Knight Meets Sharpe: Capital Asset Pricing under Ambiguity

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Abstract

This paper extends the classical mean-variance preferences to *mean-variance-ambiguity* preferences by relaxing the assumption that probabilities are known, and instead assuming that probabilities are uncertain. In general equilibrium, the two-fund separation theorem is preserved and the market portfolio is identified as efficient. Thereby, introducing ambiguity into the capital asset pricing model indicates that the *ambiguity premium* corresponds to *systematic ambiguity*, which is distinguished from systematic risk. Using the measurable closed-form *beta ambiguity*, well-known performance measures are generalized to account for ambiguity alongside risk. The introduced capital asset pricing model is empirically implementable and provides insight into empirical asset pricing anomalies. The model can be extended to other applications, including investment decisions and valuations.

Keywords and Phrases: Ambiguity index, ambiguity measurement, Knightian uncertainty, perceived probabilities, mean-variance, optimal portfolio, investment decisions, general equilibrium.

JEL Classification: D81, D89, G4, G11, G12.

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1 Introduction

The capital asset pricing model (CAPM), delivered by modern portfolio theory (MPT), has become the theoretical pillar of modern finance and is widely used in investment decisions (Barber et al., 2016). However, evidence has shown that the theoretical predictions regarding expected returns delivered by the CAPM are inconsistent with empirical findings (Fama and French, 1992, 2004). In particular, the intercept of the empirical security market line (SML) is found to be too high and its slope too flat to be justified by the theoretical CAPM.[1] A possible reason for this discrepancy is that these classical theories focus on risk, assuming away other dimensions of uncertainty in the economy.[2]

MPT (Markowitz, 1952; Tobin, 1958; Treynor, 1961) and the CAPM (Sharpe, 1964; Lintner, 1965; Mossin, 1966) are built upon the concept of mean-variance preferences, established under the expected utility paradigm (Von-Neumann and Morgenstern, 1944). The assumption underlying these theories is that the probabilities of future returns are known, establishing a unique mean-variance space upon which such preferences apply. In reality, however, probabilities of future returns are usually not precisely known, and financial decision-makers (investors) face uncertainty about these probabilities, referred to as *ambiguity* or *Knightian uncertainty* (Knight, 1921; Ellsberg, 1961).[3] The presence of ambiguity implies that the standard mean-variance preferences cannot portray a realistic picture of pricing decisions, since these standard preferences ignore information regarding the probabilities of future returns; information which is invaluable in portfolio and pricing decisions.

To portray portfolio and pricing decisions more realistically, this paper extends the standard mean-variance space to a *mean-variance-ambiguity* space, equipped with the appropriate preferences. In this space, it solves for the general equilibrium with heterogeneous investors in order to identify the set of optimal portfolios. Thereby, it reestablishes the capital market line (CML) and proves that the two-fund separation theorem is maintained in the presence of ambiguity. This framework is then utilized to introduce ambiguity into the classical CAPM, distinguishing ambiguity from risk, and *systematic ambiguity*, dominated by economy-wide characteristics, from idiosyncratic ambiguity, dominated by firm-specific characteristics. This extended model provides a closed-form solution for *beta ambiguity*, which corresponds to the systematic ambiguity associated with an asset. Analogous to the risk premium, the ambiguity premium is derived from the commonality of asset ambiguity and

[1] See, for example, Black et al. (1972), Merton (1973), and Frazzini and Pedersen (2014).

[2] The literature has been debating whether the source of this discrepancy, which generates multiple anomalies, is a missing uncertainty factor or mispricing due to biased expectations (Engelberg et al., 2018). Both reasons are affected by the presence of ambiguity.

[3] *Risk* refers to conditions in which the event to be realized is a priori unknown, but the odds of all possible events are perfectly known. *Ambiguity* refers to conditions in which not only the event to be realized is a priori unknown, but the odds of events are also not uniquely assigned.

market ambiguity, rather than from the asset's own ambiguity. In other words, investors are rewarded for systematic ambiguity and systematic risk, but not for idiosyncratic ambiguity or idiosyncratic risk. The notion of (systematic) ambiguity allows for the introduction of ambiguity into the Treynor and Sharpe ratios, as well as Jensen's alpha, thereby delivering extended performance measures. The resulting asset pricing model and performance measures can be estimated from the data and utilized in empirical studies of the cross-sectional implications of ambiguity.

In reality, investors face two tiers of uncertainty: one with respect to future returns and the other with respect to the probabilities associated with these returns. Since investors are assumed to be ambiguity averse, having a prior over probability distributions (priors), they do not compound the probability distributions of returns (beliefs) linearly with this prior when assessing expected utility. Instead, they act *as if* they overweight the probabilities of unfavorable returns and underweight the probabilities of favorable returns (Tversky and Kahneman, 1992; Wakker, 2010).[4] Expected utility is, therefore, negatively affected by both risk and ambiguity, for a given level of expected return.

