{a, c\}$, and by symmetry, choose $x_5 = a$. Due to Lemma 2.4: $x_4 \in \{b, c\}$. If $x_4 = b$ then c, d, e have the same label in Γ, that is either label 1 or the label of a. If $x_4 = c$ then $x_3 = b$ due to Lemma 2.4, and if d does not have label 1 in Γ then a, c, d, e have the same label in Γ. ∎

Proposition 5.20. *Let G be a* triangle-h graph *for $h \in \{0, 1, 2, 3\}$. Then,* $\mathrm{lcwd}_{\mathrm{inac}}(G) \geq 4$.

Proof. For the used vertex names, we refer to graph family (A) of Figure 8.

For a contradiction, suppose that $\alpha \in \mathcal{E}_{\mathrm{lin}}^{\mathrm{inac}}(3)$ is a 3-expression for G. Let $\langle x_1, \ldots, x_7 \rangle$ be the vertex ordering associated with α. Let $1 \leq p \leq 7$ be smallest such that $G[\{x_1, \ldots, x_p\}]$ contains two of the three edges ab, cd, ef. By a symmetry argument, we may assume $a, b, c, d \in \{x_1, \ldots, x_p\}$ and $x_p \in \{c, d\}$. Let α' be a subexpression of α that is a full subexpression for $G[\{a, b, c, d\}]$. We can apply the second statement of Lemma 5.19 and directly conclude that a and b have label 1 in $\mathrm{val}(\alpha')$, and this particularly implies $g \in \{x_1, \ldots, x_{p-1}\}$.

Let $\beta = \delta \oplus o(x_p)$ be a subexpression of α that is a full subexpression for $G[\{x_1, \ldots, x_p\}]$. Recall that c is a non-visible neighbour of d in $\mathrm{val}(\beta)$, so that we may assume that c has label 2 and d has label 3 in $\mathrm{val}(\beta)$. It follows: the vertices with label 2 in $\mathrm{val}(\beta)$ are neighbours of d in G, and the vertices with label 3 in $\mathrm{val}(\beta)$ are neighbours of c in G. Thus, all vertices of $\mathrm{val}(\beta)$ but c, d, g have label 1. If f is a vertex of $\mathrm{val}(\beta)$ then f has label 1, and e must be a vertex of $\mathrm{val}(\beta)$, implying $p = 7$, a contradiction. So, f is not a vertex of $\mathrm{val}(\beta)$, and g has a non-visible neighbour in $\mathrm{val}(\beta)$, so that g does not have label 1 in $\mathrm{val}(\beta)$, and since c and d are non-adjacent to f in G, g cannot have label 2 or 3 in $\mathrm{val}(\beta)$, a contradiction. Thus, $\mathrm{lcwd}_{\mathrm{inac}}(G) \geq 4$. ∎

Proposition 5.21. *Let G be a* square net *or a* square-1 graph *or a* square-2 graph. *Then,* $\mathrm{lcwd}_{\mathrm{inac}}(G) \geq 4$.

Proof. For the used vertex names, we refer to graph family (B) and graphs (B1) and (B2) of Figure 8. Observe that graph (B1) is an induced subgraph of graph (B) if g is adjacent to b and e, and graph (B2) does not have these edges.

For a contradiction, suppose that $\alpha \in \mathcal{E}_{\mathrm{lin}}^{\mathrm{inac}}(3)$ is a 3-expression for G. Let $\sigma = \langle x_1, \ldots, x_7 \rangle$ be the vertex ordering associated with α. Let $1 \leq p \leq 7$ be smallest such that $G[\{x_1, \ldots, x_p\}]$

