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Solutions

Financial Economics

A Concise Introduction to Classical and Behavioral Finance

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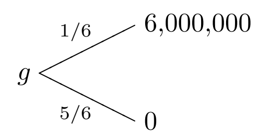
2

Decision Theory

2.1

Consider the gamble of tossing a fair die. If a 6 occurs then you win $6,000,000$ monetary units. In any of the other outcomes you don't win anything. This lottery is given as



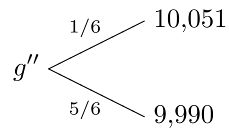
If your initial wealth is $w_0 = 10,000$ monetary units and your expected utility function is $u(x) = \lg(x)$ (x is your final wealth after playing the game, lg is the log function in base 10) then the utility of the above lottery will be

$$u(g) = \sum_{i=1}^{n=6} \lg(a_i) \cdot \frac{1}{6},$$

where a_i is the final wealth when outcome i occurred, for example outcome $a_1 = 10,000$ as you don't win or lose anything. The value of the expected utility is $u(g) = 4.46$.
If we win $1,000,000$ for sure, this is equivalent to a degenerate gamble. The expected utility of this gamble is

$$u(g') = \lg(1,010,000) = 6.0043.$$

In the third case you have to pay 10 monetary units to be able to participate in the gamble offering 61 monetary units with probability $1/6$. In this case then



To compute the certainty equivalent of this gamble we recall its definition $u(CE) = u(g'')$

$$\lg(CE) = \sum_{i=1}^{n=6} \lg(a_i) \cdot \frac{1}{6} \approx 4.00.$$

Thus $CE \approx 10^4 = 10,000$. So for small gambles as compared to your overall wealth you are practically indifferent between the gamble and its certainty equivalent.

2.2

Let a, b, c be the probability measures corresponding to the lotteries A, B, C. Then we have by asssumption

$$\int u \,\mathrm{d}a \geq \int u \,\mathrm{d}b \geq \int u \,\mathrm{d}c.$$

Let us denote the three integrals by $U(A), U(B), U(C)$. Then there is obviously a $p \in [0, 1]$ such that $U(B) = pU(A) + (1-p)U(C)$. (If you do not believe that, try $p = (U(B) - U(C))/(U(A) - U(C))$.) The utility of $pa + (1-p)c$ is now

$$\begin{aligned} U(pa + (1-p)c) &= \int u \,\mathrm{d}(pa + (1-p)c) = p \int u \,\mathrm{d}a + (1-p) \int u \,\mathrm{d}c \\ &= pU(A) + (1-p)U(C) = U(B). \end{aligned}$$

2.3

To solve this exercise we will make use of the fact that the slope of the utility function at any point ($-f$ for example) will be higher than the slope of the hypothenuse of the triangle constructed from $-f$ and $u(-f)$ thus

$$u'(-f) > \frac{u(-f)}{-f}.$$

By the Fundamental Theorem of Calculus we have the following relation:

$$\int_{-2f}^{-f} u'(s) \,\mathrm{d}s = u(-f) - u(-2f).$$

Solving for $u(-f)$ we obtain $u(-f) = u(-2f) + \int_{-2f}^{-f} u'(s)\,\mathrm{d}s$. Using the previous inequality we have the following relationship:

$$\int_{-2f}^{-f} u'(s)\,\mathrm{d}s > \int_{-2f}^{-f} u'(-f)\,\mathrm{d}s > \int_{-2f}^{-f} \frac{u(-f)}{-f}\,\mathrm{d}s,$$

$$u(-f) = u(-2f) + \int_{-2f}^{-f} u'(s)\,\mathrm{d}s > u(-2f) + \int_{-2f}^{-f} \frac{u(-f)}{-f}\,\mathrm{d}s.$$

But

$$u(-2f) + \int_{-2f}^{-f} \frac{u(-f)}{-f}\,\mathrm{d}s = u(-2f) - u(-f).$$

Then

$$2u(-f) > u(-2f).$$

As doubling the frequency is a better lottery for the agent we will choose the other one yielding him a lower utility.

2.4

The mean of the stocks is the average of +8% and −2%, thus $\mu = +3\%$. Its standard deviation is 5%, thus its variance is $\sigma^2 = 0.05^2$. The variance of the bond is of course zero. The utilities for the investor are therefore: $U_{\text{stock}} = 0.03 - \alpha 0.05^2$ and $U_{\text{bond}} = 0.02$. If we set both equal, we get $\alpha = 400$. The returns of the portfolio are now $(1-\lambda)\cdot 2\% + \lambda \cdot 8\%$ in a boom and $(1-\lambda)\cdot 2\% - \lambda \cdot 2\%$ in a recession. Mean and variance can be computed as above and we obtain $\mu(\lambda) = 0.02 + \lambda \cdot 0.01$ and $\sigma^2 = \lambda^2 \cdot 0.05^2$. The utility as function of λ is therefore

$$U_{\text{portfolio}}(\lambda) = 0.02 + \lambda \cdot 0.01 - \alpha\lambda^2 \cdot 0.05^2.$$

Taking the derivative with respect to λ to use the first order condition we obtain $\lambda_{\text{opt}} = 0.02$. (Checking the second derivative of $U_{\text{portfolio}}$ proves that this is in fact an optimum, and not, e.g., the worst case.)

2.5

The two possible choices, namely two wallets each having a card or one wallet with both cards and one empty wallet, are equivalent to two lotteries. Denote lottery A the first lottery and lottery B the second lottery. Let the loss of one wallet be independent of the loss of the other wallet (with p the probability of losing one wallet a small number) then lottery A is given as

p^2	$2p(1-p)$	$(1-p)^2$
$-2L$	$-L$	0

Lottery B is given as

p^2	$p(1-p)$	$(1-p)p$	$(1-p)^2$
$-2L$	$-2L$	0	0

For simplification, since the loss probability is small we have $p^2 \approx 0$ and hence the two lotteries are approximately

$$A \approx \begin{array}{cc} 2p & 1-2p \\ -L & 0 \end{array}$$

$$B \approx \begin{array}{cc} p & 1-p \\ -2L & 0 \end{array}$$

These two lotteries translate into the following utilities:

$$2 \cdot v(-L) \text{ versus } v(-2L)$$

where the value function $v(\cdot)$ will be concave for Bernoulli (this is the shape of the function on the entire domain) while for Kahneman it will be convex (the value function is convex over losses). Using the results of the previous exercise (or directly the definition of convexity and concavity) we see that if $v(\cdot)$ is concave then

$$2 \cdot v(-L) > v(-2L),$$

while if $v(\cdot)$ is convex then

$$2 \cdot v(-L) < v(-2L).$$

Thus Bernoulli will chose lottery A (two wallets each with one card) while Kahneman will chose lottery B (one wallet with both cards).

2.6

Idea: use simply the definition of the PT functional, plug in the probability and the prize as given in the exercise and compare the result with $v(2)$, the PT value of keeping the 2 Euro, i.e., not taking part on the lottery. If the latter is smaller, the person would take part, if it is larger, the person would not.

2.8

In PT, this can in fact happen, compare Equation (2.5). In CPT or NPT, this is not possible: if we denote the weighted probabilities by $\tilde{p}_i$ (how are they defined in CPT and in NPT?), then we can estimate

$$\sum_{i=1}^{n} \tilde{p}_i v(x_i) \leq \sum_{i=1}^{n} \tilde{p}_i \max_i v(x_i) = \max_i v(x_i).$$

2.10

To compute $CPT(p)$, we first integrate a and obtain the cumulative probability as

$$F(t) = \int_{-\infty}^{t} a(x)\,\mathrm{d}x \;=\; \begin{cases} 0 & \text{, if } t < 0, \\ \frac{1}{2}t^2 & \text{, if } 0 \le t < 1, \\ -\frac{1}{2}t^2 + 2t - 1 & \text{, if } 1 \le x < 2, \\ 1 & \text{, if } t \ge 2. \end{cases}$$

Weighting this cumulative probability yields

$$w(F(t)) = \begin{cases} 0 & \text{, if } t < 0, \\ \frac{1}{\sqrt{2}}t & \text{, if } 0 \le t < 1, \\ \left(-\frac{1}{2}t^2 + 2t - 1\right)^{1/2} & \text{, if } 1 \le t < 2, \\ 1 & \text{, if } t \ge 2. \end{cases}$$

The CPT functional can now be computed as follows:

$$\begin{aligned} CPT(p) &= \int_{-\infty}^{+\infty} v(x) \left(\frac{\mathrm{d}}{\mathrm{d}t} w(F(t))|_{t=x} \right) \mathrm{d}x \\ &= \int_{-\infty}^{0} x \left(\frac{\mathrm{d}}{\mathrm{d}t} 0|_{t=x} \right) \mathrm{d}x + \int_{0}^{1} x \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\sqrt{2}} t|_{t=x} \right) \mathrm{d}x \\ &\quad + \int_{1}^{2} x \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(-\frac{1}{2}t^2 + 2t - 1 \right)^{1/2} |_{t=x} \right) \mathrm{d}x \\ &\quad + \int_{2}^{+\infty} x \left(\frac{\mathrm{d}}{\mathrm{d}t} 1|_{t=x} \right) \mathrm{d}x \\ &= \int_{0}^{1} \frac{1}{\sqrt{2}} x\,\mathrm{d}x + \int_{1}^{2} \frac{1}{2} x \left(-\frac{1}{2}t^2 + 2t - 1 \right)^{-1/2} (2 - x)\,\mathrm{d}x. \end{aligned}$$

This expression can, in principle, be solved by hand, but it is more convenient to compute its value numerically. The result is $CPT(p) \approx 0.61$. We can compare this value with the CPT value of a sure outcome of 1 which is, for the value function $v(x) = x$, obviously 1. This implies that the sure outcome is preferred over the lottery, the person behaves risk-averse, although the value function is linear. The reason for this is the overweighting of the events with small outcomes, expressed by the weighting function $w(F) = \sqrt{F}$.

2.11

Hint: Jerome can be explained with hyperbolic discounting, but not with the other alternative explanations. Angelika's behavior can be explained taking into account her growing wealth level, but neither with hyperbolic nor classical time discounting alone.

2.12

To construct such a lottery it is enough to use a binary lottery with 50% chance each for winning X or losing Y. Let us say your initial wealth level is w. Then playing twice the lottery is equivalent to a lottery that gives you $w + 2X$ with probability 25%, $w + X - Y$ with probability 50% and $w - 2Y$ else. You just need to adjust X and Y appropriately to get a working example.

Now, what about the paradox? In Samuelson's argument he assumes that the risk attitude of the person are the same, even after playing the lottery several times. Playing the lottery several times means, however, that most likely the wealth level will have changed. That the risk attitudes are still (approximately) the same is therefore only true if either the amounts on stake are negligible small, or if the utility function has very special features (it must be linear or CARA – another nice exercise: prove this!). In Samuelson's original paper [**?**] the former is the case, in the application to investment decisions, however, this is not the case. This resolves the "paradox".

Another question that you could study: what if the investor has PT preferences and sets his reference point always to his current wealth level?

3

Two-Period Model: Mean-Variance Approach

3.1

1. The minimum–variance portfolio is the portfolio consisting of risky assets for which the portfolio variance is minimal. To find it, we solve the following problem:

$$\max_{1 \geq \lambda_1 \geq 0,\, 1 \geq \lambda_2 \geq 0} \sigma_\lambda^2 \quad \text{s.t.} \quad \lambda_1 + \lambda_2 = 1.$$

   For our example, this is equivalent to:

$$\min_{1 \geq \lambda_1 \geq 0} \begin{pmatrix} \lambda_1 & 1 - \lambda_1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ 1 - \lambda_1 \end{pmatrix},$$

   where $\lambda_2 = 1 - \lambda_1$.
   The first–order condition for this problem is:

$$4\lambda_1 - 2(1 - \lambda_1) + 2\lambda_1 - 8(1 - \lambda_1) = 0.$$

   Thus, the minimum–variance portfolio is $\lambda^* = (0.625, 0.375)$.
   The tangent portfolio is the portfolio with the maximum Sharpe ratio, i.e. the portfolio providing maximum excess return for an unit risk. An easier way to find the tangent portfolio is to maximize the investor's utility under the condition that the Two–Fund Separation Theorem holds so that $\lambda_0 + \lambda_T = 1$ where λ_T is the tangent portfolio and $\lambda_T = (\lambda_1, \lambda_2)$.

$$\max_{1 \geq \lambda_1 \geq 0,\, 1 \geq \lambda_2 \geq 0} \mu_\lambda - \frac{\gamma}{2} \sigma_\lambda^2,$$

   where the expected return of the tangent portfolio is:

$$\mu_\lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \mu_1 - R_f \\ \mu_2 - R_f \end{pmatrix}$$

   and the variance of the tangent portfolio is:

$$\sigma_\lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

The first–order condition of the problem is:

$$\begin{pmatrix} 3 \\ 5.5 \end{pmatrix} - \gamma \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0.$$

Solving for λ_1 and λ_2 we get: $\lambda_1 = 2.5/\gamma$ and $\lambda_2 = 2/\gamma$. The tangent portfolio is computed after re–normalizing the portfolio shares according to:

$$\lambda_k^T = \frac{\lambda_k^{opt}}{\sum_j \lambda_j^{opt}}$$

which gives:

$$\lambda_1^T = \frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{2.5}{5.5}$$

and

$$\lambda_2^T = \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{2}{5.5}$$

2. Using

$$\begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \mu_1^e - 2 \\ \mu_2^e - 2 \end{pmatrix}$$

we get that the implied expected returns are $\mu_1^e = \mu_2^e = 2.7$.

3. To find the Beta-factors of the risky assets, we have to calculate the covariance of the risky asset returns with the market portfolio and the variance of the market portfolio. The covariance of the asset returns with the market portfolio cab be written as:

$$\sigma_{r_1, r_M} = \lambda_1^2 \sigma_{r_1}^2 + \lambda_2 \sigma_{r_1, r_2} = 0.2$$

respectively

$$\sigma_{r_2, r_M} = \lambda_2^2 \sigma_{r_2}^2 + \lambda_1 \sigma_{r_1, r_2} = 2$$

where σ_{r_k, r_M} is the covariance of the returns of asset k, $k = 1, 2$, with the the return of the market portfolio and σ_k^2 is the variance of the returns of asset k.

The variance of the market return is:

$$\sigma_{r_M}^2 = \lambda_1^2 \sigma_{r_1}^2 + \lambda_2^2 \sigma_{r_2}^2 + 2\lambda_1 \lambda_2 \sigma_{r_1, r_2} = 1.28$$

Thus, the beta factors of the risky assets are:

$$\beta_1 = \frac{\sigma_{r_1, r_M}}{\sigma_{r_M}^2} = \frac{0.2}{1.28}$$

and

$$\beta_2 = \frac{\sigma_{r_2, r_M}}{\sigma_{r_M}^2} = \frac{2}{1.28}$$

The expected excess returns are:

$$\mu_1 - R_f = \beta_1(\mu_M - R_f) = \frac{15}{32}$$

respectively

$$\mu_2 - R_f = \beta_2(\mu_M - R_f) = \frac{75}{16}$$

4

Two-Period Model: State-Preference Approach

4.6

For assets A and B, with two possible states occurring with prob p and $(1-p)$,

$$\begin{aligned} E(A) &= pA_1 + (1-p)A_2 = 0.2 \\ E(B) &= pB_1 + (1-p)B_2 = 0.1 \\ V(A) &= p(A_1 - 0.2)^2 + (1-p)(A_2 - 0.2)^2 = 0.3 \\ V(B) &= p(B_1 - 0.1)^2 + (1-p)(B_2 - 0.1)^2 = 0.2 \\ COV(A,B) &= p(A_1 - 0.2)(B_1 - 0.1) + (1-p)(A_2 - 0.2)(B_2 - 0.1) \end{aligned}$$

Assuming $p = 0.5$, it follows that:

$$E(A) = 0.5A_1 + 0.5A_2 = 0.2 \quad (4.1)$$
$$E(B) = 0.5B_1 + 0.5B_2 = 0.1 \quad (4.2)$$
$$V(A) = 0.5(A_1 - 0.2)^2 + 0.5(A_2 - 0.2)^2 = 0.3 \quad (4.3)$$
$$V(B) = 0.5(B_1 - 0.1)^2 + 0.5(B_2 - 0.1)^2 = 0.2 \quad (4.4)$$

From (4.1) and (4.2) we express A_1 and B_1 and substitute into (4.3) and (4.4):

$$(0.2 - A_2)^2 = 0.3 \quad \textit{and} \quad (0.1 - B_2)^2 = 0.2$$

Thus we have to solve (with the determinants shown in brackets):

$$\begin{aligned} A_2^2 - 0.4A_2 - 0.26 = 0 \quad (D_A = 1.2) \\ B_2^2 - 0.2B_2 - 0.19 = 0 \quad (D_B = 0.8) \end{aligned}$$

Each equation has two roots (call it positive and negative), and we arrive to two pairs of solutions:

$$A_2^+ = 0.747 \quad \text{and} \quad A_1^+ = -0.347; \quad \text{or} \quad A_2^- = -0.347 \quad \text{and} \quad A_1^- = 0.747$$

$$B_2^+ = 0.547 \quad \text{and} \quad B_1^+ = -0.347; \quad \text{or} \quad B_2^- = -0.347 \quad \text{and} \quad B_1^- = 0.547$$

To choose a particular solution, we note that the covariance between the two returns is specified to be negative, in other words, in state s, whenever A is high, B should be low, and the other way around. Hence, the solution is either $\begin{pmatrix} -0.347 & -0.547 \\ 0.747 & -0347 \end{pmatrix}$ or $\begin{pmatrix} 0.747 & -0.347 \\ -0.347 & 0.547 \end{pmatrix}$

That the covariance is also satisfied can be checked easily.