To represent preferences for ambiguity, the standard mean-variance space is extended by adding ambiguity—the uncertainty of probabilities—as a third dimension. Suppose that rates of return are normally distributed with a density function $\phi\left(r \mid \mu, \sigma\right)$, where the mean, μ, and the standard deviation, σ, are uncertain. Then, in this three-dimensional mean-variance-ambiguity space, ambiguity, $\mho^2\left[r\right]$, is measured by the expected volatility of probabilities (Izhakian, 2020). Formally,

$$\mho^2\left[r\right] \quad = \quad \int \mathrm{E}\left[\phi\left(r \mid \mu, \sigma\right)\right] \mathrm{Var}\left[\phi\left(r \mid \mu, \sigma\right)\right] dr,$$

where the expectation, variance, and covariance of probabilities are taken using a second-order probability distribution (a distribution over a set of possible distributions). Risk, $\mathbb{V}\mathrm{ar}\left[r\right]$, in this space, is measured by the volatility of returns, taken using expected probabilities.

Preferences for risk and ambiguity are naturally reflected in the mean-variance-ambiguity space, as investors maximize their expected return for a given level of risk and ambiguity. In general equilibrium, these preferences imply that in any optimal portfolio, the relative proportion of any two risky and ambiguous assets is the same for all (heterogeneous) investors, independent of their aversion to ambiguity or to risk. This means that, in general equilibrium, every optimal portfolio is a combination of only two funds: the market portfolio and the risk-free asset (which is also ambiguity free). That is to say, the two-fund separation theorem (Tobin, 1958) holds true also in the presence of ambiguity. The proportions of these two funds, as well as the proportions and values (prices) of the assets

[4]This is explicitly formalized in cumulative prospect theory (Tversky and Kahneman, 1992), or implicitly delivered in ambiguity models (Gilboa and Schmeidler, 1989; Schmeidler, 1989; Klibanoff et al., 2005; Nau, 2006; Chew and Sagi, 2008).

comprising the market portfolio, are determined in general equilibrium by investors' preferences for risk and ambiguity. Since two-fund separation is maintained, along with the appropriate risk and ambiguity preferences, a construct analogous to the standard CML can be derived.

In the presence of ambiguity, the proportions of the assets comprising the market portfolio might be different from those in the absence of ambiguity and from those in an economy with ambiguity neutral preferences. The reason being that, in the presence of ambiguity, these proportions reflect market values that also price ambiguity. In reality, the market values (proportions) of the assets in the market portfolio are unique and observable. However, the classical CAPM faces difficulty in explaining these values. The proposed model aims to improve the theoretical explanation of these observable market values. The improvement of this model's predictions is a question for future empirical research.

The mean-variance-ambiguity framework also allows for the introduction of ambiguity into the classical CAPM.[5] In this extended model, referred to as *Capital Asset Pricing Model under Ambiguity* (ACAPM), the expected return of asset j corresponds not only to the covariation of its return, r_j, with the market-portfolio return, $r_{\mathbf{m}}$, but also to the covariation of the possible probability distributions of r_j with the possible probability distributions of $r_{\mathbf{m}}$. Formally, the expected return of asset j satisfies

$$\mathbb{E}[r_j] = r_f + \underbrace{\zeta_j^{\mathtt{P}}\left(\mathbb{E}[r_{\mathbf{m}}] - r_f\right)}_{\text{Participation Premium}} + \underbrace{\beta_j^{\mathtt{R}}\left(1-\zeta_j^{\mathtt{P}}\right)\left(\mathbb{E}[r_{\mathbf{m}}] - r_f\right)}_{\text{Risk Premium}} + \underbrace{\beta_j^{\mathtt{A}}\left(1-\zeta_j^{\mathtt{P}}\right)\left(\mathbb{E}[r_{\mathbf{m}}] - r_f\right)}_{\text{Ambiguity Premium}},$$

where

$$\zeta_j^{\mathtt{P}} = \sqrt{\frac{\mho^2[r_{\mathbf{m}}]}{1+\mho^2[r_{\mathbf{m}}]}}\mathrm{I}_{\{j\neq f\}}$$

is the zeta participation; $\mathrm{I}_{\{j\neq f\}}$ is an indicator function that takes the value one for non risk-free assets and zero otherwise;

$$\beta_j^{\mathtt{R}} = \frac{\mathrm{Cov}[r_{\mathbf{m}}, r_j]}{\mathbb{V}\mathrm{ar}[r_{\mathbf{m}}]}\frac{1+\mho^2[r_{\mathbf{m}}]}{1+\mho^2[r_{\mathbf{m}}]+\Lambda[r_{\mathbf{m}}, r_{\mathbf{m}}]}$$

is the beta risk;

$$\beta_j^{\mathtt{A}} = \frac{\Lambda[r_{\mathbf{m}}, r_j]}{1+\mho^2[r_{\mathbf{m}}]+\Lambda[r_{\mathbf{m}}, r_{\mathbf{m}}]}$$

is the beta ambiguity;

$$\Lambda[r_{\mathbf{m}}, r_j] = \int \mathrm{E}\left[\phi\left(r \mid \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\right]\mathrm{Cov}\left[\phi\left(r \mid \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right), \phi\left(r \mid \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\lambda\left(r \mid \mu_{\mathbf{m}}, \mu_j, \sigma_{\mathbf{m}}^2, \sigma_{\mathbf{m},j}\right)\right] dr;$$