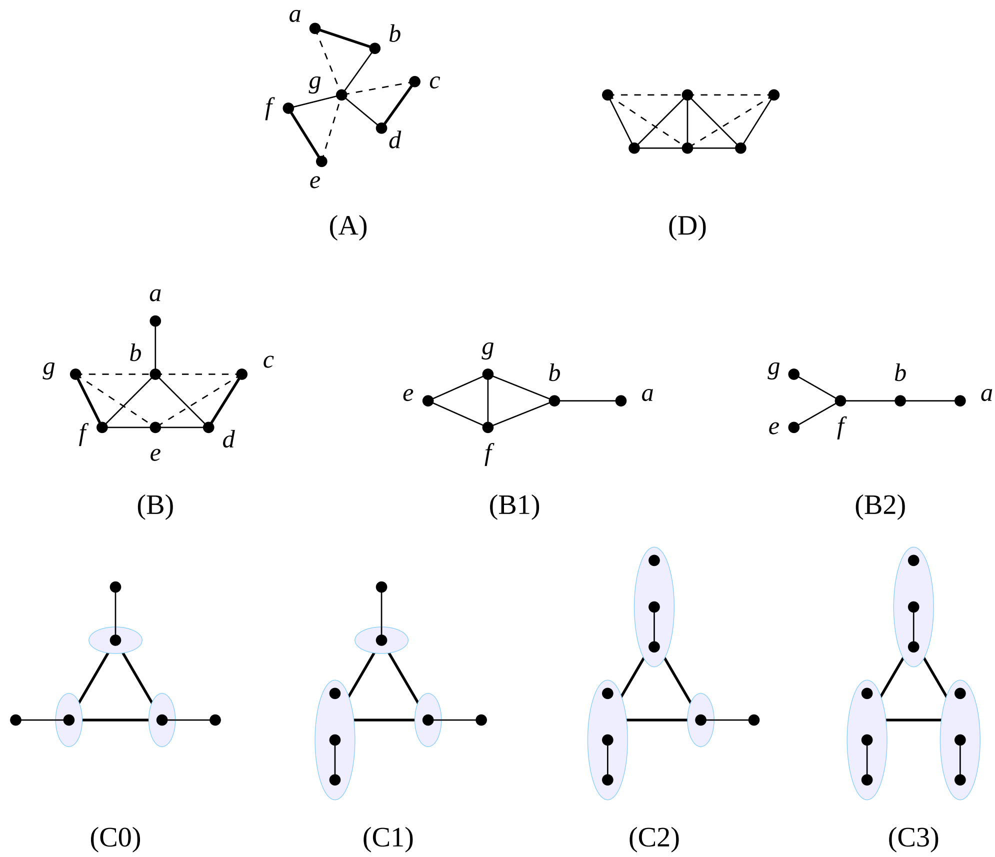

Figure 8: The figure shows the following graph families: (A) triangle graphs, (B) square graphs, (C0–3) face graphs, (D) other graphs. The vertex names are used in the lower-bound proofs. The graphs (B1) and (B2) are special induced subgraphs of graph family (B). The dashed line segments represent edges that may or may not be of the graphs.

contains the two edges cd and fg. By a symmetry argument, we may assume $x_p \in \{c, d\}$. Let $\beta = \delta \oplus o(x_p)$ be a subexpression of α that is a full subexpression for $G[\{x_1, \ldots, x_p\}]$. Then, c is a non-visible neighbour of d in $\mathrm{val}(\beta)$, so that c and d have label 2 and 3 in $\mathrm{val}(\beta)$, and due to the second statement of Lemma 5.19, f and g have label 1 in $\mathrm{val}(\beta)$. Thus, $b, e \in \{x_1, \ldots, x_{p-1}\}$, and $p \geq 6$. If $a = x_7$ then $x_p = x_6 = b$ due to Lemma 2.4, a contradiction. So, $a \in \{x_1, \ldots, x_{p-1}\}$, and $p = 7$.

Assume $x_7 = d$. Then, b, c, d, e have label 2 or 3 and a, f, g have label 1 in $\mathrm{val}(\beta)$. Let β' be a subexpression of β that is a full subexpression for $G[\{a, b, e, f, g\}]$. Observe that $G[\{a, b, e, f, g\}]$ is graph (B1) or graph (B2) of Figure 8. Let $\Gamma =_{\mathrm{def}} \mathrm{val}(\beta')$. Observe that b and e do not have label 1 or the same label as anyone of a, f, g in Γ. We apply the third statement of Lemma 5.19 to $G[\{a, b, f, e\}]$: since b and e do not have label 1 in Γ, a has label 1 and b and e have the same label in Γ and $a, b \prec_\sigma e, f$. We consider the graphs (B1) and (B2) separately.