4.8

(a)

$$E(R^1) = \frac{1}{2}(\mu_1 + \rho\sigma_1 + \mu_1 - \rho\sigma_1) = \mu_1$$

$$E(R^2) = \frac{1}{2}\,(\mu_2 - \sigma_2 + \mu_2 + \sigma_2) \; = \mu_2$$

$$V(R^1) = \frac{1}{2}(\mu_1 + \rho\sigma_1 - \mu_1)^2 + \frac{1}{2}(\mu_1 - \rho\sigma_1 - \mu_1)^2 = \rho^2\sigma_1^2$$

$$V(R^2) = \sigma_2^2$$

$$\text{COR}(R^1, R^2) = \frac{\text{cov}(R^1, R^2)}{\sqrt{V(R^1) \cdot V(R^2)}} = \frac{1}{2}(-\rho\sigma_1\sigma_2 + (-\rho\sigma_1\sigma_2))/\rho\sigma_1\sigma_2 = -1.$$

Since $\rho^2 = 1$, we have $\rho = 1$ or $\rho = -1$ and hence, $\text{COR}(R^1, R^2) = \rho$.

(b) Let $\mu_1, \mu_2 \geq 1$, i.e. assets have non-negative expected net return. We need to consider four cases in our parameter space.

A) $\mu_1 - \rho\sigma_1 < 1$, $\mu_2 - \sigma_2 < 1$; i.e. $\rho > \frac{\mu_1 - 1}{\sigma_1}$

$$\widehat{R} = \begin{pmatrix} N & \mu_2 - \sigma_2 \\ \mu_1 - \rho\sigma_1 & N \end{pmatrix}; \quad \widehat{R}^P = max\begin{pmatrix} \frac{N + \mu_2 - \sigma_2}{2}, 1 \\ \frac{N + \mu_1 - \rho\sigma_1}{2}, 1 \end{pmatrix}$$

Both outcomes are equally likely to occur, none is equal to N.

B) $\mu_1 - \rho\sigma_1 < 1$, $\mu_2 - \sigma_2 \geq 1$

$$\widehat{R} = \begin{pmatrix} N & N \\ \mu_1 - \rho\sigma_1 & N \end{pmatrix}; \quad \widehat{R}^P = max\begin{pmatrix} N, 1 \\ \frac{N + \mu_1 - \rho\sigma_1}{2}, 1 \end{pmatrix}$$

Here the probability of getting return N (given that $N > 1$) is equal to 0.5 (outcome 1).

C) $\mu_1 - \rho\sigma_1 \geq 1$, $\mu_2 - \sigma_2 < 1$; i.e. $\rho \leq \frac{\mu_1 - 1}{\sigma_1}$

$$\widehat{R} = \begin{pmatrix} N & \mu_2 - \sigma_2 \\ N & N \end{pmatrix}; \quad \widehat{R}^P = max \begin{pmatrix} \frac{N+\mu_2-\sigma_2}{2}, 1 \\ N, 1 \end{pmatrix}$$

Here the probability of getting return N (outcome 2) is equal to 0.5.

D) $\mu_1 - \rho\sigma_1 \geq 1$, $\mu_2 - \sigma_2 \geq 1$

$$\widehat{R} = \begin{pmatrix} N & N \\ N & N \end{pmatrix}; \quad \widehat{R}^P = max \begin{pmatrix} N, 1 \\ N, 1 \end{pmatrix}$$

Outcome N always attained.

4.10

(a) The state-space matrix is given by the returns of the assets (k = stocks, bonds) in the states determined by the two factors (f = oil price, growth rate). By assumption, each factor has two realizations, which we denote as "high" and "low". Hence, there are four states determining the returns of the assets:
- state 1: oil price = "high" and growth rate = "high"
- state 2: oil price = "high" and growth rate = "low"
- state 3: oil price = "low" and growth rate = "high"
- state 4: oil price = "low" and growth rate = "low"

The state-space matrix is a $s \times k$ matrix, where s denoted the number of states and k denotes the number of assets. In our example, $s = 4$ and $k = 2$. Given the returns of each asset for each of the two factors, we get that the state-space matrix is:

$$R_s^k = \begin{pmatrix} 1 & -3 \\ -2 & -1 \\ 5 & -1 \\ -3 & 2 \end{pmatrix}$$

(b) To find the factor loadings consider that the state-space matrix is the product of the factor returns in each state and the factor loadings, i.e.

$$R = [R_s^k] = [R_s^f][\beta_k^f] \tag{4.5}$$

Given $[R_s^k]$ as calculated in the previous question, for a given factor returns summarized in the matrix $[R_s^f]$ as for example

$$R_s^f = \begin{pmatrix} 7 & -3 \\ 0 & -1 \\ 7 & -1 \\ -7 & 2 \end{pmatrix}$$

we get that for (4.5) to hold the factor loadings must be

$$\beta_k^f = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \tag{4.6}$$

(c) The state-space matrix summarizes the asset returns in different states. To determine the joint distribution of the asset returns, consider the returns of each asset across the states and the joint probability of the factor combinations generating the states. In our example, the joint distribution of the asset returns is given by:

$$\begin{array}{r|rrr} 5\% & 0 & 30\% & 0 \\ 1\% & 5\% & 0 & 0 \\ -2\% & 0 & 50\% & 0 \\ -3\% & 0 & 0 & 15\% \\ \hline & -3\% & -1\% & 2\% \end{array} \tag{4.7}$$

4.11

(a) To calculate the state-space matrix we use equation (4.5) and get

$$R = \begin{bmatrix} 3 & -2 \\ 2 & -1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -8 \\ 5 & -5 \\ -7 & 4 \\ -10 & 0 \end{bmatrix}$$

(b) If we assume that the four states are equally probable then the joint distribution of the assets returns is given by

$$\begin{array}{r|rrrr} 9\% & 25\% & 0 & 0 & 0 \\ 5\% & 0 & 25\% & 0 & 0 \\ -7\% & 0 & 0 & 0 & 25\% \\ -10\% & 0 & 0 & 25\% & 0 \\ \hline & -8\% & -5\% & 0\% & 4\% \end{array}$$

4.16

(a) The utility maximization problem of the representative agent is:

$$\begin{aligned} &\max_{c_u, c_d} 0.5\ln(c_u) + 0.5\ln(c_d) \\ &\text{s.t.} c_u = \theta D(\mu+\sigma) + \delta R \\ &c_d = \theta D(\mu-\sigma)\delta R \\ &\theta S + \delta = 1 \cdot S. \end{aligned}$$

The market clearing conditions are $\theta = 1$ and $\delta = 0$. The Lagrange function is

$$L = 0.5\ln\left(\theta D(\mu+\sigma)+\delta R\right) + 0.5\ln\left(\theta D(\mu-\sigma)+\delta R\right) - \lambda\left(\theta S + \delta - S\right)$$

FOC:

$$\begin{aligned}\frac{\partial L}{\partial \theta} &= 0.5\frac{D(\mu+\sigma)}{c_u} + 0.5\frac{D(\mu-\sigma)}{c_d} - \lambda S\\ \frac{\partial L}{\partial \delta} &= 0.5\frac{R}{c_u} + 0.5\frac{R}{c_d} - \lambda.\end{aligned}$$

Dividing the two first order conditions, plug in the market clearing conditions and solve for S results in

$$S = \frac{D}{R}\frac{(\mu+\sigma)(\mu-\sigma)}{\mu} = \frac{D}{R}\frac{\mu^2-\sigma^2}{\mu}.$$

(b)

$$\frac{\partial S}{\partial \sigma} = \frac{-2D\sigma}{R\mu} < 0$$

This implies asymmetric volatility: If the volatility rises, the stock price falls.

(c) The return of the stock is:

$$\begin{aligned}R^u &= \frac{D(\mu+\sigma)}{S} = \frac{R\mu}{\mu-\sigma}\\ R^d &= \frac{D(\mu-\sigma)}{S} = \frac{R\mu}{\mu+\sigma}.\end{aligned}$$

The expected value and the volatility can then be calculated (note that $p = 0.5$):

$$\begin{aligned}\mu(R) &= \frac{R\mu^2}{(\mu-\sigma)(\mu+\sigma)}\\ \mu(R^2) &= \frac{R^2\mu^2(\mu^2+\sigma^2)}{(\mu-\sigma)^2(\mu+\sigma)^2}\\ \sigma(R) &= \sqrt{\mu(R^2)-\mu(R)^2} = \frac{R\mu\sigma}{(\mu-\sigma)(\mu+\sigma)}.\end{aligned}$$

(d) The stock price can be rewritten to

$$S = \frac{D}{R}\frac{(\mu+\sigma)(\mu-\sigma)}{\mu} = \frac{\overline{D}}{\sigma(R)}.$$

Also here the impact or the volatility to the stock price is negative:

$$\frac{\partial S}{\partial \sigma(R)} = -\frac{\overline{D}}{\sigma^2(R)} < 0.$$

4.17

(a) In order to maximize the utility, agent i chooses a strategy $\theta^i = \left(\theta_1^i, \theta_2^i\right)^T$,i.e., agent i buys θ_j^i of asset j subject to his initial endowment. Formally, this can be described by the optimization problems

$$\begin{aligned}&\text{maximize } \frac{1}{3}\ln\left(1+\theta_1^1\right)+\frac{1}{3}\ln\left(\frac{1}{3}+1\theta_2^1\right)\\&\text{subject to } q'\theta^1 \leq 0 \text{ and } w^1+\theta^1 A > 0\end{aligned} \tag{4.8}$$

for agent 1 and

$$\begin{aligned}&\text{maximize } \frac{1}{3}\ln\left(1+\theta_1^2\right)+\frac{1}{3}\ln\left(\frac{1}{3}+\theta_2^2\right)\\&\text{subject to } q^T\theta^2 \leq 0 \text{ and } w^2+\theta^2 A > 0\end{aligned} \tag{4.9}$$

for agent 2.

In the optimization problems, the initial endowment the price vector , the probabilities and the payoffs are fixed. The price of the first asset is be normalized to 1. Finally, due to the budget restriction, we replace θ_2^i by $\frac{-\theta_1^i}{q^1}$. This simplifies the optimization problem to a maximization of a function depending on θ_1^i. Differentiating the function with respect to θ_1^i and setting the resulting term equal to 0 gives two equations which can be solved. This gives

$$\begin{aligned}\theta_1^1 &= \frac{q^1\frac{1}{3}-1}{2},\\ \theta_1^2 &= \frac{q^1\frac{1}{3}-1}{2}.\end{aligned} \tag{4.10}$$

In order to "clear away" any excess supply and excess demand, the quantity demanded and the quantity supplied should be equal. In our setup, this means that $\theta_1^1 = -\theta_1^2$ and $\theta_2^1 = -\theta_2^2$ hold. Together with (4.10), these four conditions uniquely determine θ_j^i for $i, j = 1, 2$. Furthermore, combining (4.10) with $\theta_1^1 = -\theta_1^2$, we deduce the explicit form

$$q^1 = \frac{2}{\frac{2}{3}} = 3$$

of the price of asset 2.

(b) In order to exclude bad market behaviour in the sense of arbitrage, it is enough to have existence of an equivalent martingale measure. It is well known, that a tree with three branches defines an incomplete model, i.e., either there is no martingale measure or there are more than one and hence infinitely many. In our setting, an equivalent martingale measure is a vector $\pi = (\pi_1, \pi_2, \pi_3)^T$ where $\pi_i > 0$ and such that $\pi^T A = q$ holds, i.e.,

$$\begin{bmatrix} 1 & 3 \end{bmatrix} = [\pi_1 \quad \pi_2 \quad \pi_3] \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$
$$\pi_1 = 1$$
$$\pi_2 + \pi_3 = 4$$
$$\sum_s \pi_s = 5.$$

Risk neutral probabilities are then

$$\pi_s^* = \frac{\pi_s}{\sum_s \pi_s} = \frac{\pi_s}{5}.$$

Then,

$$\pi_1^* = \frac{1}{5}$$
$$\pi_2^* + \pi_3^* = \frac{\pi_2}{5} + \frac{\pi_3}{5}.$$

(c) Aggregate endowment:

$$w = \left[2 \quad 5 + \frac{1}{3} \quad 5 + \frac{1}{3}\right]$$

Likelihood ratio is then

$$l_s = \frac{\pi_s^*}{p_s},$$

where $p_s = 1/3$, for all $s = 1, 2, 3$. Then we have

$$l_1 = \frac{1/5}{1/3} = \frac{3}{5}$$
$$l_2 + l_3 = \frac{\frac{\pi_2 + \pi_3}{5}}{1/3} = \frac{3(\pi_2 + \pi_3)}{5}.$$

As we see

$$\pi_1 \leq \max\{\pi_2, \pi_3\}.$$

This implies for all possible likelihood ratio processes that would give the correct prices, either l_2 or l_3 must be higher than l_1, and state 2 and state 3 has higher aggregate endowment.

4.18

(a) The rank of A is two. This can be seen by the last two rows of A, which are a triangular matrix. But there are three states of the world. With two (independent) assets and three states it is not possible to generate any payoff stream. Therefore the market is incomplete.

(b) By the FTAP there is no arbitrage, if strictly positive state prices can be found. For example $\pi = (0.25, 0.5, 0.25)'$. With that $\pi >> 0$ it is indeed true that $A'\pi = q$. Therefore π is indeed a vector of strictly positive state prices and therefore there is no arbitrage.
A more formal way to find a π (there are infinitely many) is to solve the linear equation system $A'\pi = q$ for π by the Gauss algorithm.