[5]Several studies derived a consumption-based CAPM (Breeden, 1979; Duffie and Zame, 1989). Other studies extend the CAPM by introducing various risk factors, including skewness (Kraus and Litzenberger, 1976; Harvey and Siddique, 2000), stochastic volatility in intertemporal settings (Campbell et al., 2018), probability weights (Barberis and Huang, 2008), liquidity risk (Acharya and Pedersen, 2005; Liu, 2006), long-run risk in aggregate consumption (Ai and Kiku, 2013), extrapolated past prices (Barberis et al., 2015), and index investments (Baruch and Zhang, 2017). This paper introduces a different aspect of uncertainty, which relates to probabilities rather than to outcomes.

and

$$\lambda\left(r \mid \mu_{\mathbf{m}}, \mu_j, \sigma^2_{\mathbf{m}}, \sigma_{\mathbf{m},j}\right) = \frac{r-\mu_{\mathbf{m}}}{\sigma^2_{\mathbf{m}}}\left(\frac{\sigma_{\mathbf{m},j}}{\sigma^2_{\mathbf{m}}}\left(r-\mu_{\mathbf{m}}\right)+\mu_j\right)-\frac{\sigma_{\mathbf{m},j}}{\sigma^2_{\mathbf{m}}}.$$

The risk-free rate of return is denoted r_f, and the expectation, variance and covariance of returns are taken using the expected probabilities.

In the ACAPM, beta risk and beta ambiguity are independent of individual attitudes toward risk and ambiguity, reflecting only beliefs (information). Beta risk captures the effect of systematic risk, measured by the covariation of asset return with the market return; it departs from the standard theory due to the uncertainty regarding the probabilities used to assess risk. Beta ambiguity captures the effect of systematic ambiguity, measured by the covariation of asset return probabilities and market return probabilities. Extending other models, within which the ambiguity premium is attributed to the asset's own ambiguity and does not consider the relation between asset ambiguity and market ambiguity, the current model shows that only the systematic component of ambiguity, rather than the total ambiguity, is the relevant determinant of the asset's expected return. In addition, the model introduces a fixed *participation* premium that rewards for bearing the fundamental ambiguity in the economy. Different than risk, where a marginal exposure to the market portfolio implies a marginal exposure to risk, a marginal exposure to the market portfolio exposes the investor to a discrete level of ambiguity, the reward for which is the participation premium. A special case occurs when probabilities are known (or investors are ambiguity neutral), in which case the ACAPM reduces to the classical CAPM.

Existing empirical findings are inconsistent with the predictions of the classical CAPM. Specifically, the slope of the SML is found to be flatter, and the intercept higher than predicted by the traditional theory. As Figure 1 illustrates, the ACAPM delivers a new structure of the SML that, in addition to risk, accounts for ambiguity. This new structure may provide improved identification of idiosyncratic risk, overpricing, and underpricing. Thereby, it may help explain some of the empirical inconsistencies and anomalies related to the standard CAPM, including the fact that expected returns may differ from the risk-free rate even for assets having no systematic risk (the zero-beta anomaly, Black et al., 1972; Merton, 1973); the empirical SML being too flat to be explained by the theoretical prediction of the CAPM (the beta anomaly, Black et al., 1972; Frazzini and Pedersen, 2014); the idiosyncratic volatility being (negatively) priced in sharp contrast to the prediction of the CAPM (the idiosyncratic volatility anomaly, Ang et al., 2006; Liu et al., 2018); and the additional positive premia associated with firms with small market capitalization and high book-to-market equity ratio (the size and value anomalies, Fama and French, 1992). Since the ACAPM can be estimated using trading data, it can be utilized in cross-sectional empirical tests of the rates of return, which may address the aforementioned anomalies.