(B1) Since b does not have the same label as f or g in Γ, the first statement of Lemma 5.19 shows that f and g have the same label in Γ and $f, g \prec_\sigma b$. This contradicts the above $b \prec_\sigma f$.

(B2) Since $a, b \prec_\sigma f$ and b does not have label 1 in Γ, the third statement of Lemma 5.19 shows that b and g have the same label in Γ, a contradiction.

Thus, if $x_7 = d$ then we obtain the desired contradiction. Analogously, if $x_7 = c$ and c is adjacent to b and e in G.

It remains to consider the case when $x_7 = c$ and c is non-adjacent to b and e in G. Then, $N_G(c) = \{d\}$, and $x_6 = d$ due to Lemma 2.4. Let $\beta'' = \delta'' \oplus o(x_6)$ be a subexpression of α that is a full subexpression for $G{-}c$. No vertex of $\mathrm{val}(\delta'')$ has label o, and thus, a, f, g have label 1 and b and e do not have label 1 in $\mathrm{val}(\delta'')$, and we can apply the above arguments to obtain the desired contradiction. ∎

Proposition 5.22. *Let G be a* triangle net *or a* face-h graph *for $h \in \{1, 2, 3\}$. Then,* $\mathrm{lcwd}_{\mathrm{inac}}(G) \geq 4$.

Proof. For illustrations of the graphs, we refer to graphs (C0), (C1), (C2), (C3) of Figure 8, where graph (C0) is a triangle net. The vertex set of G is partitioned into three *sides*, each of which is the set consisting of the three grouped vertices or the single grouped vertex and its pendant vertex. We distinguish the sides as *side 1*, *side 2*, *side 3*.

For a contradiction, suppose that $\alpha \in \mathcal{E}^{\mathrm{inac}}_{\mathrm{lin}}(3)$ is a 3-expression for G. Let $\langle x_1, \ldots, x_n \rangle$ be the vertex ordering associated with α. If $|N_G(x_n)| \geq 2$ then let $p =_{\mathrm{def}} n$, and if $|N_G(x_n)| = 1$ then $N_G(x_n) = \{x_{n-1}\}$ due to Lemma 2.4 and let $p =_{\mathrm{def}} n - 1$. Without loss of generality, we may assume that x_p is from side 1. If $p = n - 1$ then x_{n-1} and x_n are the two vertices from side 1. Let K be the set of the vertices from side 2 and 3, and let $A =_{\mathrm{def}} K \cap N_G(x_p)$ and $B =_{\mathrm{def}} K \setminus N_G(x_p)$. Note $|B| \leq 2$.

Let $\alpha' = \delta \oplus o(x_p)$ be a subexpression of α that is a full subexpression for $G[\{x_1, \ldots, x_p\}]$, and let $\Gamma =_{\mathrm{def}} \mathrm{val}(\alpha')$. Since x_p has a non-visible neighbour in Γ, we may assume $o = 2$. The vertices in A are non-visible neighbours of x_p in Γ and thus have label 3 in Γ. The vertices in B are pendant vertices of G and thus are non-adjacent to some vertex in A, so that no vertex from B can have label 2 in Γ, and thus, the vertices in B have label 1 in Γ.

- If $|B| = 0$ then $G[A]$ is a co-$2P_3$, and since the vertices of $G[A]$ have the same label in Γ that is not label 1, $[\Gamma][A] = G[A]$ and δ can be made into a linear 2-expression for $G[A]$, and $\mathrm{lcwd}(G[A]) \leq 2$, a contradiction to Theorem 2.1.

- If $|B| = 1$ then $G[A \cup B]$ is the graph of the fourth statement of Lemma 5.19, which directly yields a contradiction.

- If $|B| = 2$ then $G[A \cup B]$ is a chordless path (a, b, c, d), where $A = \{b, c\}$ and $B = \{a, d\}$, and the third statement of Lemma 5.19 yields a contradiction.