(c) The initial endowment of the representative investor in terms of assets is $\overline{\theta} = (1, 1)'$. The utility maximization problem is then

$$\max_{\theta} \ \ln(c_1) + \ln(c_2) + \gamma \ln(c_3)$$
$$\text{s.t. } c = A\theta = \begin{pmatrix} \theta_1 + 4\theta_2 \\ \theta_1 + 2\theta_2 \\ \theta_1 \end{pmatrix}$$
$$q'\theta = q'\overline{\theta} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}' \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3.$$

Note that $q = (1, 2)'$ and $q'\theta = \theta_1 + 2\theta_2 = 3$. Plug in the consumption into the maximization problem, the Lagrange function is then:

$$L = \ln(\theta_1 + 4\theta_2) + \ln(\theta_1 + 2\theta_2) + \gamma \ln(\theta_1) - \lambda(\theta_1 + 2\theta_2 - 3)$$

The FOCs are:

$$\frac{\partial L}{\partial \theta_1} = \frac{1}{\theta_1 + 4\theta_2} + \frac{1}{\theta_1 + 2\theta_2} + \frac{\gamma}{\theta_1} - \lambda = 0$$
$$\frac{\partial L}{\partial \theta_2} = \frac{4}{\theta_1 + 4\theta_2} + \frac{2}{\theta_1 + 2\theta_2} - 2\lambda = 0.$$

Market clearing implies $\theta_1 = \theta_2 = 1$. From the second market clearing condition we get:

$$\lambda = \frac{2}{5} + \frac{1}{3} = \frac{11}{15}.$$

Plugging that into the first FOC results in:

$$\frac{\gamma}{\theta_1} = \lambda - \frac{1}{\theta_1 + 4\theta_2} - \frac{1}{\theta_1 + 2\theta_2}$$
$$\gamma = \frac{11}{15} - \frac{1}{5} - \frac{1}{3} = \frac{3}{15}.$$

(d) The initial endowment of the rep investor in terms of initial wealth is:

$$w = A\overline{\theta} = \begin{pmatrix} 1 & 4 \\ 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$$

For the initial wealth the third asset does not need to be taken into account because it is of zero net supply. The utility maximization problem for the representative investor is

$$\begin{aligned} &\max_c \ \ln(c_1) + \ln(c_2) + 2\ln(c_3) \\ &\text{s.t. } \pi' w = \pi' c \end{aligned}$$

The market clearing condition is $c = w = (5, 3, 1)'$. The Lagrange function is:

$$L = \ln(c_1) + \ln(c_2) + 2\ln(c_3) - \lambda\left(\pi'(c - w)\right)$$

The first order conditions are:

$$\begin{aligned} \frac{\partial L}{\partial c_1} &= \frac{1}{c_1} - \lambda\pi_1 = 0 \\ \frac{\partial L}{\partial c_2} &= \frac{1}{c_2} - \lambda\pi_2 = 0 \\ \frac{\partial L}{\partial c_1} &= \frac{2}{c_3} - \lambda\pi_3 = 0 \end{aligned}$$

By taking $\lambda\pi_s$ on one side of the equation and dividing the last two FOCs by the first one we get by using the market clearing condition:

$$\frac{\pi_2}{\pi_1} = \frac{c_1}{c_2} = \frac{5}{3} \qquad\qquad \frac{\pi_3}{\pi_1} = \frac{2c_1}{c_3} = \frac{10}{1}$$

By norming $\pi_1 = 3$ we get for $\pi = (3, 5, 30)'$. The normed state prices, π^*, are then:

$$\pi^* = \frac{\pi}{1'\pi} = \frac{1}{38}\begin{pmatrix} 3 \\ 5 \\ 30 \end{pmatrix}$$

(e) Yes, the state prices π are strictly positive. By FTAP there is no arbitrage. *Alternative solution:* The utility function of the representative investor is strictly monotonic. If there would be an arbitrage opportunity the representative investor would use it infinitely often to get an infinite utility. This would not be in line with the market clearing condition. Therefore if there is an equilibrium, there cannot be arbitrage.

4.19

(a) No, the quadratic utility function (with $\gamma > 0$) starts to decrease at one $c > 0$. In other words less consumption is preferred over more consumption. But in the case of strongly monotonic preferences it must hold that more is better.

(b) Note that: $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$. We can write

$$\begin{aligned} U^R &= \mathbb{E}\left((c-1) - \frac{\gamma}{2}(c-1)^2\right) \\ &= \mathbb{E}(c) - 1 - \frac{\gamma}{2}\left[\mathbb{E}(c^2) - 2\mathbb{E}(c) + 1\right] \\ &= (1+\gamma)\mathbb{E}(c) - (1+0.5\gamma) - 0.5\gamma\left[\mathrm{Var}(c) + \mathbb{E}(c)^2\right] \\ &= -0.5\gamma\,\mathrm{Var}(c) - 0.5\gamma\mathbb{E}(c)^2 + (1+\gamma)\mathbb{E}(c) - (1+0.5\gamma). \end{aligned}$$

(c) The utility maximization problem of the representative investor is:

$$\begin{aligned} \max_{\theta}\ & \frac{1}{2}\left[c_u - 1 - \frac{\gamma}{2}(c_u-1)^2\right] + \frac{1}{2}\left[c_d - 1 - \frac{\gamma}{2}(c_d-1)^2\right] \\ \text{s.t. }\ & c_u = \theta_S(\mu+\sigma) + \theta_B R \\ & c_d = \theta_S(\mu-\sigma) + \theta_B R \\ & S\theta_S + \theta_B = S \end{aligned}$$

The market clearing conditions are $\theta_S = 1$ and $\theta_B = 0$. Plugging in the consumption, the Lagrange function becomes:

$$\begin{aligned} L = & \frac{1}{2}\left[\theta_S(\mu+\sigma) + \theta_B R - 1 - \frac{\gamma}{2}\left(\theta_S(\mu+\sigma) + \theta_B R - 1\right)^2\right] \\ & + \frac{1}{2}\left[\theta_S(\mu-\sigma) + \theta_B R - 1 - \frac{\gamma}{2}\left(\theta_S(\mu-\sigma) + \theta_B R - 1\right)^2\right] \\ & - \lambda\left(S\theta_S + \theta_B - S\right) \end{aligned}$$

The first order conditions are:

$$
\begin{aligned}
\frac{\partial L}{\partial \theta_S} &= \frac{1}{2}\left[(\mu+\sigma) - \gamma\left(\theta_S(\mu+\sigma) + \theta_B R - 1\right)(\mu+\sigma)\right] \\
&+ \frac{1}{2}\left[(\mu-\sigma) - \gamma\left(\theta_S(\mu-\sigma) + \theta_B R - 1\right)(\mu-\sigma)\right] - \lambda S = 0 \\
\frac{\partial L}{\partial \theta_B} &= \frac{1}{2}\left[R - \gamma\left(\theta_S(\mu+\sigma) + \theta_B R - 1\right) R\right] \\
&+ \frac{1}{2}\left[R - \gamma\left(\theta_S(\mu-\sigma) + \theta_B R - 1\right) R\right] - \lambda \\
&= R\left[1 - \gamma\left(\theta_S \mu + \theta_B R - 1\right)\right] - \lambda = 0
\end{aligned}
$$

Solving the first FOC for λS gives us

$$\lambda S = \mu - \gamma\mu\left(\theta_S\mu + \theta_B R - 1\right) - \gamma\sigma^2$$

Solving the second first order condition for λ gives us

$$\lambda = R\left[1 - \gamma\left(\theta_S\mu + \theta_B R - 1\right)\right].$$

Plugging λ into the equation of S it results:

$$S = \frac{\mu - \gamma\mu\left(\theta_S\mu + \theta_B R - 1\right) - \gamma\sigma^2}{R\left[1 - \gamma\left(\theta_S\mu + \theta_B R - 1\right)\right]} = \frac{\mu}{R} - \frac{\gamma\sigma^2}{R\left[1 - \gamma\left(\theta_S\mu + \theta_B R - 1\right)\right]}$$

Using the market clearing conditions $\theta_S = 1$ and $\theta_B = 0$ we get:

$$S = \frac{\mu}{R} - \frac{\gamma\sigma^2}{R\left[1 - \gamma\left(\mu - 1\right)\right]}.$$

4.20

(a) The utility maximization problem of the representative agent is

$$
\begin{aligned}
\max_{c_s} \; & \sum_{s=1}^{S} \text{prob}_s \, u(c_s) \\
\text{s.t.} \; & \sum_{s=1}^{S} \pi_s c_s = w_0
\end{aligned}
$$

The Lagrange function is then

$$L = \sum_{s=1}^{S} \text{prob}_s \, u(c_s) - \lambda\left(\sum_s \pi_s c_s - w_0\right)$$

The first order conditions are for $s \in \{1, \ldots, S\}$:

$$\frac{\partial L}{\partial c_s} = \text{prob}_s \, u'(c_s) - \lambda \pi_s = 0$$

Summing over all first order conditions and solving for λ results in

$$\lambda = \frac{\sum_s \text{prob}_s \, u'(c_s)}{\sum_\tau \pi_\tau}$$

Plug this into the first order condition and solve for π_s. By plugging in π_s and the utility function it is obtained

$$l_s = \frac{\pi_s^*}{\text{prob}_s} = \frac{\frac{\pi_s}{\sum_\tau \pi_\tau}}{\text{prob}_s} = \frac{1 - \gamma c_s}{1 - \gamma \sum_\tau \text{prob}_\tau \, c_\tau} = \frac{1 - \gamma c_s}{1 - \gamma \mathbb{E}[C]}$$

(b) The investor invests a fraction λ^k of his initial wealth into asset k. By market clearing the representative investor consumes what he gets from the assets:

$$c_s = \sum_{k=1}^{K} R_s^k \lambda^k w_0 = R_s^M \cdot 1$$

(c) From the script it is known

$$\begin{aligned} R_f &= \mathbb{E}\left[R^k\right] + \text{cov}\left(R^k, l\right) = \mathbb{E}\left[R^k\right] + \text{cov}\left(R^k, \frac{1 - \gamma C}{1 - \gamma \mathbb{E}[C]}\right) \\ &= \mathbb{E}\left[R^k\right] - \frac{\gamma}{1 - \gamma \mathbb{E}[C]} \text{cov}\left(R^k, C\right) = \mathbb{E}[R_k] - \frac{\gamma}{1 - \gamma \mathbb{E}[R^M]} \text{cov}\left(R^k, R^M\right) \end{aligned}$$

Set $k = M$ and rewrite the expression

$$\frac{\gamma}{1 - \gamma \mathbb{E}[R^M]} = \frac{\mathbb{E}\left[R^M\right] - R_f}{\text{cov}(R^M, R^M)}$$

Plug this in for a general k and it results

$$\begin{aligned} \mathbb{E}[R_k] - R_f &= \frac{\mathbb{E}\left[R^M\right] - R_f}{\text{cov}(R^M, R^M)} \text{cov}\left(R^k, R^M\right) = \frac{\text{cov}\left(R^k, R^M\right)}{\sigma^2(R^M)} \left(\mathbb{E}\left[R^M\right] - R_f\right) \\ &= \beta_k \left(\mathbb{E}\left[R^M\right] - R_f\right) \end{aligned}$$

4.21

(a) Note that all problems represent the Cobb-Douglas utility function, where we can always normalize preference weights to sum up to unity. I.e., the following utility functions represent identical preference relation (and, hence, can be interchanged freely when determining optimal consumption/investment decisions):

$$\max\ U^i = a_1^i \ln(c_1^i) + a_2^i \ln(c_2^i) \quad \text{and} \quad \max\ U^i = \alpha_1^i \ln(c_1^i) + \alpha_2^i \ln(c_2^i)$$

where $\alpha_s^i \equiv a_s^i / \sum_{s'=0}^{S} a_{s'}^i$. The general setting of these problems is as follows

$$\max U^i(c_s^i) = \sum_{s=0}^{S} \alpha_s^i \ln(c_s^i)$$

$$\text{s.t.} \ \sum_{s=0}^{S} \pi_s c_s^i \leq \sum_{s=0}^{S} \pi_s w_s^i$$

$$\sum_{i=1}^{I} c_s^i = \sum_{i=1}^{I} w_s^i$$

where c_s^i is the consumption by agent i in state s, π_s is the state price of state s, and w_s^i is the wealth by agent i in state s, and $\sum_{s=1}^{S} \alpha_s^i = 1$. These three equation represent the utility maximization of the investors, the budget constraint of the investors and the market clearing condition for every state.
Let $L = \sum_{s=0}^{S} \alpha_s^i \ln(c_s^i) + \lambda(\sum_{s=0}^{S} \pi_s w_s^i - \sum_{s=0}^{S} \pi_s c_s^i)$. The first-order condition of the problem is then:

$$\frac{\alpha_s^i}{c_s^i} - \pi_s \lambda = 0$$

$$\iff c_s^i = \frac{\alpha_s^i}{\pi_s \lambda} \tag{4.11}$$

From (4.11) we have the ratio of consumption by agent i and agent j in state s is:

$$\frac{c_s^i}{c_s^j} = \frac{\alpha_s^i}{\alpha_s^j}$$

And the ratio of consumption in state s and state z by agent i:

$$\frac{c_s^i}{c_z^i} = \frac{\alpha_s^i \pi_z}{\alpha_s^i \pi_s}$$

To obtain the ratio of state prices, we define:

$$\alpha_1^i \equiv \alpha^i = c_1^i \pi_1 \lambda \quad \text{and} \quad \alpha_2^i \equiv 1 - \alpha^i = c_2^i \pi_2 \lambda$$

Use the fact that alpha's sum to unity (and by substituting (2), which is equality at optimum) to get the multiplier as $\lambda = (\pi_1 w_1^i + \pi_2 w_2^i)^{-1}$ and upon substituting this into (6), express optimal consumptions as

$$c_s^i = \frac{\alpha_s^i}{\pi_s}(\pi_1 w_1^i + \pi_2 w_2^i)$$

By the Walras' Law, clearing in any one of the states s is sufficient, e.g.

$s = 1$:

$$\begin{aligned}
&\sum_{i=1}^{2} c_1^i = \frac{1}{\pi_1}\sum_{i=1}^{2} \alpha^i(\pi_1 w_1^i + \pi_2 w_2^i) \\
\Longleftrightarrow\ &\sum_{i=1}^{2} w_1^i = \frac{1}{\pi_1}\sum_{i=1}^{2} \alpha^i(\pi_1 w_1^i + \pi_2 w_2^i) \text{ (Demand=Supply)} \\
\Longleftrightarrow\ &-\sum_{i=1}^{2} \alpha^i \pi_2 w_2^i = \sum_{i=1}^{2} \alpha^i \pi_1 w_1^i - \pi_1 \sum_{i=1}^{2} w_1^i \\
\Longleftrightarrow\ &\frac{\pi_1}{\pi_2} = \frac{\sum_{i=1}^{2} \alpha^i w_2^i}{\sum_{i=1}^{2} (1-\alpha^i) w_1^i}
\end{aligned} \tag{4.12}$$

The above is the general setting for such problems. Now to the specific exercises:

$$\begin{aligned}
&\max U^1(c) = \ln(c_0^1) + 2\ln(c_1^1) \\
&\max U^2(c) = 2\ln(c_0^2) + \ln(c_1^2) \\
&\text{s.t.} \sum_{i=1}^{2} c_0^i + \sum_{i=1}^{2} \pi_1 c_1^i \leq \sum_{i=1}^{2} w_0^i + \sum_{i=1}^{2} \pi_1 w_1^i \\
&\sum_{i=1}^{I} c_s^i = \sum_{i=1}^{I} w_s^i, \text{ for s=0,1}
\end{aligned}$$

Applying (4.11) and (4.12), we have $\pi_1 = 1$ (upon normalizing $\pi_0 = 1$) and $C = \begin{bmatrix} 2/3 & 4/3 \\ 4/3 & 2/3 \end{bmatrix}$ where C_{is} denotes the consumption of agent i at state s. Here s=0 (initial state) and 1 (future state).

(b) The equilibrium is

$$\begin{aligned}
&\max U^1(c) = \ln(c_1^1) + \ln(c_2^1)\\
&\max U^2(c) = \ln(c_1^2) + \ln(c_2^2)\\
&\text{s.t. } \sum_{i=1}^{2} (\pi_1 c_1^i + \pi_2 c_2^i) \leq \sum_{i=1}^{2} (\pi_1 w_1^i + \pi_2 w_2^i)\\
&\sum_{i=1}^{2} c_s^i = \sum_{i=1}^{2} w_s^i, \text{ for s=1,2}
\end{aligned}$$

Applying (4.11) and (4.12), we have $\frac{\pi_1}{\pi_2} = 1$ (e.g., choose $\pi_1 = \pi_2 = 1$) and $C = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ where C_{is} denotes the consumption of agent i at state s.