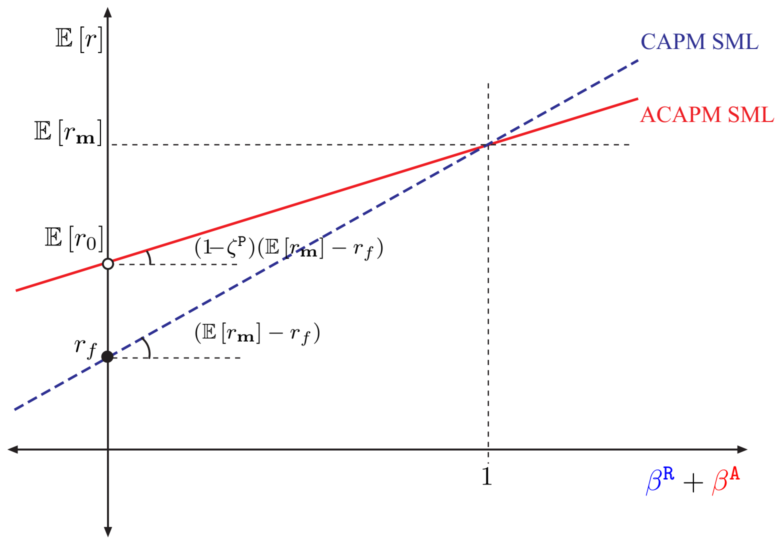

Figure 1: **The security market line**

In this figure, the blue dashed line describes the SML as predicted by the classical CAPM; i.e., the expected return as a function of $\beta^{\mathtt{R}}$. The red solid line describes the SML as predicted by the ACAPM; i.e., the expected return as a function of $\beta^{\mathtt{R}} + \beta^{\mathtt{A}}$. The market return, $\mathbb{E}[r_{\mathbf{m}}]$, and the risk-free rate of return, r_f, are the empirically observable ones in the economy.

The new formulations of the CML and the SML in the mean-variance-ambiguity space allow for the extension of the classical performance measures. The Sharpe (1966) ratio, which measures the reward in terms of excess return per unit of the total (systematic and idiosyncratic) risk borne, can be extended to measure, for any risky and ambiguous asset j, the reward per unit of risk and ambiguity borne:

$$\frac{\mathbb{E}[r_j] - r_f}{\mathbb{S}\text{td}[r_j]\sqrt{1+\mho^2[r_j]}}.$$

The Treynor (1965) ratio, which measures the reward per unit of systematic risk borne, can be extended to measure, for any risky and ambiguous asset j, the reward per unit of systematic risk and ambiguity borne:

$$\frac{\mathbb{E}[r_j] - r_f}{\zeta_j^{\mathtt{P}} + \left(1-\zeta_j^{\mathtt{P}}\right)\left(\beta_j^{\mathtt{R}} + \beta_j^{\mathtt{A}}\right)}.$$

In this framework, Jensen's (1968) alpha, which measures the abnormal return over the theoretical expected return, is written:

$$r_j - r_f - \zeta_j^{\mathtt{P}}\left(\mathbb{E}[r_{\mathbf{m}}] - r_f\right) - \left(\beta_j^{\mathtt{R}} + \beta_j^{\mathtt{A}}\right)\left(1-\zeta_j^{\mathtt{P}}\right)\left(\mathbb{E}[r_{\mathbf{m}}] - r_f\right).$$

These extended performance measures are empirically applicable to capital budgeting estimations and to the evaluation of professionally managed portfolios.

The implications of ambiguity have been studied in different aspects of asset pricing and portfolio selection, including the equity premium (Izhakian and Benninga, 2011; Ui, 2011; Zimper, 2012), market participation (Cao et al., 2005; Easley and O'Hara, 2009), zero trade (Dow and Werlang, 1992), portfolio inertia (Simonsen and Werlang, 1991; Illeditsch, 2011), portfolio choice (Pflug and Wozabal, 2007; Garlappi et al., 2007; Gollier, 2011; Boyle et al., 2012), learning (Leippold et al., 2008; Ju and Miao, 2012; Groneck et al., 2016), asset supply (Bianchi et al., 2018); demand uncertainty (Ilut et al., 2020); the term structure of interest rates (Gagliardini et al., 2009), and credit default swaps spreads (Augustin and Izhakian, 2020).[6] Adding to these studies, which consider the ambiguity of an asset independently of the surrounding ambiguity in the market, the current paper studies the pricing of asset ambiguity relative to the surrounding market ambiguity.

In related studies, Chen and Epstein (2002), and Epstein and Ji (2013), introduce ambiguity into the consumption CAPM using dynamic recursive max-min preferences (Gilboa and Schmeidler, 1989), assuming a representative investor; Epstein and Ji (2013) also assume ambiguity only with respect to the covariance matrix.[7] In these models, an asset's beta is subject to the investor's attitudes toward ambiguity and risk. Adding to this literature, the current paper considers heterogeneous investors and accounts for both ambiguous means and ambiguous covariance matrices, through ambiguous probabilities. Contrary to other studies, in the current paper, beta ambiguity is independent of the investors' (heterogenous) attitudes toward ambiguity and risk. Therefore, the ACAPM can be tested empirically using the methodology of estimating ambiguity from the data, suggested in recent literature (e.g., Izhakian and Yermack, 2017; Brenner and Izhakian, 2018).

The ACAPM provides a theoretical foundation for cross-sectional empirical tests of the rates of return. Prior studies have focused mainly on the implications of ambiguity for the time-series of asset prices (e.g., Izhakian and Benninga, 2011; Ui, 2011). Adding to these studies, the current paper focuses on the implications of cross-sectional asset ambiguity (relative to the surrounding market ambiguity) for asset returns. The extended performance measures that the current paper introduces provide a theoretical foundation for better assessment of portfolio performances.

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[6]The implications of ambiguity for asset pricing is surveyed by Guidolin and Rinaldi (2013).

[7]Kogan and Wang (2003) consider a representative investor with max-min preferences, who constructs her subjective set of priors around a reference prior based upon her aversion to ambiguity. Therefore, in their model, the asset's beta is subject to the investor's attitude toward ambiguity. In addition, their model assumes a known covariance matrix and a known reference prior. Nevertheless, Epstein and Ji (2013) highlight the importance of an ambiguous covariance matrix.