We conclude: $\mathrm{lcwd}_{\mathrm{inac}}(G) \geq 4$. ∎

Proposition 5.23 ([17, 18]).

1) Let G be a triangle-0 graph *or a* triangle net *or a* square net*. Then,* $\mathrm{lcwd}(G) \geq 4$*.*

2) Let G be a full house with antenna *or a* full domino *or a* diamond net*. Then,* $\mathrm{lcwd}_{\mathrm{inac}}(G) \geq 4$*.*

Proof. The first statement is due to [17]: the three graphs are not AT-free, and thus, they are not cocomparability graphs, so that $\mathrm{lcwd}(G) \geq 4$.

The second statement is due to [18]: the disjoint union of any of the three graphs has linear clique-width at least 4. The statement follows from an application of Proposition 3.3. Note that these are the graphs of graph family (D) of Figure 8. ∎

Finally, we can give the characterisation result.

Theorem 5.24. *Let G be a graph. The following statements about G are equivalent:*

1) G is a sequence two-chain graph

2) $\mathrm{lcwd}_{\mathrm{inac}}(G) \leq 3$

3) G does not contain a graph in $\mathcal{D} \cup \mathcal{F}$ as an induced subgraph.

Proof. If G is a sequence two-chain graph then $\mathrm{lcwd}_{\mathrm{inac}}(G) \leq 3$ due to Lemma 5.18.

If $\mathrm{lcwd}_{\mathrm{inac}}(G) \leq 3$ then $\mathrm{cwd}_{\mathrm{inac}}(G) \leq 3$ due to Lemma 2.2, and G is a distance-hereditary graph due to Proposition 4.8, and G does not contain a graph in $\mathcal{D}$ as an induced subgraph due to Theorem 4.6. If $\mathrm{lcwd}_{\mathrm{inac}}(G) \leq 3$ then $\mathrm{lcwd}_{\mathrm{inac}}(H) \leq 3$ for every induced subgraph H of G due to Lemma 2.3, and G does not contain a graph in $\mathcal{F}$ as an induced subgraph due to Propositions 5.20 and 5.21 and 5.22 and 5.23.

If G does not contain a graph in $\mathcal{D} \cup \mathcal{F}$ as an induced subgraph then G is a distance-hereditary graph due to Theorem 4.6, and each connected component of G is a distance-hereditary graph clearly, and each connected component of G does not contain a graph in $\mathcal{F}$ as an induced subgraph, so that each connected component of G is a sequence two-chain graph due to Proposition 5.17, and the disjoint union of sequence two-chain graphs is a sequence two-chain graph, so that G is a sequence two-chain graph. ∎

Corollary 5.25. *Let G be a disconnected graph. Assume* $\mathrm{lcwd}(G) \geq 4$*, and assume* $\mathrm{lcwd}(H) \leq 3$ *for every proper induced subgraph H of G. Then, G is the disjoint union of exactly two graphs in $\mathcal{D} \cup \mathcal{F}$.*

Proof. Let $C_1, \ldots, C_r$ be the connected components of G, where $r \geq 2$, and we may assume $\mathrm{lcwd}_{\mathrm{inac}}(C_1) \geq \cdots \geq \mathrm{lcwd}_{\mathrm{inac}}(C_r)$. Due to Proposition 3.3, $\mathrm{lcwd}_{\mathrm{inac}}(C_1) \geq \mathrm{lcwd}_{\mathrm{inac}}(C_2) \geq 4$, and $\mathrm{lcwd}(C_1 \oplus C_2) \geq 4$. The minimality assumption of the lemma about G implies $r = 2$ and $\mathrm{lcwd}_{\mathrm{inac}}(H) \leq 3$ for every proper induced subgraph H of C_1 and C_2. Due to Theorem 5.24, C_1 and C_2 contain a graph in $\mathcal{D} \cup \mathcal{F}$ as an induced subgraph and no proper induced subgraph of C_1 and C_2 contains a graph in $\mathcal{D} \cup \mathcal{F}$ as an induced subgraph, so that C_1 and C_2 must be (isomorphic to) a graph in $\mathcal{D} \cup \mathcal{F}$. ∎

6 Conclusions

As the most notable result, we gave a characterisation of the graphs of linear clique-width at most 3 with an inactive label by their set of forbidden induced subgraphs. These graphs are grouped as graph families (A), (B), (C), (D), (E) in Figures 8 and 3. Especially graph families (A), (B) and (D) are defined through a base graph and some edges may be added. It is an interesting observation that each optional edge may or may not be added, and the resulting graph is a forbidden graph, however not minimal always. This property may be worthy of exploring in the future.