(c) The equilibrium is

$$\begin{aligned}
&\max U^1(c) = 2\ln(c_1^1)\ln(c_2^1)\\
&\max U^2(c) = \ln(c_1^2) + 2\ln(c_2^2)\\
&\text{s.t. } \sum_{i=1}^{2} (\pi_1 c_1^i + \pi_2 c_2^i) \leq \sum_{i=1}^{2} (\pi_1 w_1^i + \pi_2 w_2^i)\\
&\sum_{i=1}^{2} c_s^i = \sum_{i=1}^{2} w_s^i, \text{ for s=1,2}
\end{aligned}$$

Applying (4.11) and (4.12), we have $\frac{\pi_1}{\pi_2} = 1$ (e.g., choose $\pi_1 = \pi_2 = 1$) and $C = \begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{bmatrix}$.

(d) The motive for trade in a) is timing preference (saving vs. spending).
The motive for trade in b) is risk preference (insurance).
The motive for trade in c) is distinct belief distribution (betting).

4.22

We need to prove: If there exists a $\pi \in \mathbb{R}^s_{++}$ such that $q_k = \sum_{s=1}^S A_s^k \pi_s$, $k = 0, \ldots, K$, then there is no $\theta \in \mathbb{R}^{K+1}$ such that $\begin{bmatrix} -q' \\ A \end{bmatrix} \theta > 0$.

We start from $q_k = \sum_{s=1}^S A_s^k \pi_s$, i.e. $\pi' A = q'$. Suppose that there exists an arbitrage opportunity, i.e., there is some $\theta \in \mathbb{R}^{K+1}$ with $\begin{bmatrix} -q' \\ A \end{bmatrix} \theta > 0$. Then, in particular, the first line of this expression must be positive, i.e., $q'\theta < 0$. We put this together with the above formula $q' = \pi' A$ and get $\pi' A\theta < 0$. Since π is positive in all components, this means that $A\theta < 0$, but this conflicts with the assumed positivity of $\begin{bmatrix} -q' \\ A \end{bmatrix} \theta$. Hence, our assumption that there exists an arbitrage opportunity must be wrong.

4.23

Via the FTAP we know that if there is no arbitrage there exists a strictly positive normed state price vector π^* (i.e. $1'\pi^* = 1$) such that $R\pi^* = R^f \cdot 1$. With $R_s^k = A_s^k / q_k$ and R^f the risk free rate.

In the case where the number of states is equal to the number of asset this exercise is quite easy to solve: If R has full rank then $R\pi^* = R^f \cdot 1$ implies that $\pi^* = R^{-1} R^f \cdot 1$. If $\pi^* > 0$ then there is via the FTAP no arbitrage.

In our case the problem is more complicated, because we have more states than assets. We need still to check, if there is a $\pi^* > 0$ such that $R\pi^* = R^f \cdot 1$. Because the $\pi^* R^f = R^f$ this relation can be rewritten to $(R - R^f)\pi^* = 0$. But since R is not a quadratic matrix anymore the solution has to be a different one. $R\pi^* = R^f \cdot 1$ implies that $((R - R^f)\pi^*)'((R - R^f)\pi^*) = 0$. Therefore we have no arbitrage if:

$$0 = \min_{\pi^* > 0,\ 1'\pi^* = 1} ((R - R^f)\pi^*)'((R - R^f)\pi^*)$$

This minimization problem can be solved by numerical minimization. If it turns out that the minimum is indeed zero, then we have no arbitrage. The solution in Excel can be found on the web. We obtain $((R - R^f)\pi^*)'((R - R^f)\pi^*) = 1.79 \cdot 10^{-6}$. This is extremely close to zero such that there is no arbitrage.

4.24

(a) The payoffs of a call is replicated by a portfolio of stocks and bonds. Since there is no arbitrage the price of the call must be the price of the replication portfolio. The price of the call option, C_0 is then given by the following equation system:

$$\begin{aligned} C_0 &= nS + mB \\ C_u &= n(uS) + mRB \\ C_d &= n(dS) + mRB \end{aligned}$$

By solving the last two equation for n and m results in

$$n = \frac{C_u - C_d}{S(u - d)} \qquad\qquad m = \frac{uC_d - dC_u}{RB(u - d)}$$

For C_0 it is obtained

$$C_0 = nS + mB = \frac{1}{R}\frac{C_u(R-d) + C_d(u-R)}{u-d}$$

(b) From the Fundamental Theorem of Asset Pricing we know that there exist positive state prices π (not normed) such that

$$\begin{aligned} B &= \pi_u RB + \pi_d RB \\ S &= \pi_u uS + \pi_d dS \\ C_0 &= \pi_u C_u + \pi_d C_d \end{aligned}$$

Solving the first two equations for π_u and π_d we get

$$\pi_u = \frac{R-d}{R(u-d)} \qquad \pi_d = \frac{u-R}{R(u-d)}$$

For C_0 it is obtained

$$C_0 = \pi_u C_u + \pi_d C_d = \frac{1}{R}\frac{C_u(R-d) + C_d(u-R)}{u-d}$$

(c) Plug in the formula (we determined before) and we get for the price of the put option

$$P_0 = \frac{1}{R}\frac{P_u(R-d) + P_d(u-R)}{u-d} = \frac{300}{11} \approx 27.27$$

where P_u is the payoff of the put in the upper state and P_d the payoff of the put the lower state.

4.25

(a) The barrier option cannot be hedged since the rank of the matrix $[A, A_3]$ is three. This can be checked by calculating the determinant of the first three rows of the matrix $[A, A_3]$. If this determinant is nonzero, the matrix of these three rows has full rank. Therefore there is no linear combination of the stock and the bond which can hedge the barrier option.

(b) The price bounds of the barrier option are determined via

$$\begin{aligned} \overline{q}(y) &= \max_\pi \pi' y \qquad && \text{s.t. } A'\pi = q \text{ and } \ \pi \geq 0 \\ \underline{q}(y) &= \min_\pi \pi' y \qquad && \text{s.t. } A'\pi = q \text{ and } \ \pi \geq 0 \end{aligned}$$

This problem has to be solved numerically. A solution with Excel is also on the web. The solutions are

$$\overline{q}(y) = 1.03 \qquad \underline{q}(y) = 0.76$$

(c) Also this problem has to be solved numerically. A solution with Excel is also on the web. A possible solution is

$$\alpha = 0.7 \qquad \alpha^{+} = 0.4 \qquad \alpha^{-} = 1 \qquad \beta = 1.5 \qquad RP = 1.2$$

4.26

(a) One way to get the price bounds of asset $y = A_3$ is to solve the following optimization problems

$$\overline{q}(y) = \min_{\theta} q'\theta \qquad \text{s.t } A\theta \geq y$$
$$\underline{q}(y) = \max_{\theta} q'\theta \qquad \text{s.t } A\theta \leq y$$

The conditions that $A\theta \geq y$ are $\theta_1 \geq 2$ and $\theta_2 \geq -1$. To obtain these conditions look first at state two of the problem then at state one. On that set the minimum is obviously

$$\overline{q}(y) = 2q_2 - 1q_1 = 1.55$$

The conditions that $A\theta \leq y$ are $\theta_1 \leq 0$ and $\theta_2 \leq 1$. To obtain these conditions look first at state two of the problem then at state one. On that set the maximum is obviously

$$\underline{q}(y) = 0q_1 + 1q_2 = 0.25$$

(b) An alternative way to determine the bounds is

$$\overline{q}(y) = \max_{\pi} \pi' y \qquad \text{s.t. } A'\pi = q \text{ and } \pi \geq 0$$
$$\underline{q}(y) = \min_{\pi} \pi' y \qquad \text{s.t. } A'\pi = q \text{ and } \pi \geq 0$$

Plug in the values it is obtained:

$$\begin{aligned}\overline{q}(y) &= \max_{\pi \geq 0} \begin{pmatrix}\pi_1\\ \pi_2\\ \pi_3\end{pmatrix}' \begin{pmatrix}1\\ 0\\ 1\end{pmatrix} \qquad \text{s.t. } \pi_1 = 0.25 \text{ and } \pi_2 + \pi_3 = 0.65\\ &= \max_{\pi_2 \geq 0} 0.9 - \pi_2\end{aligned}$$

The optimal state prices are then

$$\pi_2 = 0 \qquad \pi_3 = 0.65 \qquad \pi_1 = 0.25$$

Therefore

$$\overline{q}(y) = y'\pi = 0.9$$

In the same way we have for the lower bound

$$\underline{q}(y) = \min_{\pi_3 \geq 0} 0.25 + \pi_3 = 0.25$$

4.27

Suppose non-negative payoffs and short sales constraints: $A_s^k > 0$ and $\theta_s^k \geq 0$. Proof that there is no $\theta \geq 0$ such that $q'\theta \leq 0$ and $A\theta > 0$ is equivalent to $q \gg 0$.

First we prove that if there is no $\theta \geq 0$ such that $q\theta \leq 0$ and $A\theta > 0$, then $q \gg 0$. That is, under positive payoffs and shortsale constraints, no-arbitrage implies positive prices.

Suppose if there exists $q^k \leq 0$ (i.e., the price for asset k is less than or equal to 0), then we can find $\theta = [0, 0, ..., 1, 0, ..., 0]$, where only the k-th element is 1, and all other elements are 0. In this case, there exists $\theta \geq 0$, so that $q'\theta \leq 0$ $A\theta > 0$, which conflicts with the no-arbitrage condition. Hence with frictions, no-arbitrage implies positive prices.

Second we prove that if $q \gg 0$, then there is no $\theta \geq 0$ such that $q'\theta \leq 0$ and $A\theta > 0$. That is, positive prices imply no-arbitrage under positive payoffs and shortsale constraints.

Assume if there is an arbitrage opportunity, i.e., we can find $\theta \geq 0$ such that $q'\theta \leq 0$ and $A\theta > 0$. Then θ must be less than or equal to 0, in order to satisfy $q'\theta \leq 0$. This conflicts with $q \gg 0$ (i.e., positive price). So under shortsale constraints, positive prices are arbitrage-free.

4.28

We consider an economy with two states: a boom ($s = 1$) and a recession ($s = 2$). The probability of a boom, denoted by α, is commonly known to the agents. We assume $\alpha = \frac{5}{7}$. There are two assets, a bond ($k = 1$) and a stock ($k = 2$), with payoff matrix

$$A = \begin{pmatrix} \frac{1}{2} & 2 \\ 1 & 0 \end{pmatrix}.$$

The agents have logarithmic expected utility functions given by

$$U^i\left(c_1^i, c_2^i\right) := \alpha \log\left(c_1^i\right) + (1-\alpha)\log\left(c_2^i\right),$$

where c_s^i denotes consumption of agent $i \in \{1,2\}$ in state $s \in \{1,2\}$. The first (second) agent owns one unit of the first (second) asset, i.e. $\bar{\theta}^1 = (1,0)'$, $w^1 = \left(w_1^1, w_2^1\right)' = \left(\frac{1}{2}, 1\right)'$ and $\bar{\theta}^2 = (0,1)'$, $w^2 = \left(w_1^2, w_2^2\right)' = (2,0)'$. There are no other endowments.

(a) Since A is a square matrix the financial market is complete if and only if the determinant of A is not equal to zero. We have

$$\det(A) = -2 \neq 0.$$

Hence, the financial market is complete.

(b) We know form a previous exercise session that in equilibrium

$$\frac{\pi_2}{\pi_1} = \frac{(1-\alpha)\, w_1^1 + (1-\alpha)\, w_1^2}{\alpha w_2^1 + \alpha w_2^2} = \frac{(1-\alpha)\left(w_1^1 + w_1^2\right)}{\alpha\left(w_2^1 + w_2^2\right)}$$

and

$$c_1^i = \frac{\alpha\left(\pi_1 w_1^i + \pi_2 w_2^i\right)}{\pi_1}$$
$$c_2^i = \frac{(1-\alpha)\left(\pi_1 w_1^i + \pi_2 w_2^i\right)}{\pi_2}$$

hold. Plugging in the values for α and w_s^i, $i, s \in \{1,2\}$, we find $\frac{\pi_2}{\pi_1} = 1$ and

$$c^1 = \left(c_1^1, c_2^1\right)' = \left(\frac{15}{14}, \frac{3}{7}\right)', \quad c^2 = \left(c_1^2, c_2^2\right)' = \left(\frac{10}{7}, \frac{4}{7}\right)'.$$

Note that the market clearing conditions $c_1^1 + c_1^2 = w_1^1 + w_1^2$ and $c_2^1 + c_2^2 = w_2^1 + w_2^2$ are satisfied.

(c) We know that in equilibrium

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = A' \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}$$

holds. From this we can derive an expression for $\frac{q_2}{q_1}$. Using the result $\frac{\pi_2}{\pi_1} = 1$ from part (a) we find $\frac{q_2}{q_1} = \frac{4}{3}$. Let $i \in \{1,2\}$. In equilibrium, net asset demand $\theta^i = \left(\theta_1^i, \theta_2^i\right)'$ is implicitly given by the budget constraint $c^i = w^i + A\theta^i$. Hence, we have $\theta^i = A^{-1}\left(c^i - w^i\right)$. Using

$$A^{-1} = -\frac{1}{2}\begin{pmatrix} 0 & -2 \\ -1 & \frac{1}{2} \end{pmatrix}$$

we find

$$\theta^2 = -\theta^1 = \frac{1}{7}(4, -3)'.$$

In terms of asset demand $\hat{\theta}^i := \bar{\theta}^i + \theta^i$ we have

$$\hat{\theta}^1 = \frac{3}{7}(1,1)' \quad \text{and} \quad \hat{\theta}^2 = \frac{4}{7}(1,1)'.$$

(d) Recall that

$$\lambda_k^i := \frac{q_k \hat{\theta}_k^i}{\pi' w^i}, \quad i \in \{1,2\},$$

and

$$R := A\begin{pmatrix} \frac{1}{q_1} & 0 \\ 0 & \frac{1}{q_2} \end{pmatrix}.$$

Assuming

$$\pi_1 = \pi_2 = 1, \qquad q_1 = \frac{3}{2}, \quad \text{and} \quad q_2 = 2$$

we find

$$\lambda^1 = \lambda^2 = \frac{1}{7}(3,4)'.$$

and

$$R = \begin{pmatrix} \frac{1}{3} & 1 \\ \frac{2}{3} & 0 \end{pmatrix}.$$

(e) The 2×2 matrix of factor loadings β is implicitly defined by $R = F\beta$. Hence, we find

$$\beta = F^{-1}R = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{5}{6} & \frac{1}{2} \end{pmatrix}.$$

(f) The factor prices q^F are

$$q^F = F'\pi = \begin{pmatrix} -0.5 \\ 1.5 \end{pmatrix}$$

The payoff of the portfolio built by the factors should have the same payoffs as the the portfolio $\hat{\theta}^i$. Therefore we have for the factor allocation θ_F^i

$$A\hat{\theta}^i = F\theta_F^i$$

From that we get

$$\theta_F^i = F^{-1}A\hat{\theta}^i = \frac{1}{4}\begin{pmatrix} 3 & -4 \\ 5 & 4 \end{pmatrix}\hat{\theta}^i$$

From that the factor allocation for the investors can be determined

$$\theta_F^1 = \frac{1}{28}\begin{pmatrix}-3\\27\end{pmatrix} \qquad \theta_F^2 = \frac{1}{7}\begin{pmatrix}-1\\9\end{pmatrix}$$

4.29

We are given the same utility function across agents i:

$$U^1(c_0^i, c_1^1) = \ln(c_0^i) + \frac{1}{1+\delta}\ln(c_1^i), \quad i = 1, ..., I$$

where $w_1^i = (1+g)w_0^i$, g being the growth factor.
Determine the real interest rate $r = r(\delta, g)$

We renormalize the utility as follows (mind that multiplying a utility function with a positive scalar does not alter the preference relation, and, hence, yields the same optimal choices):

$$V^i \equiv \frac{1+\delta}{2+\delta}U^i = \frac{1+\delta}{2+\delta}\ln(c_0^i) + \frac{1}{2+\delta}c_1^i$$

The new weights sum up to one. We apply the standard formula with $\alpha \equiv \frac{1+\delta}{2+\delta}$

$$\frac{\pi_1}{\pi_0} = \frac{\sum_i(1-\alpha)w_0^i}{\sum_i \alpha w_1^i} = \frac{(1-\alpha)\sum_i w_0^i}{\alpha(1+g)\sum_i w_0^i} = \frac{1}{(1+\delta)(1+g)}$$

As before, we are concerned with the ratio of state prices, and not with particular absolute values, so we can normalize π_0 to be one.