2 The equilibrium model

2.1 Ambiguity

Ambiguity, or Knightian uncertainty, provides the basis for a rich literature in decision theory. This literature takes a variety of approaches for modeling decision making under ambiguity (Gilboa and Schmeidler, 1989; Schmeidler, 1989; Klibanoff et al., 2005; Nau, 2006; Chew and Sagi, 2008). One important concept of these models is that, in the presence of ambiguity, ambiguity-averse decision-makers act *as if* they overweight the probabilities of unfavorable outcomes and underweight the probabilities of favorable outcomes, thereby lowering the perceived expected utility. In particular, the higher the degree of ambiguity or the aversion to ambiguity, the lower the perceived expected utility.

Ambiguity may affect asset pricing due to the role it plays in investment decisions. Investment decisions are made based upon perceived expected utility, which is estimated using subjective perceived probabilities. When an asset's perceived expected utility is relatively low, investors are reluctant to hold it, reducing its equilibrium price. To illustrate, consider an asset whose payoff is determined by a flip of an unbalanced coin, for which the investors do not know the odds of heads or tails. The payoff of the asset is \$100 for heads and \$0 for tails. Suppose now that new information increases the assessed degree of ambiguity about the coin. As investors are ambiguity averse, they lower their perceived probabilities of favorable (good) payoffs and raise their perceived probabilities of unfavorable (bad) payoffs. As a result, the expected utility falls, so that investors find this asset less attractive, and may prefer to reduce their holding in the asset, decreasing its equilibrium price. Instead, suppose that the good payoff increases to \$200. In this case, both risk and expected payoff increase, such that the investors may find this asset more (or less) attractive, which may increase (or decrease) its price. However, in this case, ambiguity has not changed, as investors have no reason to change the assessed probabilities (beliefs) or the assessed degree of ambiguity, since no new information about probabilities has been obtained.

This example illustrates that ambiguity is *outcome independent* up to a state space partition. That is, if the outcomes associated with events change, while the induced partition of the state space into events (set of events) remains unchanged, then the degree of ambiguity remains unchanged, since all probabilities remain unchanged. This is a critical insight, since outcome dependence enforces risk dependence. Furthermore, since ambiguity is outcome independent, the related preferences must also be outcome independent and apply exclusively to probabilities; otherwise, when outcomes are changed, investors would change their perceived probabilities (beliefs) of events even though no new information about the probabilities of events has been obtained.

2.2 The economy

Following Mossin (1966), consider a single-good frictionless exchange economy with one risk-free asset, indexed $j = 0$, and n risky and ambiguous real assets (firms), indexed $j = 1, \ldots, n$.[8] Each investor brings to the market her present holdings of assets, and an exchange takes place. Then, assets' payoffs are realized, and consumption takes place. The consumption good is perishable, and the only way to transfer consumption between individuals is through the capital market. Prices and payoffs of assets are denominated in units of the single consumption good. Prices of *all* assets, including the risk-free asset, are endogenously determined in general equilibrium.[9]

The payoff of the risk-free asset is one in every state of nature, $y_0 = 1$; and is, therefore, also ambiguity free.[10] The payoff vector of the risk-free, and the risky and ambiguous assets is $\mathbf{y} = [y_0, y_1, \ldots, y_n]'$. These payoffs are characterized by a vector of their means, $\boldsymbol{\mu}_{\mathbf{y}}$, and a symmetric, positive definite covariance matrix of rank $n+1$, $\boldsymbol{\Sigma}_{\mathbf{yy}}$. The implication of $\boldsymbol{\Sigma}_{\mathbf{yy}}$ being full ranked (nonsingular) is that there are no redundant assets.[11] Since ambiguity is present, the distribution of $\mathbf{y}$ is not unique. Instead, there is a set $\mathcal{P}$ of (joint) probability measures (priors), where each $\mathrm{P} \in \mathcal{P}$ is associated with a (joint) probability density function $\varphi(\cdot)$, according to which $\mathbf{y}$ may be distributed. Since $y_0 = 1$ is constant (the risk-free asset), all $\mathrm{P} \in \mathcal{P}$ agree on its probability. A second-order probability distribution, ξ, determines which $\mathrm{P} \in \mathcal{P}$ is realized.[12] As a consequence, $\boldsymbol{\mu}_{\mathbf{y}}$ and $\boldsymbol{\Sigma}_{\mathbf{yy}}$ are not unique, but jointly distributed according to ξ. All individuals are assumed to have an identical perception of $\mathcal{P}$ and ξ (homogenous beliefs, symmetric information). The price vector of the assets is $\mathbf{p} = [p_0, p_1, \ldots, p_n]'$. A portfolio of assets is a vector $\mathbf{x} = [x_0, x_1, \ldots, x_n]'$ of number of shares. The number of shares outstanding of each asset is perfectly divisible.