Our list of forbidden induced subgraphs opens an interesting relationship between clique-width and rank-width. Oum showed that the graphs of rank-width at most 1 are the distance-hereditary graphs [25]. So, the graphs of rank-width at most 1 and the graphs of clique-width at most 3 with an inactive label coincide (Proposition 4.8). What about their linear restrictions? Interestingly, the two resulting graph classes do not coincide. Recently, Adler, Farley, Proskurowski characterised the minimal forbidden induced subgraphs of graphs of linear rank-width at most 1 [1], and it turns out that the graphs of linear rank-width at most 1 are a proper subclass of the graphs of linear clique-width at most 3 with an inactive label.

We were able to give the complete list of disconnected minimal forbidden induced subgraphs for the graphs of linear clique-width at most 3 (Corollary 5.25). This list is neither empty nor trivial. The result and the relationship to graphs of linear clique-width at most 3 with an inactive label was established in Proposition 3.3. We can extend our list of minimal forbidden induced subgraphs for graphs of linear clique-width at most 3 to some connected graphs, by applying Proposition 3.4: taking the disjoint union of any two graphs from $\mathcal{D} \cup \mathcal{F}$ and adding an edge between the two graphs yields a graph of linear clique-width at least 4. Similarly through the vertex join operation of Proposition 3.5.

For the lower-bound proofs, we applied arguments that were based on subexpressions. These arguments are reminiscent of clique-width and linear clique-width characterisations [19, 15, 22, 17], that are not applicable directly due to the special role of inactive labels. Is it possible to characterise clique-width and linear clique-width with an inactive label in a similar fashion? This is indeed the case, as Puppe showed by adapting the existing characterisations [27].

Finally, a computational remark. Let k be an integer with $k \geq 3$, and assume an algorithm for recognising graphs of linear clique-width at most k. We can apply this algorithm to recognise graphs of linear clique-width at most k with an inactive label. Let H_k be a graph with $\mathrm{lcwd}(H_k) \leq k$ and $\mathrm{lcwd}_{\mathrm{inac}}(H_k) > k$. According to the second statement of Proposition 3.3, for every graph G, $\mathrm{lcwd}_{\mathrm{inac}}(G) \leq k$ if and only if $\mathrm{lcwd}(G \oplus H_k) \leq k$. It follows that the decision or computation problem for linear clique-width with an inactive label is at most as hard as that problem for linear clique-width, applying Proposition 3.4, even when restricted to connected graphs. Of course, the argument relies on the graph H_k, that we do not know here but that does exist (see [19] for an example).

References

[1] I. Adler, A. M. Farley, A. Proskurowski. Obstructions for linear rank-width at most 1. *Discrete Applied Mathematics*, to appear.

[2] I. Adler and M. M. Kanté. Linear Rank-Width and Linear Clique-Width of Trees. *WG 2013*, 2013.

[3] H.-J. Bandelt and H. M. Mulder. Distance-Hereditary Graphs. *Journal of Combinatorial Theory, Series B*, 41:182–208, 1986.

[4] A. Brandstädt, F. Dragan, H.-O. Le, R. Mosca. New Graph Classes of Bounded Clique-Width. *Theory of Computing Systems*, 38:623–645, 2005.

[5] A. Brandstädt, J. Engelfriet, H.-O. Le, V. V. Lozin. Clique-Width for 4-Vertex Forbidden Subgraphs. *Theory of Computing Systems*, 39:561–590, 2006.