Next, we apply the No Arbitrage argument in this environment without uncertainty:

$$0 = \begin{pmatrix}-1 & 1+r\end{pmatrix}\begin{pmatrix}\pi_0\\ \pi_1\end{pmatrix}$$
$$0 = -\pi_0 + \pi_1(1+r)$$
$$r = \delta g + \delta + g$$

4.30

(a) By no arbitrage (and the FTAP) we have

$$R^f = \pi^* u + (1-\pi^*)d$$
$$\pi^* = \frac{R^f - d}{u-d}$$

(b) The utility maximization problem of the representative agent is:

$$\max_{c_u, c_d} p\log(c_u) + (1-p)\log(c_d)$$
$$\text{s.t. } \pi^* c_u + (1-\pi^*)c_d = \pi^* u + (1-\pi^*)d$$

The Market clearing conditions are

$$c_u = u \qquad\qquad c_d = d$$

The Lagrange function is then

$$L = p\log(c_u) + (1-p)\log(c_d) - \lambda\left(\pi^* c_u + (1-\pi^*)c_d - (\pi^* u + (1-\pi^*)d)\right)$$

The first order conditions are then:

$$\frac{\partial L}{\partial c_u} = \frac{p}{c_u} - \lambda\pi^* = 0$$
$$\frac{\partial L}{\partial c_d} = \frac{1-p}{c_d} - \lambda(1-\pi^*) = 0$$
$$\frac{1-p}{c_d\pi^*} = \lambda = \frac{1-p}{c_d(1-\pi^*)}$$
$$\pi^* = \frac{pc_d}{(1-p)c_u + pc_d}$$

By market clearing we obtain

$$\pi^* = \frac{pd}{(1-p)u + pd}$$

(c) Equate π^* from the last two parts and we obtain:

$$\frac{R^f - d}{u-d} = \pi^* = \frac{pd}{(1-p)u + pd}$$
$$R^f = \frac{ud}{(1-p)u + pd}$$

(d) This problem can be solved analogously to the last three parts. The results are:

$$\pi^* = \frac{pu^{-\gamma}}{(1-p)u^{-\gamma} + pd^{-\gamma}}$$
$$R^f = \frac{pu^{1-\gamma} + (1-p)d^{1-\gamma}}{(1-p)u^{-\gamma} + pd^{-\gamma}}$$

Note that the solution of the previous three parts are just a special case of this solution (i. e. $\gamma = 1$).

4.31

The optimization problem of the representative agent is:

$$\max_{c_0, c_s} u(c_0) + \delta \sum_s p_s u(c_s)$$
$$\text{s.t. } w_0 + \frac{1}{1+r_f} \sum_s \pi_s^* w_s = c_0 + \frac{1}{1+r_f} \sum_s \pi_s^* c_s$$

Solve the constraint for c_s plug it into the maximization problem and use $\pi_s^* = p_s l_s$:

$$\max_{c_s} u\left(w_0 + \frac{1}{1+r_f} \sum_s p_s l_s (w_s - c_s) \right) + \delta \sum_s p_s u(c_s)$$

The first order conditions are:

$$\frac{\partial U}{\partial c_s} = u'(c_0)\left(-\frac{p_s l_s}{1+r_f} \right) + \delta p_s u'(c_s) = 0$$

For l_s we obtain:

$$l_s = \frac{u'(c_s)}{u'(c_0)} \delta (1+r_f)$$

(a) For $\eta = \delta \frac{1+r_f}{u'(c_0)}$ we have:

$$l_s = \eta(1 - \gamma c_s)$$

(b) For $\eta = \delta \frac{1+r_f}{u'(c_0)}$ we have:

$$l_s = \eta c_s^{-\rho}$$

(c) For $\eta = \delta \frac{1+r_f}{u'(c_0)}$ we have:

$$l_s = \eta \alpha e^{-\alpha c_s}$$

(d) For $\eta = \delta \frac{1+r_f}{u'(c_0)}$ we have:

$$l_s = \eta \begin{cases} \alpha^+ (c - RP)^{\alpha^+ - 1} & \text{if } c > RP \\ \beta \alpha^- (RP - c)^{\alpha^- - 1} & \text{if } c \leq RP \end{cases}$$

(e) For $\eta = \delta \frac{1+r_f}{u'(c_0)}$ we have:

$$l_s = \eta \begin{cases} 1 - 2\alpha^+ (c - RP) & \text{if } c > RP \\ \beta (1 - 2\alpha^- (RP - c)) & \text{if } c \leq RP \end{cases}$$

4.32

From the derivation of the CAPM (in a previous exercise session) we know

$$l_s = \frac{1 - \gamma c_s}{1 - \gamma \mathbb{E}(c)} = \frac{1}{1 - \gamma \mathbb{E}(R^M)} - \frac{\gamma}{1 - \gamma \mathbb{E}(R^M)} R_s^M$$

and

$$\frac{\gamma}{1 - \gamma \mathbb{E}(R^M)} = \frac{\mathbb{E}\left(R^M - R^f\right)}{\mathrm{Var}(R^M)}$$

Plug this into l_s, take the expectation of both sides of the equation and use $\mathbb{E}(l) = 1$, we obtain

$$\begin{aligned} 1 = \mathbb{E}(l) &= \frac{1}{1 - \gamma \mathbb{E}(R^M)} - \frac{\mathbb{E}\left(R^M - R^f\right)}{\mathrm{Var}(R^M)} \mathbb{E}(R^M) \\ \frac{1}{1 - \gamma \mathbb{E}(R^M)} &= 1 + \frac{\mathbb{E}\left(R^M - R^f\right)}{\mathrm{Var}(R^M)} \mathbb{E}(R^M) \\ l_s &= 1 + \frac{\mathbb{E}\left(R^M - R^f\right)}{\mathrm{Var}(R^M)} \mathbb{E}(R^M) - \frac{\mathbb{E}\left(R^M - R^f\right)}{\mathrm{Var}(R^M)} R_s^M \\ &= a - b R_s^M \end{aligned}$$

with

$$\begin{aligned} a &= 1 + \frac{\mathbb{E}\left(R^M - R^f\right)}{\mathrm{Var}(R^M)} \mathbb{E}(R^M) \\ b &= \frac{\mathbb{E}\left(R^M - R^f\right)}{\mathrm{Var}(R^M)} \end{aligned}$$

Since all states are equally likely we have $p_s = p = 1/S$. Therefore it is obtained:

$$\pi_s^* = l_s p_s = \frac{a - b R_s^M}{S} =: \pi_s^{CAPM}$$

We should find now a $\overline{\pi}_s^*$ which minimizes the distance to the CAPM and for that no arbitrage holds. I.e.

$$\begin{aligned}
&\min_{\overline{\pi}_s^*} = \sum_s (\overline{\pi}_s^* - \pi_s^{CAPM})^2 \\
&\text{s.t. } \sum_s \overline{\pi}_s^* = 1 && \overline{\pi}_s^* > 0 \text{ for all } s \\
&\sum_s \overline{\pi}_s^* (R_s^k - R_f) = 0 && \text{for all } k
\end{aligned}$$

A numerical solution of that problem with Excel can be found on the website.

4.33

(a) Look at the Arbitrage Pricing Model:

$$R_t^{e,k} = \beta_k'(f_t - \mathbb{E}(f)) + \epsilon_t^k$$

where ϵ_t^k is white noise. It can be easily checked that $\beta^k = \operatorname{Var}(f)^{-1}\operatorname{cov}(f, R^{e,k})$ is just the OLS estimator of β_k of the APT.

(b) Using $R^f = \mathbb{E}(R^k) + \operatorname{cov}(l, R^k)$, $l = 1 + b'(f - \mathbb{E}(f))$ and $R^{e,k} = R^k - R_f$ it can be written:

$$\begin{aligned}
\mathbb{E}(R^{e,k}) &= -\operatorname{cov}(1 + b'(f - \mathbb{E}(f)), R^{e,k}) = -b'\operatorname{cov}(f, R^{e,k}) \\
&= -b\operatorname{Var}(f)\operatorname{Var}(f)^{-1}\operatorname{cov}(f, R^{e,k}) \\
&= \lambda'\beta
\end{aligned}$$

(c) The estimation can be done as follows:

1. β_k can be estimated be OLS from the following equation:

$$R_t^{e,k} = \beta_k'(f_t - \mathbb{E}(f)) + \epsilon_t^k$$

where the average is taken for $\mathbb{E}(f)$.

2. λ can be estimated by OLS out of

$$\mathbb{E}\left(R_t^{e,k}\right) = \lambda'\beta_k + \eta^k$$

with η^k white noise. For $\mathbb{E}\left(R_t^{e,k}\right)$ the average and for β_k the estimator from the last step is taken.

3. Since $\lambda = -\operatorname{Var}(f)b$, b is obtained via

$$b = -\operatorname{Var}(f)^{-1}\lambda$$

where the empirical covariance matrix is used for $\operatorname{Var}(f)$ and the estimated λ from the last step is used.

4.34

(a) A financial markets equilibrium is a list of portfolio strategies $\theta^{i,*}$, $i = 1, \dots, I$ and a price system q^k, $k = 1, \dots, K$ such that for all $i = 1, \dots, I$

$$\begin{aligned} \theta^{i,*} &= \arg\max_{\theta^i} U^i(c^i) \\ \text{s.t. } q'\theta^i &\leq 0 \\ c^i &\leq w^i + A\theta^i \end{aligned}$$

and markets clear

$$0 = \sum_i \theta^i \quad \text{and} \sum_i c_s^i = \sum_i w_s^i$$

(b) It has to be checked that the investors have optimized their portfolios and the market clearing is satisfied. The market clearing condition can be easily checked by plugging into $\sum_i \theta^{i,*} = 0$.
By substituting the endowments of investor 1 and payoffs of the assets(i.e. $\mathbf{c}^1 = \mathbf{w}^1 + A\theta^1$) , we obtain the max problem and BC of the investor 1:

$$\max_{\theta_1^1, \theta_2^1} U^1 = ln(c_1^1) + ln(c_2^1)$$

$$\begin{aligned} \text{State 0} &: \; q_1\theta_1^1 + q_2\theta_2^1 \leq 0 \\ \text{State 1} &: \; c_1^1 \leq \theta_1^1, \quad c_1^1 \geq 0 \\ \text{State 2} &: \; c_2^1 \leq \theta_2^1 + 1, \quad c_2^1 \geq 0 \\ \text{State 3} &: \; c_3^1 \leq \theta_2^1 + 2, \quad c_3^1 \geq 0 \end{aligned}$$

The maximization problem of investor 1 becomes then:

$$\begin{aligned} \max_{\theta^1} \ln(c_1^1) + \ln(c_2^1) &= \max_{\theta^1} \ln(\theta_1^1) + \ln(\theta_2^1 + 1) \\ \text{s.t. } q'\theta^1 &\leq 0 \text{ and } \mathbf{c}^1 \geq 0 \end{aligned}$$

The Lagrange function of this problem is

$$L = \ln(\theta_1^1) + \ln(\theta_2^1) - \lambda \left(q_1\theta_1^1 + q_2\theta_2^1 \right)$$

The first order conditions are then

$$\frac{\partial L}{\partial \theta_1^1} = \frac{1}{\theta_1^1} - \lambda q_1 = 0$$
$$\frac{\partial L}{\partial \theta_2^1} = \frac{1}{\theta_2^1 + 1} - \lambda q_2 = 0$$

From that it is obtained:

$$\frac{q_1}{q_2} = \frac{\theta_2^1 + 1}{\theta_1^1} \quad \text{and} \quad \frac{q_1}{q_2} = \frac{-\theta_2^1}{\theta_1^1}$$

Combining the two gives us

$$\theta_2^1 = -\frac{1}{2} \quad \text{and} \quad \theta_1^1 = \frac{q_2}{2q_1}$$

By plugging in the values it can be verified that this equation is satisfied. From that it can be concluded that investor 1 maximizes his utility. For investor 2 this has also to be verified. The maximization problem of investor 2 can be derived in the same way as in the case of investor 1:

$$\max_{\theta_1^2, \theta_2^2} U^2 = ln(c_2^2) + ln(c_3^2)$$

$$\begin{aligned} \text{State 0} &: \ q_1\theta_1^2 + q_2\theta_2^2 \leq 0 \\ \text{State 1} &: \ c_1^2 \leq \theta_1^2 + 2, \quad c_1^2 \geq 0 \\ \text{State 2} &: \ c_2^2 \leq \theta_2^2 + 1, \quad c_2^2 \geq 0 \\ \text{State 3} &: \ c_3^2 \leq \theta_2^2, \quad c_3^2 \geq 0 \end{aligned}$$

$$\max_{\theta^2} \ln(c_1^2) + \ln(c_2^2) = \max_{\theta^2} \ln(\theta_2^2 + 1) + \ln(\theta_2^2)$$
$$\text{s.t. } q'\theta^2 \leq 0 \text{ and } \mathbf{c}^2 \geq 0$$

Here we know already from market clearing $\theta_2^2 = 1/2$ as $\theta_2^1 + \theta_2^2 = 0$ must hold at equilibrium.

Investing in asset 1 brings no utility to agent 2. Therefore he invests as few as possible in asset 1. Since c_2^1 must be at least zero, therefore $\theta_1^2 = -2$.

In this case $c_2^1 = 0$. In other words, as Investor 2 does not care about c_1^2, he is indifferent between giving it and keeping it. However, Investor 1 cares about how much he consumes in state 1 so he will push it to reach the (herr Constrained Pareto) efficient allocation. A (constrained) Pareto efficient allocation is an allocation that there is not other allocation (within the market restrictions) that at least one investor con do better without making at least some worse. Here, Investor 1 could do better by increasing her consumption in state 1 without making the Investor 2 worse as he does not care about it, utility U^2 does not depend on it(increasing in consumption in state 1). Thus, we can see if the Investor 2 sell 1 of asset 1 he will have the same utility as he sells 2 of asset 1, and as Investor 1 will increase if he gets 2 of asset 1 compared to getting 1 of asset 1, at equilibrium he will get 2 of asset 1. $\theta_1^2 = -2$, and market clearing implies $\theta_1^1 = 2$.

Because of $q'\theta^2 = 0$, $\theta_1^2 = -2$ and $\theta_2^2 = 1/2$

$$\frac{-theta_2^2}{\theta_1^2} = \frac{q_1}{q_2} = \frac{-\theta_2^1}{\theta_1^1} = \frac{1}{4}$$

By normalizing the first price to 1, we reach the prices $\mathbf{q} = (1,4)'$. Therefore c^{*1}, c^{*2}, $\theta^{1,*}$, $\theta^{2,*}$ and q^* are indeed an equilibrium.

(c) If the matrix $[A, A_3]$ has full rank, then asset 3 cannot be replicated by the other assets. Since $\det([A, A_3]) = 1$ the matrix has full rank and asset 3 can indeed not be replicated by the other assets.

(d) Since there is no arbitrage, strictly positive state prices must exist (FTAP) such that:

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} = q = \pi A = \begin{pmatrix} \pi_1 \\ \pi_2 + \pi_3 \end{pmatrix}$$

The state prices are then $\pi_1 = 1$ and $\pi_2 = 4 - \pi_3$ with $\pi_3 \in (0,4)$. The last restriction is due to the fact that otherwise some state prices would not be positive anymore.

Since $q_3 = A_3'\pi = \pi_3$, we know that $q_3 \in (0,4)$ and is therefore not uniquely determined.