Let double-struck capital font ($\mathbb{E}[\cdot]$ and $\mathbb{V}\mathrm{ar}[\cdot]$) denote expectation or variance taken using the expected probabilities, and regular straight font ($\mathrm{E}[\cdot]$ and $\mathrm{Var}[\cdot]$) denote expectation or variance taken

[8] In particular, there are no costs for transactions or information acquisitions.

[9] Therefore, the risk-free rate of return is endogenously determined. Note that in Sharpe (1964) the risk-free rate of return is exogenously given.

[10] The following notational conventions are used. All vectors are column vectors. The transpose operation is denoted by a single quotation mark. Bold lowercase (Greek or upright Roman) letters denote vectors. Bold uppercase (Greek or upright Roman) letters denote matrices. Constants and variables are italicized, operators are in regular font (followed by square parentheses), and sets are in capital calligraphic font. No special notation is used to distinguish random variables from their realizations. The context should clarify the intention.

[11] The case of two perfectly correlated payoffs with different means implies that one could short one asset, long the other asset, and create an infinite expected payoff with no risk and no ambiguity. However, such a case is a violation of the law of one price, which is ruled out in equilibrium, as proved later.

[12] Formally, there is a probability space $(\mathcal{S}, \mathcal{E}, \mathrm{P})$, where $\mathcal{S}$ is an infinite state space; $\mathcal{E}$ is a σ-algebra of subsets of the state space (a set of events); a λ-system $\mathcal{H} \subset \mathcal{E}$ contains the events with an unambiguous probability (i.e., events with a known, objective probability); and $\mathrm{P} \in \mathcal{P}$ is an additive probability measure. The set of all probability measures $\mathcal{P}$ is assumed to be endowed with an algebra $\Pi \subset 2^{\mathcal{P}}$ of subsets of $\mathcal{P}$ that satisfies the structure required by Kopylov (2010). Π is equipped with a unique countably-additive probability measure ξ that assigns each subset $A \in \Pi$ with a probability $\xi(A)$.

using the second-order probabilities, ξ.[13] With these notations in place, the expected portfolio payoff, taken using the expected probabilities, is $\mathbf{x}'\mathrm{E}[\boldsymbol{\mu}_{\mathbf{y}}] = \mathbf{x}'\mathbb{E}[\mathbf{y}]$ and, by Lemma 1 in Appendix A.1, the variance of the portfolio payoff is $\mathbf{x}'\mathrm{E}[\boldsymbol{\Sigma}_{\mathbf{yy}}]\,\mathbf{x} + \mathbf{x}'\boldsymbol{\Sigma}_{\boldsymbol{\mu}_{\mathbf{y}}\boldsymbol{\mu}_{\mathbf{y}}}\mathbf{x}$. Since all investors have the same information (symmetric information), they have homogeneous beliefs and thus homogeneous expectations, variances and covariances (Sharpe, 1964; Lintner, 1965; Mossin, 1966).

2.3 The decision theory framework

To develop a general equilibrium model of the three-way relation between ambiguity, risk, and expected return, the proposed model rests on a key requirement that is necessary to differentiate ambiguity from risk: preferences for ambiguity that are outcome independent. To this end, this paper utilizes the theoretical framework of Expected Utility with Uncertain Probabilities (EUUP, Izhakian, 2017), as preferences for ambiguity in this framework are outcome independent. A by-product of the EUUP model is a model-derived outcome-independent (up to a state space partition) and risk-independent measure of ambiguity that is rooted in an axiomatic decision theory (Izhakian, 2020).[14]

The EUUP model assumes two tiers of uncertainty: one with respect to outcomes, and the other with respect to the probabilities of these outcomes. A decision-maker in this framework applies two differentiated phases of the decision process, each reflecting one of these tiers. In the first phase, she forms a representation of her perceived probabilities for all the events relevant to her decision, as the certainty equivalent probabilities of the uncertain probabilities. In the second phase, she assesses the expected utility of each alternative using these perceived probabilities. Since preferences for ambiguity apply exclusively to the probabilities of events (independently of their associated outcomes), aversion to ambiguity takes the form of aversion to mean-preserving spreads in *probabilities*.[15]

Investors have heterogeneous distinct preferences for ambiguity and for risk. As is common, investors are assumed to be risk averse with constant absolute risk aversion (Kraus and Litzenberger,

[13] Formally, the expectation and the variance of the probability of y occurring are defined, respectively, by $\mathrm{E}[\varphi(y)] = \int_{\mathcal{P}} \varphi(y)\, d\xi$ and $\mathrm{Var}[\varphi(y)] = \int_{\mathcal{P}} \big(\varphi(y) - \mathrm{E}[\varphi(y)]\big)^2 d\xi$. The expected payoff is defined by the double expectation (with respect to probabilities and payoffs) $\mathbb{E}[y] = \int_{\mathcal{P}} \left(\int_{\mathcal{S}} y d\varphi\right) d\xi = \int \mathrm{E}[\varphi(y)]\, y dy$. The covariance of payoffs y and z is defined by $\mathbb{C}\mathrm{ov}[y,z] = \int_{\mathcal{P}} \left(\int_{\mathcal{S}} (y - \mathbb{E}[y])(z - \mathbb{E}[z])\, d\varphi\right) d\xi = \int\int \mathrm{E}[\varphi(y,z)]\big(y - \mathbb{E}[y]\big)\big(z - \mathbb{E}[z]\big) dydz$.