[6] A. Brandstädt, H.-O. Le, R. Mosca. Gem- and co-gem-free graphs have bounded clique-width. *International Journal of Foundations of Computer Science*, 15:163–186, 2004.

[7] D. G. Corneil, M. Habib, J.-M. Lanlignel, B. A. Reed, U. Rotics. Polynomial-time recognition of clique-width ≤ 3 graphs. *Discrete Applied Mathematics*, 160:834–865, 2012.

[8] D. G. Corneil, H. Lerchs, L. Stewart Burlingham. Complement reducible graphs. *Discrete Applied Mathematics*, 3:163–174, 1981.

[9] B. Courcelle, J. Engelfriet, G. Rozenberg. Handle-rewriting hypergraph grammars. *Journal of Computer and System Sciences*, 46:218–270, 1993.

[10] B. Courcelle, J. A. Makowsky, U. Rotics. Linear time solvable optimization problems on graphs of bounded clique-width. *Theory of Computing Systems*, 33:125–150, 2000.

[11] B. Courcelle and S. Olariu. Upper bounds to the clique width of graphs. *Discrete Applied Mathematics*, 101:77–114, 2000.

[12] M. R. Fellows, F. A. Rosamond, U. Rotics, S. Szeider. Clique-Width is NP-Complete. *SIAM Journal on Discrete Mathematics*, 23:909–939, 2009.

[13] M. C. Golumbic and U. Rotics. On the Clique-Width of Some Perfect Graph Classes. *International Journal of Foundations of Computer Science*, 11:423–443, 2000.

[14] F. Gurski. Characterizations for co-graphs defined by restricted NLC-width or clique-width operations. *Discrete Mathematics*, 306:271–277, 2006.

[15] F. Gurski. Linear layouts measuring neighbourhoods in graphs. *Discrete Mathematics*, 306:1637–1650, 2006.

[16] P. L. Hammer and F. Maffray. Completely separable graphs. *Discrete Applied Mathematics*, 27:85–99, 1990.

[17] P. Heggernes, D. Meister, C. Papadopoulos. Graphs of linear clique-width at most 3. *Theoretical Computer Science*, 412:5466–5486, 2011.

[18] P. Heggernes, D. Meister, C. Papadopoulos. Characterising the linear clique-width of a class of graphs by forbidden induced subgraphs. *Discrete Applied Mathematics*, 160:888–901, 2012.

[19] P. Heggernes, D. Meister, C. Papadopoulos, U. Rotics. Clique-width of path powers in linear time: a new characterisation of clique-width. Submitted manuscript, 2012.
A preliminary version was presented at *CSR 2011*, Springer LNCS, 6651:233–246, 2011.

[20] E. Howorka. A characterization of distance-hereditary graphs. *The Quarterly Journal of Mathematics*, 28:417–420, 1977.

[21] V. Lozin. Minimal Classes of Graphs of Unbounded Clique-Width. *Annals of Combinatorics*, 15:707–722, 2011.

[22] V. Lozin and D. Rautenbach. The relative clique-width of a graph. *Journal of Combinatorial Theory, Series B*, 97:846–858, 2007.

[23] D. Meister and U. Rotics. Clique-width of full bubble model graphs. Submitted manuscript, 2013.
A full preliminary version is available as a technical report, no. 13-1, Technical Reports in Mathematics and Computer Science, University of Trier, 2013.

[24] D. Meister and U. Rotics. Minimal forbidden induced subgraphs of graphs of bounded clique-width and bounded linear clique-width. *CoRR*, arXiv:1306.2114, 2013.

[25] S. Oum. Rank-width and vertex-minors. *Journal of Combinatorial Theory, Series B*, 95:79–100, 2005.

[26] S. Oum and P. Seymour. Approximating clique-width and branch-width. *Journal of Combinatorial Theory, Series B*, 96:514–528, 2006.

[27] M. Puppe. *A modified definition of clique-width.* Bachelor's thesis, University of Trier, 2012.

[28] M. Rao. Clique-width of graphs defined by one-vertex extensions. *Discrete Mathematics*, 308:6157–6165, 2008.