(e) Since the market is complete, any allocation of the endowments is feasible. Therefore the equilibrium can be restated into the following form:

$$\begin{aligned} &\max_{c^i} U^i(c^i) \\ &\quad \text{s.t } \pi' c^i = \pi' w^i \\ &\quad\quad c^i \geq 0 \end{aligned}$$

and the market clears

$$\sum_i c^i = \sum_i w^i$$

The Lagrange function of the maximization problem of the first investor is

$$L = \ln(c_1^1) + \ln(c_2^1) - \lambda(\sum_s \pi_s c_s^1 - \sum_s \pi_s w_s^1)$$

The first order conditions of this problem are:

$$\frac{\partial L}{\partial c_1^1} = \frac{1}{c_1^1} - \lambda\pi_1 = 0$$
$$\frac{\partial L}{\partial c_2^1} = \frac{1}{c_2^1} - \lambda\pi_2 = 0$$

Taking $\lambda\pi_i$ on one side and divide both equations results in

$$\frac{\pi_1}{\pi_2} = \frac{c_2^1}{c_1^1}$$

Since c_3^1 has no influence to the utility, it is chosen as small as possible: i.e. $c_3^1 = 0$. By market clearing we know that $c_3^2 = w_3^1 + w_3^2 - c_3^1 = 2$. Thus we have

$$c_3^1 = 0, \quad c_2^1 = c_1^1 \frac{\pi_1}{\pi_2}$$

By analogous methods it results for agent 2:

$$\frac{\pi_2}{\pi_3} = \frac{c_3^2}{c_2^2} \qquad\qquad c_1^2 = 0$$

From market clearing we have also $c_3^2 = 2$ and $c_1^1 = 2$.
From the market clearing for the second state we have

$$c_2^1 + c_2^2 = w_2^1 + w_2^2 = 2 = 2\frac{\pi_1}{\pi_2} = 2\frac{\pi_3}{\pi_2}$$

This leads to

$$\pi_1 + \pi_3 = \pi_2$$

Rewriting the BC of one of the investors(the other will yield the same result):

$$\pi_1(c_1^1 - 0) + \pi_2(\frac{2\pi_1}{\pi_2} - 1) + \pi_3(c_3^1 - 2) = 0$$

which yields a second equation for state prices

$$4\pi_1 - 2\pi_3 = \pi_2$$

Making use of the two equations for state prices we have

$$\pi_2 = \pi_1 + \pi_3 = 4\pi_1 - 2\pi_3$$

,
we find $\pi_1 = \pi_3$.
From that we know the state prices $\pi = (1, 2, 1)'$ by normalizing the first state price to 1. The consumption can be calculated to portfolios via $c^i = A\theta^i$ and the prices are $q = \pi' A$. For the equilibrium we obtain then

$$\theta^1 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \qquad \theta^2 = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} \qquad q = \frac{1}{2}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

(e) The equilibrium of the representative investor is given by θ^*, q such that

$$\begin{aligned} \theta^* &= \arg\max_{\theta} \frac{1}{5}\ln(c_1) + \frac{1}{5}\ln(c_2) + \frac{3}{5}\ln(c_3) \\ \text{s.t. } & A\theta + \mathbf{w} = \mathbf{c} \\ & \mathbf{q}\theta \leq 0 \\ & \mathbf{c} \geq 0 \end{aligned}$$

and the market clearing condition

$$\theta = 0$$

From the budget constraint (i.e. from $A\theta + w = c$) we obtain

$$c_1 = 2 + \theta_1 \qquad c_2 = c_3 = 2 + \theta_2$$

The Lagrange function for the representative investor and its first order conditions are then

$$\begin{aligned} L &= \frac{1}{5}\ln(2+\theta_1) + \frac{1}{5}\ln(2+\theta_2) + \frac{3}{5}\ln(2+\theta_2) - \lambda(q'\theta) \\ \frac{\partial L}{\partial \theta_1} &= \frac{1}{5}\frac{1}{2+\theta_1} - \lambda q_1 = 0 \\ \frac{\partial L}{\partial \theta_2} &= \frac{1}{5}\frac{1}{2+\theta_2} + \frac{3}{5}\frac{1}{2+\theta_2} - \lambda q_2 = 0 \end{aligned}$$

From that and by market clearing (i.e. $\theta = 0$)

$$\frac{q_1}{q_2} = \frac{\frac{1}{5}\frac{1}{2+\theta_1}}{\frac{4}{5}\frac{1}{2+\theta_2}} = \frac{1}{4}$$

Norming $q_1 = 1$ we get $q = (1, 4)'$. In other words the representative agent is able to replicate the price system of the other two agents.

(f) Now we add the the financial innovation A_3 to the economy of the representative agent. From the new budget constraint (i.e. $[A, A_3]\theta + w = c$)

$$c_1 = 2 + \theta_1 \qquad c_2 = 2 + \theta_2 \qquad c_3 = 2 + \theta_2 + \theta_3$$

The Lagrange function and the FOC are then

$$L = \frac{1}{5}\ln(2+\theta_1) + \frac{1}{5}\ln(2+\theta_2) + \frac{3}{5}\ln(2+\theta_2+\theta_3) - \lambda(q'\theta)$$
$$\frac{\partial L}{\partial \theta_1} = \frac{1}{5}\frac{1}{2+\theta_1} - \lambda q_1 = 0$$
$$\frac{\partial L}{\partial \theta_2} = \frac{1}{5}\frac{1}{2+\theta_2} + \frac{3}{5}\frac{1}{2+\theta_2+\theta_3} - \lambda q_2 = 0$$

$$\frac{q_1}{q_2} = \frac{\frac{1}{5\cdot 2}}{\frac{4}{5\cdot 2}} = \frac{1}{4}$$
$$\frac{q_1}{q_3} = \frac{\frac{1}{5\cdot 2}}{\frac{3}{5\cdot 2}} = \frac{1}{3}$$

By market clearing (i.e. $\theta = 0$) and Norming $q_1 = 1$ results in $q = (1, 4, 3)'$.

Equivalently, and easier as the financial markets are complete now we can use AD BC!

$$maxU = \frac{1}{5}\ln(c_1) + \frac{1}{5}\ln(c_2) + \frac{3}{5}\ln(c_2)$$
$$s.t$$
$$\sum_s \pi_s c_s \leq \sum_s \pi_s w_s$$

FOC's yield

$$\frac{\pi_1}{\pi_2} = \frac{5c_2}{5c_1} \quad \text{and} \quad \frac{\pi_2}{\pi_3} = \frac{5c_3}{15c_2}$$

Normalizing $\pi_1 = 1$ leads to $\pi^R = (113)$ and this leads to asset prices $\mathbf{q}'^R = (143)$ which was $\mathbf{q}'^{Eq} = (131)$

This price system is for sure different to the price system in the equilibrium with the two agents in part (e). In other words if you change the market structure (but the endowment remains constant), a representative agent stops doing his job.

4.35

(a) The economy is the usual exchange economy with logarithmic preferences. **Short skecth of derivation of the standard formula for log utility (Cobb Douglas)**:

First, note that maximizing Cobb- Douglas $u = x_1^{\alpha_1} x_2^{\alpha_2}$ is equivalent to maximizing monotone transformation of Cobb-Douglas. Thus maximizing $f(u) = ln(u)$ will give the same allocation $ln(u) = \alpha_1 ln(x_1) + \alpha_2 ln(x_2)$. Take the general case with these type of utilities under complete markets:

$$\max_{\mathbf{c}^i} U^i = \sum_{s=1}^{S} \alpha_s^i ln(c_s^i)$$

s.t.

$$\sum_{s=1}^{S} \pi_s c_s^i \leq \sum_{s=1}^{S} \pi_s w_s^i$$

and markets clear

$$\sum_{i=1}^{I} c_s^i = \sum_{i=1}^{I} w_s^i \quad \forall s = 1, ..., S.$$

The Lagrangian then is

$$L = \sum_{s=1}^{S} \alpha_s^i ln(c_s^i) - \lambda^i \left(\sum_{s=1}^{S} \pi_s (c_s^i - w_s^i) \right)$$

FOC yields $\forall s = 1, ..., S$

$$c_s^i = \frac{\alpha_s^i}{\pi_s \lambda^i}$$

and substituting this into the budget constraint:

$$\sum_{s=1}^{S} \pi_s \frac{\alpha_s^i}{\pi_s \lambda^i} = \sum_{s=1}^{S} \pi_s w_s^i$$

this leads

$$\frac{1}{\lambda^i}\sum_{s=1}^{S}\alpha_s^i = \sum_{s=1}^{S}\pi_s w_s^i$$

Here let us denotes $\alpha_A^i := \sum_{s=1}^{S}\alpha_s^i$. The we have an expression for the lagrange multiplier:

$$\lambda^i = \frac{\alpha_A^i}{\sum_{s=1}^{S}\pi_s w_s^i}$$

and substituting this into consumption allocation

$$c_s^i = \frac{\alpha_s^i}{\pi_s\lambda^i} = \frac{\alpha_s^i\left(\sum_{s=1}^{S}\pi_s w_s^i\right)}{\alpha_A^i\pi_s}$$

Market clearing we can apply we sum across investors $\forall s = 1, ..., S$

$$\sum_{i=1}^{I} c_s^i = \sum_{i=1}^{I}\frac{\alpha_s^i\left(\sum_{s=1}^{S}\pi_s w_s^i\right)}{\alpha_A^i\pi_s} = \sum_{i=1}^{I} w_s^i$$

Rewriting it:

$$\frac{1}{\pi_s}\sum_{i=1}^{I}\frac{\alpha_s^i\left(\sum_{s=1}^{S}\pi_s w_s^i\right)}{\alpha_A^i} = \sum_{i=1}^{I} w_s^i$$

we see that we found and expression for π_s that does not depend on consumption allocation but depends on only endowments and weights for states(α_s).

$$\pi_s = \frac{1}{\sum_{i=1}^{I} w_s^i}\sum_{i=1}^{I}\frac{\alpha_s^i}{\alpha_A^i}\left(\sum_{s=1}^{S}\pi_s w_s^i\right)$$

Taking the ratio of the two state prices will give:

$$\frac{\pi_{s_1}}{\pi_{s_2}} = \frac{\sum_{i=1}^{I} w_{s_2}^i}{\sum_{i=1}^{I} w_{s_1}^i}\frac{\sum_{i=1}^{I}\frac{\alpha_{s_1}^i}{\alpha_A^i}\left(\sum_{s=1}^{S}\pi_s w_s^i\right)}{\sum_{i=1}^{I}\frac{\alpha_{s_2}^i}{\alpha_A^i}\left(\sum_{s=1}^{S}\pi_s w_s^i\right)}$$

Now we can apply this standard formula for our problem with

$$\alpha^1 = 0.75 \quad \text{and} \quad \alpha^2 = 0.25$$

Inv 1 owns the first asset so she gets the endowment of payoff :

$$w_1^1 = w_2^1 = 1$$

Inv 2 owns the second asset so he gets the endowment of payoff:

$$w_1^2 = 2 \quad \text{and} \quad w_2^2 = 0.5$$

$$\frac{\pi_1}{\pi_2} = \frac{\sum_{i=1}^{2} \alpha^i w_2^i}{\sum_{i=1}^{2} (1-\alpha^i) w_1^i} = \frac{1}{2}$$

By norming $\pi_1 = 1$ we get

$$\pi = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Furthermore we know that in this kind of problem

$$c_s^i = \frac{\alpha_s^i}{\pi_s}(\pi_1 w_1^i + \pi_2 w_2^i)$$

For the equilibrium it results that

$$\pi = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad c^1 = \frac{1}{8}\begin{pmatrix} 18 \\ 3 \end{pmatrix} \qquad c^2 = \frac{1}{8}\begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

For the asset prices it results

$$q = A'\pi = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

(b) We are looking for the parameter γ such that the representative investor is able to replicate the asset prices from (a). The utility maximization problem of the representative agent is

$$\max_c \gamma \ln(c_1) + (1-\gamma)\ln(c_2)$$
$$\text{s.t. } \pi' c = \pi' w,$$

the market clearing condition is $c = w$ and $w = (3, 1.5)'$. The Lagrange function of this problem is

$$L = \gamma \ln(c_1) + (1-\gamma)\ln(c_2) - \lambda(\pi' c - \pi' w)$$

The first order conditions are then

$$\frac{\partial L}{\partial c_1} = \frac{\gamma}{c_1} - \lambda \pi_1 = 0$$
$$\frac{\partial L}{\partial c_2} = \frac{1-\gamma}{c_2} - \lambda \pi_2 = 0$$

Solving these equations for π_1/π_2 and then using the market clearing condition (i.e. $c = w$) it results

$$\frac{\pi_1}{\pi_2} = \frac{c_1\gamma}{c_2(1-\gamma)} = \frac{w_1\gamma}{w_2(1-\gamma)} = \frac{\gamma}{2(1-\gamma)}$$

From (a) we know that $\pi_1/\pi_2 = 0.5$. The representative investor generates the same q as the two investors, if this equation is satisfied. The remaining task is to find γ such that this condition is satisfied

$$\frac{1}{2} = \frac{\pi_1}{\pi_2} = \frac{\gamma}{2(1-\gamma)}$$

From that we get $\gamma = 0.5$.

(c) The new aggregated endowment is now $w = (4, 1.5)'$ and $\gamma = 0.5$. Using the results from (b), we can write

$$\frac{\pi_1}{\pi_2} = \frac{w_1\gamma}{w_2(1-\gamma)} = \frac{3}{8}$$

To make asset prices comparable the length of π is normed to 1. For π and q we get

$$\pi = \frac{1}{\sqrt{3^2+8^2}}\begin{pmatrix}3\\8\end{pmatrix} \qquad q = \pi' A = \frac{1}{\sqrt{73}}\begin{pmatrix}11\\13\end{pmatrix} = \begin{pmatrix}1.29\\1.52\end{pmatrix}$$

(d) By the standard result we get with the new endowment

$$\frac{\pi_1}{\pi_2} = \frac{\sum_{i=1}^{2}\alpha^i w_2^i}{\sum_{i=1}^{2}(1-\alpha^i)w_1^i} = \frac{7}{20}$$

Norm the length of π to 1, then we get

$$\pi = \frac{1}{\sqrt{449}}\begin{pmatrix}7\\20\end{pmatrix} \qquad q = A'\pi = \frac{1}{\sqrt{449}}\begin{pmatrix}27\\31\end{pmatrix} = \begin{pmatrix}1.27\\1.46\end{pmatrix}$$

If endowments change, the *old* representative agents suggests inaccurate prices, so we need a new representative agent with new weights.

4.36

The given prices are equilibrium prices, if the agents maximize their utility under the budget constraint (given these prices) and the market clearing condition must be satisfied. The optimization problem of the agents is:

$$\max_{x^i} U^i(x^i)$$
$$\text{s.t. } p'w^i = p'x^i$$

and the market clearing condition is $x^1 + x^2 = w^1 + w^2$. The Lagrange function of agent 1 is:

$$L^1 = -\frac{1}{2}\left(\frac{1}{\left(x_1^1\right)^2} + \frac{12^3}{37^3\left(x_2^1\right)^2}\right) - \lambda(p_1 x_1^1 + p_2 x_2^1 - p_1 \cdot 1)$$

The first order conditions are:

$$\frac{\partial L^1}{\partial x_1^1} = -0.5\frac{-2}{\left(x_1^1\right)^3} - \lambda p_1 = 0$$
$$\frac{\partial L^1}{\partial x_2^1} = -0.5\frac{-2 \cdot 12^3}{\left(37 x_2^1\right)^3} - \lambda p_2 = 0$$

Dividing the two budget constraints, norming $p_2 = 1$ and defining $q = p_1^{\frac{1}{3}}$, we obtain:

$$x_2^1 = \frac{12}{37} q x_1^1$$

Plug that into the budget constraint $p_1 x_1^1 + 1 \cdot x_2^1 = 1 \cdot p_1$ it results:

$$x_1^1 = \frac{37q^2}{37q^2 + 12} \qquad x_2^1 = \frac{12q^3}{37q^2 + 12}$$

Analogously it results for the second agent:

$$x_1^2 = \frac{12}{12q^3 + 37} \qquad x_2^2 = \frac{37}{12q^2 + 37}$$

Plug that into the budget constraint for state 1 (i.e. in $x_1^1 + x_1^2 = 1$):

$$1 = \frac{37q^2}{37q^2 + 12} + \frac{12}{12q^3 + 37}$$
$$0 = 12^2 q^3 - 12 \cdot 37 q^2 + 37 \cdot 12 q - 12^2$$

Numerically solving this polynom results in

$$q^* = 1 \qquad \overline{q} = \frac{3}{4} \qquad \widetilde{q} = \frac{4}{3}$$

By plugging in or by Walras Law also the second market clearing condition is satisfied.