[14] Outcome-independent preferences for ambiguity are necessary for the separation of ambiguity from risk, as well as attitudes from beliefs (Izhakian, 2020). While making a significant contribution to the literature, the risk-independent measurement of ambiguity poses a challenge for other frameworks since they do not separate ambiguity from attitudes toward ambiguity (Gilboa and Schmeidler, 1989; Schmeidler, 1989) or preferences for ambiguity are outcome dependent and therefore risk dependent (Klibanoff et al., 2005; Nau, 2006; Chew and Sagi, 2008).

[15] In Rothschild and Stiglitz (1970), aversion to risk takes the form of aversion to mean-preserving spreads in *outcomes*.

1973; Brennan, 1979; Acharya and Pedersen, 2005):

$$\mathrm{U}^i(c) \quad = \quad \frac{e^{-\gamma^i k} - e^{-\gamma^i c}}{\gamma^i}, \tag{1}$$

where c is consumption; $\gamma^i > 0$ is investor i's coefficient of (absolute) risk aversion; and k is a *reference point* that satisfies $\mathrm{U}^i(k) = 0$. Consumption (outcome) lower than k is unfavorable, consumption higher than k is favorable, and k is relatively close to the expected consumption, $\mathbb{E}[c]$.[16]

Similarly, investors are assumed to be ambiguity averse with constant absolute ambiguity aversion:[17]

$$\Upsilon^i(\mathrm{P}(c)) \quad = \quad -\frac{e^{-\eta^i \mathrm{P}(c)}}{\eta^i}, \tag{2}$$

where $0 < \eta^i \leq \frac{1}{\mathrm{Var}[\varphi(c)]}$ is investor i's coefficient of (absolute) ambiguity aversion.[18] As investors are averse to ambiguity, they do not compound the set of priors $\mathcal{P}$ and the prior ξ over $\mathcal{P}$ in a linear way (compound lotteries), but instead they use Υ to aggregate these probabilities in a non-linear way to form their perceived probabilities. Intuitively, in the EUUP model, ambiguity aversion is exhibited when an investor prefers an outcome with the expectation of its uncertain probability rather than with the uncertain probability itself.[19]

The assumptions regarding the investor's preference representation (constant absolute risk and ambiguity aversion) are made, without loss of generality, for tractability only. As shown below, these assumptions imply a natural closed-form solution for asset allocations, prices, and expected returns. Moreover, as shown below, since the two-fund separation theorem holds true also in the presence of ambiguity, other preference representations (for risk and ambiguity) are supported.

Within the EUUP model the perceived probabilities are formed by the certainty equivalent probabilities of uncertain probabilities. That is, the perceived probability is the minimum (maximum) unique certain probability value that an individual is willing to accept in exchange for the uncertain probability of a given favorable (unfavorable) event. Using the perceived probabilities, following

[16]Close in the sense that the third and higher absolute moments of c around k are of a smaller order than the second absolute central moment, and are therefore negligible.

[17]This assumption is supported by experimental evidence (Baillon and Placido, 2019).

[18]Ambiguity aversion takes the form of a concave Υ, ambiguity loving takes the form of a convex Υ, and ambiguity neutral takes the form of a linear Υ. The condition on Υ bounds the level of ambiguity aversion (the concavity of Υ) to ensure that the approximated perceived probabilities are nonnegative and satisfy set monotonicity with respect to set-inclusion.

[19]Note that risk aversion is exhibited when an investor prefers the expectation of the uncertain outcome over the uncertain outcome itself.

Izhakian (2020, Theorem 2), the expected utility can be formed by

$$
\begin{aligned}
\mathrm{V}^{i}(c) \;=\; & \int_{c\leq k} \mathrm{U}^{i}(c) \underbrace{\mathrm{E}\left[\varphi(c)\right]\left(1+\eta^{i}\mathrm{Var}\left[\varphi(c)\right]\right)}_{\text{Perceived Probability of Unfavorable Outcome}} dc \; + \\
& \int_{c\geq k} \mathrm{U}^{i}(c) \underbrace{\mathrm{E}\left[\varphi(c)\right]\left(1-\eta^{i}\mathrm{Var}\left[\varphi(c)\right]\right)}_{\text{Perceived Probability of Favorable Outcome}} dc + R_2(c),
\end{aligned} \tag{3}
$$

where the remainder $R_2(c) = o\big(\int \mathrm{E}\left[|\varphi(c) - \mathrm{E}[\varphi(c)]|^3\right] cdc\big)$ as $\int |\varphi(c) - \mathrm{E}[\varphi(c)]|\, dc \to 0$.[20]

When investors are ambiguity neutral (Υ is linear), they compound probabilities linearly, and Equation (3) reduces to the conventional expected utility. The same holds true when ambiguity is not present ($\mathrm{Var}[\varphi(c)] = 0$). In contrast, when investors are ambiguity averse (Υ is concave), they do not aggregate probabilities linearly; instead, they overweight the probabilities of the unfavorable outcomes and underweight the probabilities of favorable outcomes. In particular, the higher the ambiguity or the aversion to ambiguity, the higher the perceived probabilities of unfavorable outcomes and the lower the perceived probabilities of favorable outcomes. As a result, when ambiguity increases, the expected utility assessed using the perceived probabilities decreases.