The equilibria of the given economy are

$$\begin{aligned} x^{1,*} &= \frac{1}{49}\begin{pmatrix}37\\49\end{pmatrix} & x^{2,*} &= \frac{1}{49}\begin{pmatrix}12\\37\end{pmatrix} & p^* &= \begin{pmatrix}1\\1\end{pmatrix} \\ \overline{x}^1 &= \frac{1}{175}\begin{pmatrix}111\\27\end{pmatrix} & \overline{x}^2 &= \frac{1}{175}\begin{pmatrix}64\\148\end{pmatrix} & \overline{p} &= \begin{pmatrix}\frac{3^3}{4^3}\\1\end{pmatrix} \\ \widetilde{x}^1 &= \frac{1}{175}\begin{pmatrix}148\\64\end{pmatrix} & \widetilde{x}^2 &= \frac{1}{175}\begin{pmatrix}27\\111\end{pmatrix} & \widetilde{p} &= \begin{pmatrix}\frac{4^3}{3^3}\\1\end{pmatrix} \end{aligned}$$

4.37

(a) Assume that w is your income and the game L pays with 50% $2w$ and with 50% $(1-x)w$. You chose x sucht that you are indifferent between the sewure payment w and L.
The answer depends clearly on your preferences. For this exercise $x = 25\%$ has been chosen.

(b) A utility function with constant relative risk aversion (CRRA) is

$$u(x) = \frac{c^{1-\alpha}}{1-\alpha}$$

Since $\frac{-cu''(c)}{u'(c)} = \alpha$, the CRRA is α. Therefore the α, which makes the investor indifferent between w and L, has to be determined:

$$\begin{aligned} U(w) &= U(L) \\ \frac{w^{1-\alpha}}{1-\alpha} &= 0.5\frac{(2w)^{1-\alpha}}{1-\alpha} + \frac{((1-x)w)^{1-\alpha}}{1-\alpha} \\ 2 &= 2^{1-\alpha} + (1-x)^{1-\alpha} \end{aligned}$$

α has to be determined numerically. For $x = 25\%$ it is $\alpha = 2.91$.

(c) For the mean variance investor the problem must be rewritten in terms of returns. We get an new game L' which pays in 50% of the cases $2w/w - 1 = 1$ and in the other 50% of the cases $(1-x)w/w - 1 = -x$. The secure alternative has an expected value and a standard deviation of zero. Expected value and variance of L' are

$$\mu(L') = \frac{1}{2}(1-x) \qquad \mu\left(L'^2\right) = \frac{1}{2}(1+x^2) \qquad \sigma^2(L') = \frac{1}{4}(1+x)^2$$

The utility function of a mean variance investor is $U(X) = \mu(X) - \gamma/2\sigma^2(X)$. The mean variance investor must be indifferent between L' and the secure alternative. Therefore we get

$$\begin{aligned} U(0) &= U(L') \\ \mu(0) - \frac{\gamma}{2}\sigma^2(0) &= \mu(L') - \frac{\gamma}{2}\sigma^2(L') \\ 0 &= \frac{1}{2}(1-x) - \frac{\gamma}{2}\frac{(1+x)^2}{4} \\ \gamma &= \frac{4(1-x)}{(1+x)^2} = \frac{3 \cdot 4^2}{5^2} = 1.92 \end{aligned}$$

(d) Also for the prospect utility maximizer the returns are used once more. Also here the investor is indifferent between the secure alternative and L':

$$\begin{aligned} U(0) &= U(L') \\ v(0) &= \frac{1}{2}v(1) + \frac{1}{2}v(-x) \\ -\beta(0-0)^{\alpha^-} &= \frac{1}{2}(1-0)^{\alpha^+} + \frac{1}{2}(-\beta)(0+x)^{\alpha^-} \\ 0 &= 1 - \beta x^{\alpha^-} \\ x &= \beta^{\frac{-1}{\alpha^-}} = 0.40 \end{aligned}$$

(e) Let r_s be the return of the stocks, σ_s the standard deviation of the stocks and r_0 the return of the bonds (plus the bond is risk free). Then the maximization problem of the mean variance investor becomes

$$\lambda^* = \arg\max_{\lambda} r_0 + \lambda(r_s - r_0) - \frac{\gamma}{2}\lambda^2\sigma_s^2$$

The FOC of this problem is

$$(r_s - r_0) - \gamma\lambda^*\sigma_s^2 = 0$$

Given is $r_s - r_0 = 6.4\%$, $\sigma_s = 21\%$ and $\lambda^* = 0.5$. The above equation can be solved for γ and it results:

$$\gamma = \frac{r_s - r_0}{\lambda\sigma_s^2} = \frac{0.064 \cdot 2}{0.21^2} = 2.90$$

(f) The CRRA investor must be indifferent between the risky lottery and the secure alternative. In this case some background wealth is added

$$\begin{aligned}
U(w+0.5w) &= U(L+0.5w)\\
\frac{(w+0.5w)^{1-\alpha}}{1-\alpha} &= \frac{1}{2}\frac{(2w+0.5w)^{1-\alpha}}{1-\alpha}+\frac{1}{2}\frac{((1-x)w+0.5w)^{1-\alpha}}{1-\alpha}\\
1.5^{1-\alpha} &= \frac{1}{2}2.5^{1-\alpha}+\frac{1}{2}(1.5-x)^{1-\alpha}
\end{aligned}$$

This equation has to be solved numerically. For $x = 25\%$ we get $\alpha = 4.25$.

4.38

(a) Summing over the budget constraint of all investors we get:

$$\sum_i\left(\sum_k p_k D_{ki}\right) = \sum_i\left(\sum_k p_k S_{ki}\right)$$
$$\sum_k p_k D_k = \sum_k p_k S_k$$

The last equation is called the Walras identity. Rewriting the Walras identity results in:

$$\sum_k p_k\left(D_k - S_k\right) = 0$$
$$\sum_k p_k Z_k = 0$$

(b) Assume that $K-1$ markets are cleared i.e.

$$\begin{array}{cccc}
D_1 = S_1, & D_2 = S_2, & \dots & D_{K-1} = S_{K-1}\\
p_1 D_1 = p_1 S_1, & p_2 D_2 = p_2 S_2, & \dots & p_{K-1} D_{K-1} = p_{K-1} S_{K-1}
\end{array}$$

Add them up

$$\sum_{k=1}^{K-1} p_k D_k = \sum_{k=1}^{K-1} p_k S_k$$

Subtract that from the Walras identity:

$$\sum_{k=1}^{K} p_k D_k - \sum_{k=1}^{K-1} p_k D_k = \sum_{k=1}^{K} p_k S_k - \sum_{k=1}^{K-1} p_k S_k$$
$$p_K D_K = p_K S_K$$
$$D_K = S_K$$

Therefore if $K-1$ markets are cleared also the K-th market is cleared.

4.39

(a) Since the utility functions are strictly increasing, the budget constraints are satisfied with an equal sign. Otherwise the agent would throw away something from which he could gain utility. Writing the budget constraints in vector form :

$$c^{i,*} - w^i = \begin{pmatrix} -q^{*\,\prime} \\ A \end{pmatrix} \theta^{i,*}$$

Summing over all consumer results in

$$\sum_i c^{i,*} - w^i = \begin{pmatrix} -q^{*\,\prime} \\ A \end{pmatrix} \sum_i \theta^{i,*} = 0$$

The last equal sign follows from the market clearing in the asset market i.e. from $\sum_i \theta^i = 0$. Now we get market clearing of the consumption markets:

$$\sum_i c^{i,*} = \sum_i w^i$$

(b) The budget constraint for all states in the second period can be written as

$$c_1^{i,*} - w_1^i = A\theta^{i,*}$$

where are $c_1^{i,*}$ and w_1^i are the consumption vector and the initial endowment without $c^{0,*}$ and w^0. From the fact that there are no redundant assets we know that $\text{rank}(A) = K$. Therefore we know for any $x \in \mathbb{R}^K$:

$$Ax = 0 \Rightarrow x = 0$$

Summing over all budget constraints and using the market clearing condition of the consumption goods imply:

$$\sum_i c_1^{i,*} - w_1^i = A \sum_i \theta^{i,*} = 0$$

$A\left(\sum_i \theta^{i,*}\right) = 0$ and $\operatorname{rank}(A) = K$ implies that $\sum_i \theta^{i,*} = 0$. I.e. the asset market clears.
Furthermore the market clearing condition of $s = 0$ has not been used for that proof. Summing over $c_0^i + q^{*\prime}\theta^i = w_0^i$ and using $\sum_i \theta^{i,*} = 0$ implies:

$$\sum_i w_0^i = \sum_i c_0^i + q^{*\prime} \sum_i \theta^i = \sum_i c_0^i$$

Which is just the market clearing condition of the first period. In other words only the market clearing conditions of the markets of the consumption good in the second period need to be checked. The rest happens automatically.

5

Multiple-Periods Model

5.5

(a) The asset price in the second period is zero. This is the case, since the economy ends in $t = 3$ and therefore a stock bought at $t = 3$ does not pay out anything anymore.

(b) Define the bond as asset 0 and the stock as asset 1. The price of asset k in period t in state s is $q_{t,s}^{k}$. The amount of asset k the representative investor holds in t, s is $\theta_{t,s}^{k}$. The dividend asset k pays in t, s is $D_{t,s}^{k}$. The maximization problem of the representative agent is then:

$$\begin{aligned}
\max_{\theta}\ & \ln(c_0) + p_u \ln(c_{1,u}) + p_d \ln(c_{1,d}) \\
& + p_{uu} \ln(c_{2,uu}) + p_{ud} \ln(c_{2,ud}) + p_{du} \ln(c_{2,du}) + p_{dd} \ln(c_{2,dd}) \\
\text{s.t. } & c_0 = \sum_{k=0}^{1} \left(D_0^k + q_0^k\right) \theta_{-1}^k - \sum_{k=0}^{1} q_0^k \theta_0^k \\
& c_{1,u} = \sum_{k=0}^{1} \left(D_{1,u}^k + q_{1,u}^k\right) \theta_0^k - \sum_{k=0}^{1} q_{1,u}^k \theta_{1,u}^k \\
& c_{1,d} = \sum_{k=0}^{1} \left(D_{1,d}^k + q_{1,d}^k\right) \theta_0^k - \sum_{k=0}^{1} q_{1,d}^k \theta_{1,d}^k \\
& c_{2,uu} = \sum_{k=0}^{1} \left(D_{2,uu}^k + q_{2,uu}^k\right) \theta_{1,u}^k \\
& c_{2,ud} = \sum_{k=0}^{1} \left(D_{2,ud}^k + q_{2,ud}^k\right) \theta_{1,u}^k \\
& c_{2,du} = \sum_{k=0}^{1} \left(D_{2,du}^k + q_{2,du}^k\right) \theta_{1,d}^k \\
& c_{2,du} = \sum_{k=0}^{1} \left(D_{2,du}^k + q_{2,du}^k\right) \theta_{1,d}^k
\end{aligned}$$

Plugging in the consumption from the budget constraint into the utility-maximization, the following first order conditions result:

$$\begin{aligned}
\frac{\partial U}{\partial \theta_0^k} &= \frac{-q_0^k}{c_0} + p_u \frac{D_{1,u}^k + q_{1,u}^k}{c_{1,u}} + p_d \frac{D_{1,d}^k + q_{1,d}^k}{c_{1,d}} = 0 \\
\frac{\partial U}{\partial \theta_{1,u}^k} &= p_u \frac{-q_{1,u}^k}{c_{1,u}} + p_{uu} \frac{D_{2,uu}^k}{c_{2,uu}} + p_{ud} \frac{D_{2,ud}^k}{c_{2,ud}} = 0 \\
\frac{\partial U}{\partial \theta_{1,d}^k} &= p_d \frac{-q_{1,d}^k}{c_{1,d}} + p_{du} \frac{D_{2,du}^k}{c_{2,du}} + p_{dd} \frac{D_{2,dd}^k}{c_{2,dd}} = 0
\end{aligned}$$

Rewriting the first order conditions we get the following equations for the asset prices:

$$q_0^k = c_0 \left[p_u \frac{D_{1,u}^k + q_{1,u}^k}{c_{1,u}} + p_d \frac{D_{1,d}^k + q_{1,d}^k}{c_{1,d}} \right]$$
$$q_{1,u}^k = \frac{c_{1,u}}{p_u} \left[p_{uu} \frac{D_{2,uu}^k}{c_{2,uu}} + p_{ud} \frac{D_{2,ud}^k}{c_{2,ud}} \right]$$
$$q_{1,d}^k = \frac{c_{1,d}}{p_d} \left[p_{du} \frac{D_{2,du}^k}{c_{2,du}} + p_{dd} \frac{D_{2,dd}^k}{c_{2,dd}} \right]$$

The market clearing condition with one representative agent is that $\theta_{t,s}^k = 1$ for all t, k and s. Plugging that into the budget constraint implies:

$$c_{t,s} = \sum_k \left(D_{t,s}^k + q_{t,s}^k \right) \theta_{t-1,s^-}^k - \sum_k q_{t,s}^k \theta_{t,s}^k$$
$$c_{t,s} = \sum_k \left(D_{t,s}^k + q_{t,s}^k \right) \cdot 1 - \sum_k q_{t,s}^k \cdot 1$$
$$c_{t,s} = \sum_k D_{t,s}^k$$

I.e. the representative agent consumes just the dividends. Therefore we get:

$$c_0 = 1 + 1 = 2 \qquad c_{1,u} = 1 + 2 = 3 \qquad c_{1,d} = 1 + 0.5 = 1.5$$
$$c_{2,uu} = 1 + 2 = 3 \qquad c_{2,ud} = 1 + 0.5 = 1.5$$
$$c_{2,du} = 1 + 1.5 = 2.5 \qquad c_{2,dd} = 1 + 0.5 = 1.5$$

The probabilities of the different states are: $p_u = p_d = 0.5$ and $p_{uu} = p_{ud} = p_{du} = p_{dd} = 0.5 \cdot 0.5 = 0.25$. Now we have all ingredients to plug into the equation for the prices from before. We get

$$q_0^0 = \frac{61}{30} \qquad q_{1,u}^0 = \frac{3}{2} \qquad q_{1,d}^0 = \frac{4}{5}$$
$$q_0^1 = \frac{59}{30} \qquad q_{1,u}^1 = \frac{3}{2} \qquad q_{1,d}^1 = \frac{7}{10}$$

(c) Yes the market is complete, since the payoff matrix of all submarkets is complete. The payoff matrix of the submarket in period 0 are the payoffs (price plus dividend) of the different assets in the following period i.e.

$$A_0 = \begin{pmatrix} q_{1,u}^0 + D_{1,u}^0 & q_{1,u}^1 + D_{1,u}^1 \\ q_{1,d}^0 + D_{1,d}^0 & q_{1,d}^1 + D_{1,d}^1 \end{pmatrix} = \begin{pmatrix} 2.5 & 3.5 \\ 1.8 & 1.2 \end{pmatrix}$$

For the payoff matrix of the submarkets in the next period we have

$$A_{1,u} = \begin{pmatrix} D^0_{2,uu} & D^1_{2,uu} \\ D^0_{2,ud} & D^1_{2,ud} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0.5 \end{pmatrix}$$
$$A_{1,d} = \begin{pmatrix} D^0_{2,du} & D^1_{2,du} \\ D^0_{2,dd} & D^1_{2,dd} \end{pmatrix} = \begin{pmatrix} 1 & 1.5 \\ 1 & 0.5 \end{pmatrix}$$

The determinants of A_0, $A_{1,u}$ and $A_{1,d}$ are all different of zero. Therefore the rank of all these three matrices is 2. This is smaller or equal to the number of the assets (which is 2). Therefore the market is complete.

(d) As in the two period model the state prices, $\pi_{t,s}$, can be seen as the prices in the following equilibrium model with one representative agent:

$$\begin{aligned} \max_c \;& \ln(c_0) + p_u \ln(c_{1,u}) + p_d \ln(c_{1,d}) \\ & + p_{uu} \ln(c_{2,uu}) + p_{ud} \ln(c_{2,ud}) + p_{du} \ln(c_{2,du}) + p_{dd} \ln(c_{2,dd}) \\ \text{s.t.} \;& \sum c_{t,s}\pi_{t,s} = \sum w_{t,s}\pi_{t,s} \end{aligned}$$

with the market clearing condition $c_{t,s} = w_{t,s}$. The initial endowment in period t in state s is $w_{t,s} = \sum_k D^k_{t,s}$. This problem can be solved via Lagrange and the FOC is:

$$\frac{\partial L}{\partial c_{t,s}} = p_s \frac{1}{c_{t,s}} - \lambda \pi_{t,s} = 0$$

Taking $\lambda\pi_{t,s}$ on the other side and dividing by the FOC of $t = 0$ and norming $\pi_0 = 1$ results in:

$$\begin{aligned} \frac{p_s \frac{1}{c_{t,s}}}{\frac{1}{c_0}} &= \frac{\lambda \pi_{t,s}}{\lambda \pi_0} \\ \pi_{t,s} &= p_s \frac{c_0}{c_{t,s}} \end{aligned}$$

Plug in the numbers results in:

$$\begin{aligned} \pi_{1,u} &= \frac{1}{3} & \pi_{2,uu} &= \frac{1}{6} & \pi_{2,ud} &= \frac{1}{3} \\ \pi_{1,d} &= \frac{2}{3} & \pi_{2,du} &= \frac{1}{5} & \pi_{2,dd} &= \frac{1}{3} \end{aligned}$$

(e) In part (d) we have shown that there are strictly positive state prices in the market. By the FTAP this implies no arbitrage.

(f) The value of an European Call is:

$$C_0^E = \sum_s \pi_{2,s} \left(D_{2,s} - K\right)^+ = \frac{1}{6} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{5} \cdot \frac{1}{2} + \frac{1}{3} \cdot 0 = \frac{4}{15}$$

(g) The American option is more complicated. The option can be exercised at any point in time. It needs to be decided at every point in time (and in every state), if it is more valuable to exercise the option or to wait and then to exercise the option. In the following the value of exercising the option (discounted to $t = 0$) and the value of waiting (discounted to $t = 0$) is calculated:

Node $1, u$:

– Exercising:

$$\pi_{1,u} \cdot \left(2 + \frac{3}{2} - 1,0\right)^+ = \frac{5}{6}$$

– Waiting:

$$\pi_{2,uu} \cdot 1 + \pi_{2,ud} \cdot 0 = \frac{1}{6}$$

⇒ Exercising is more valuable. The discounted value of the option at $t = 0$ for the node (t,s)=(1,u) is $C_{1,u}^A = \frac{5}{6}$.

Node $1, d$:

– Exercising:

$$\pi_{1,d} \cdot \left(\frac{7}{10} + \frac{5}{10} - 1,0\right)^+ = \frac{2}{15}$$

– Waiting:

$$\pi_{2,du} \cdot \left(\frac{3}{2} - 1,0\right)^+ + \pi_{2,dd} \cdot \left(\frac{1}{2} - 1,0\right)^+ = \frac{1}{10}$$

⇒ Exercising is more valuable. The discounted value of the option at $t = 0$ is $C_{1,d}^A = \frac{2}{15}$.

Node $t = 0$:

– Exercising:

$$\pi_0 \cdot \left(\frac{59}{30} + 1 - 1,0\right)^+ = \frac{59}{30}$$

– Waiting:

$$C_{1,u} + C_{1,d} = \frac{5}{6} + \frac{2}{15} = \frac{29}{30}$$

⇒ The option to exercise is more valuable. The value of the option at $t = 0$ is $C_0^A = \frac{59}{30}$.

(h) The riskfree asset can be priced as follows:

$$\left(\frac{1}{1+r_{f,t}}\right)^t = \sum_s \pi_{t,s}$$

With that we get for the riskfree rates:

$$1 + r_{f,1} = 1 \qquad\qquad 1 + r_{f,2} = \sqrt{\frac{30}{31}}$$

(i) No arbitrage implies automatically the results. The returns of the stock (inclusive the dividends) in the first period are:

$$R^1_{1,u} = \frac{7 \cdot 15}{59} \qquad\qquad R^1_{1,d} = \frac{3 \cdot 12}{59}$$

i. Plug into the formula and we obtain:

$$\begin{aligned}\frac{1}{1+r_{f,1}}\mathbb{E}_{\pi^*_0}\left(D^1_1 + q^1_1\right) &= 1 \cdot \frac{1}{3}\left(\frac{3}{2} + 2\right) + 1 \cdot \frac{2}{3}\left(\frac{7}{10} + \frac{1}{2}\right) \\ &= \frac{59}{30} = q^1_0\end{aligned}$$

ii.

$$\begin{aligned}\mathbb{E}_{\pi^*_0}\left(R^1_1\right) &= \frac{1}{3} \cdot \frac{7 \cdot 15}{59} + \frac{2}{3} \cdot \frac{3 \cdot 12}{59} \\ &= 1 = R_{f,1}\end{aligned}$$

iii. For the likelihood ratio process we have:

$$l_{t,s} = \frac{\pi^*_{t,s}}{p_{t,s}} \qquad\qquad l_{1,u} = \frac{2}{3} \qquad\qquad l_{1,d} = \frac{4}{3}$$

Furthermore we get

$$\begin{aligned}\mathbb{E}_P(R^1_1) &= \frac{141}{2 \cdot 59} \\ \text{cov}_P(R^1_1, l_1) &= \mathbb{E}_P(R^1_1 l_1) - \mathbb{E}_P(R^1_1)\mathbb{E}_P(l_1) = \frac{-23}{2 \cdot 59}\end{aligned}$$

Plug that in:

$$R_{f,1} - \text{cov}_P(R^1_1, l_1) = 1 - \frac{-23}{2 \cdot 59} = \frac{141}{2 \cdot 59} = \mathbb{E}_P(R^1_1)$$

5.6

Because of no arbitrage it must hold:

$$(1+r_{t_0,t_1})^{t_1}\,(1+f(t_0,t_1,t_2))^{t_2-t_1} = (1+r_{t_0,t_2})^{t_2}$$

where r_{t_0,t_1} is the annual interest rate of a bond which has in t_0 a maturity of $t_1 - t_2$ and $f(t_0,t_1,t_2)$ is the forward rate between t_1 and t_2 from a forward traded at t_0. For the forward rate we get:

$$1+f(t_0,t_1,t_2) = \frac{(1+r_{t_0,t_2})^{\frac{t_2}{t_2-t_1}}}{(1+r_{t_0,t_1})^{\frac{t_1}{t_2-t_1}}}$$

This implies $f(0,5,8) = 8.4\%$.

5.7

The maximization problem of the representative agent is:

$$\begin{aligned}\max_{c_t,c_{t+1},s_t}\ & \ln(c_t) + \frac{1}{1+\delta}\ln(c_{t+1})\\ \text{s.t. } & p_t c_t + s_t = p_t w_t\\ & p_{t+1}c_{t+1} = p_{t+1}w_{t+1} + (1+r)s_t\end{aligned}$$

s_t is the amount of money, which the representative investors is saving in t. By solving the budget constraints for the consumption the optimization problem becomes:

$$\max_{s_t}\ \ln\left(w_t - \frac{s_t}{p_t}\right) + \frac{1}{1+\delta}\ln\left(w_{t+1} + (1+r)\frac{s_{t+1}}{p_{t+1}}\right)$$

The FOC is:

$$\frac{\partial U}{\partial s_t} = \frac{1}{c_t}\frac{-1}{p_t} + \frac{1}{1+\delta}\frac{1}{c_{t+1}}\frac{1+r}{p_{t+1}} = 0$$

Plug in the market clearing conditions (i.e. $c_t = w_t$ and $c_{t+1} = w_{t+1}$) and then solve for $1+r$ lead to:

$$1+r = (1+\delta)\frac{w_{t+1}p_{t+1}}{w_t p_t} = (1+\delta)(1+g_{t,t+1})$$

$g_{t,t+1}$ is the nominal growth rate between t and $t+1$.

5.8

(a) The utility maximization problem of the representative investor is:

$$\begin{aligned}\max_{c_0,c_1,c_2,s_{01},s_{02},s_{12}} \ & \ln(c_0) + \frac{1}{1+\delta}\ln(c_1) + \frac{1}{(1+\delta)^2}\ln(c_2) \\ \text{s.t. } & p_0c_0 + s_{01} + s_{02} = p_0w_0 \\ & p_1c_1 + s_{12} = p_1w_1 + (1+r_{01})s_{01} \\ & p_2c_2 = p_2w_2 + (1+f_{12})s_{12} + (1+r_{02})^2 s_{02}\end{aligned}$$

s_{01}, s_{02} and s_{12} are the investments into the bonds and the forward. The market clearing conditions are $c_0 = w_0$, $c_1 = w_1$ and $c_2 = w_2$. This problem can be solved analogously to exercise 5.7. We get

$$\begin{aligned}1 + r_{01} &= (1+\delta)(1+g_{01}) \qquad 1 + r_{02} = (1+\delta)\sqrt{1+g_{02}} \\ 1 + f_{12} &= (1+\delta)(1+g_{12})\end{aligned}$$

$\overline{g}_{t\ t+1}$ is the nominal growth rate between t and $t+1$. Till here we know the interest rates in $t = 0$. In $t = 1$ the economy is exactly the same as in exercise 5.7. r_{12}, the interest rate realized in $t = 1$, is then:

$$1 + r_{12} = (1+\delta)(1+g_{12})$$

(b) The utility maximization problem of the representative investor is:

$$\begin{aligned}\max_{c_0,c_1,c_2,s_{01},s_{02},s_{12}} \ & \ln(c_0) + \frac{1}{1+\beta}\left(\frac{1}{1+\delta}\ln(c_1) + \frac{1}{(1+\delta)^2}\ln(c_2)\right) \\ \text{s.t. } & p_0c_0 + s_{01} + s_{02} = p_0w_0 \\ & p_1c_1 + s_{12} = p_1w_1 + (1+r_{01})s_{01} \\ & p_2c_2 = p_2w_2 + (1+f_{12})s_{12} + (1+r_{02})^2 s_{02}\end{aligned}$$

The market clearing conditions are $c_0 = w_0$, $c_1 = w_1$ and $c_2 = w_2$. This problem can be solved in the same way as in (a). We get

$$\begin{aligned}1 + r_{01} &= (1+\beta)(1+\delta)(1+g_{01}) \quad 1 + r_{02} = \sqrt{1+\beta}\ (1+\delta)\sqrt{1+g_{02}} \\ 1 + f_{12} &= (1+\delta)(1+g_{12})\end{aligned}$$

$g_{t\ t+1}$ is the nominal growth rate between t and $t+1$. Till here we know the interest rates in $t = 0$. In $t = 1$ the economy is exactly the same as in exercise 5.7. r_{12}, the interest rate realized in $t = 1$, is then:

$$1 + r_{12} = (1+\beta)(1+\delta)(1+g_{12})$$

The utility function to obtain that result was: $\ln(c_1) + \frac{1}{1+\beta}\frac{1}{1+\delta}\ln(c_2)$.

(c) i. The utility maximization problem of the representative investor is:

$$\max\ \ln(c_0) + q\left(\frac{1}{1+\delta}\ln(c_{1u}) + \frac{1}{(1+\delta)^2}\ln(c_{2u})\right)$$
$$+ (1-q)\left(\frac{1}{1+\delta}\ln(c_{1u}) + \frac{1}{(1+\delta)^2}\ln(c_{2u})\right)$$
$$\text{s.t. } p_0c_0 + s_{01} + s_{02} = p_0w_0$$
$$p_{1u}c_{1u} + s_{12} = p_{1u}w_{1u} + (1+r_{01})s_{01}$$
$$p_1c_{1d} + s_{12} = p_{1d}w_{1d} + (1+r_{01})s_{01}$$
$$p_{2u}c_{2u} = p_{2u}w_{2u} + (1+f_{12})s_{12} + (1+r_{02})^2s_{02}$$
$$p_{2d}c_{2d} = p_{2d}w_{2d} + (1+f_{12})s_{12} + (1+r_{02})^2s_{02}$$

s_{01}, s_{02} and s_{12} are the investments into the bonds and the forward. The market clearing conditions are $c_0 = w_0$, $c_1 = w_1$ and $c_2 = w_2$. This problem can be solved analogously to exercise 5.7. We get

$$1+r_{01} = \frac{1+\delta}{\mathbb{E}\left(\frac{1}{1+g_{01}}\right)} \qquad 1+r_{02} = \frac{1+\delta}{\sqrt{\mathbb{E}\left(\frac{1}{1+g_{02}}\right)}}$$
$$1+f_{12} = (1+\delta)\frac{\mathbb{E}\left(\frac{1}{1+g_{01}}\right)}{\mathbb{E}\left(\frac{1}{1+g_{02}}\right)}$$

$g_{t\ t+1}$ is the nominal growth rate between t and $t+1$. Till here we know the interest rates in $t=0$. In $t=1$ the economy is exactly the same as in exercise 5.7. r_{12s}, the interest rate realized in $t=1$, if the economy is in state s, is then:

$$1+r_{12u} = (1+\delta)(1+g_{12u}) \qquad 1+r_{12d} = (1+\delta)(1+g_{12d})$$
$$\mathbb{E}(1+r_{12}) = (1+\delta)\mathbb{E}(1+g_{12})$$

ii. First some helpful calculations:

$$\mathbb{E}\left(\frac{1}{1+g_{01}}\right) = 0.5\frac{9}{10} + 0.5\frac{21}{20} = \frac{39}{40}$$
$$\mathbb{E}\left(\frac{1}{1+g_{02}}\right) = 0.5\left(\frac{9}{10}\right)^2 + 0.5\cdot 1 = \frac{181}{200}$$
$$1+g_{01d} = \frac{1+g_{02d}}{1+g_{01d}} = \frac{21}{20}$$

For the different interest rates it is obtained:

$$\begin{aligned} 1 + r_{01} &= 1.1 \cdot \frac{40}{39} = 1.128 \\ 1 + r_{02} &= 1.1\sqrt{\frac{200}{181}} = 1.156 \\ 1 + f_{12} &= 1.1 \cdot \frac{195}{181} = 1.185 \end{aligned}$$

The in $t = 1$ realized interest rates are depending on the state

$$\begin{aligned} 1 + r_{12u} &= 1.1 \cdot \frac{10}{9} = 1.222 \\ 1 + r_{12d} &= 1.1 \cdot \frac{21}{20} = 1.155 \\ \mathbb{E}\left(1 + r_{12}\right) &= 1.1 \cdot \frac{389}{360} = 1.189 \end{aligned}$$

(d) With the pure rational case without uncertainty an increasing or decreasing term structure can be obtained by choosing the right growth rates. But there is no forward rate bias (i.e. $f(0,1,2) = f_{12} = r_{12}$).
In the case with hyperbolic discounting, the term structure can also be explained. Plus there is a negative (but constant) forward rate bias (i.e. $f_{12} - r_{12} < 0$.
In the rational case with uncertainty, the shape of the term structure is also explained. The numerical example shows that in the upper state the realized interest rate rises and we have a negative forward rate bias. In the down state just the opposite happens. This is in line with the empirical observations. Since the model determines just one forward rate, it is not able to tell anything about the persistence of the forward rate bias. Empirical evidence shows that the forward rate bias has for several years the same sign.

References