The notion of the variance of probabilities in Equation (3) allows for the degree of ambiguity to be measured by the expected volatility of probabilities (Izhakian, 2020):

$$
\mho^2[c] \;=\; \int \mathrm{E}\left[\varphi(c)\right] \mathrm{Var}\left[\varphi(c)\right] dc. \tag{4}
$$

The measure $\mho^2$ (mho^2) is outcome independent and risk independent, always positive, and attains its minimum value, 0, only when all probabilities are perfectly known.[21]

2.4 Asset allocation decision

Suppose a large number of investors, labeled $i = 1, 2, \ldots, m$. Each investor brings to the market her present holdings of assets, $\bar{\mathbf{x}}^i = \left[\bar{x}_0^i, \bar{x}_1^i, \ldots, \bar{x}_n^i\right]'$. Thus, the budget set of investor i is

$$
\mathcal{B}^i \;=\; \left\{\mathbf{x}^i \in \mathbb{R}^{n+1} \mid \left(\bar{\mathbf{x}}^i - \mathbf{x}^i\right)' \mathbf{p} = 0\right\}. \tag{5}
$$

[20] The remainder of order $o\big(\int \mathrm{E}\left[|\varphi(c) - \mathrm{E}[\varphi(c)]|^3\right] cdc\big)$ means that the order of the error is three times smaller than the order of consumption, c, and tends to zero as $\int |\varphi(c) - \mathrm{E}[\varphi(c)]|\, dc \to 0$. This is equivalent to a cubic expansion, i.e., a remainder of order $o\left(\mathbb{E}\left[|c - \mathbb{E}[c]|^3\right]\right)$, in which the fourth and higher absolute central moments of consumption are of a strictly smaller order than the third absolute central moment as $|c - \mathbb{E}[c]| \to 0$, and are therefore negligible.

[21] Some studies interpret the volatility of the volatility or the volatility of the mean as measures of ambiguity. These measures, however, are outcome dependent, and therefore risk dependent. Moreover, $\mho^2$ solves some major issues that arise from the use of the volatility of the volatility or the volatility of the mean as measures of ambiguity. For example, comparing two assets with different degrees of ambiguity but each with a constant volatility, or two assets with different degrees of ambiguity but each with a constant mean.

Subject to her budget constraint, each investor chooses the portfolio of assets, $\mathbf{x}^i$, that maximizes her expected utility $\mathrm{V}^i(\cdot)$ as defined in Equation (3). Accordingly, investor i solves the maximization problem

$$\max_{\mathbf{x}^i \in \mathcal{B}^i} \quad \mathrm{V}^i\left(\mathbf{x}^{i\prime}\mathbf{y}\right). \tag{6}$$

Investors are heterogeneous in the sense that each investor i may have a different aversion to risk, γ^i, and a different aversion to ambiguity, η^i. Thus, for each investor i, the solution for the maximization problem in Equation (6) depends on her risk and ambiguity aversion. The solution also depends on prices, which determine the budget constraint. However, for brevity, the notation does not show this dependency. A *general equilibrium* is a vector of prices $\mathbf{p}$, under which each investor i solves the maximization problem in Equation (6), such that the market clears for all assets. That is,

$$\sum_i \mathbf{x}^i \quad = \quad \sum_i \bar{\mathbf{x}}^i. \tag{7}$$

2.5 Mean-variance-ambiguity preferences

To solve the maximization problem in Equation (6), and thereby the equilibrium prices, investors' preferences can be represented as mean-variance-ambiguity preferences. To this end, the next theorem identifies the certainty equivalent utility in Equation (3).[22]

Theorem 1. *Suppose that $\mathbb{E}[c] > 0$. The expected utility then satisfies*

$$\mathrm{V}^i(c) \quad = \quad \mathrm{V}^i\left(\mathbb{E}[c] - \mathcal{K}\right), \tag{8}$$

where $\mathbb{E}[c] - \mathcal{K}$ is the certainty equivalent consumption;

$$\mathcal{K} \quad = \quad \underbrace{\gamma^i \frac{1}{2} \mathbb{V}\mathrm{ar}[c]}_{\textit{Risk Premium}} \;+\; \underbrace{\eta^i \frac{1}{2\mathbb{E}[c]} \mathbb{V}\mathrm{ar}[c]\, \mho^2[c]}_{\textit{Ambiguity Premium}} \;+\; R_2(c)$$

is the risk and ambiguity premium; and the remainder $R_2(c) = o\left(\mathbb{E}\left[|c - \mathbb{E}[c]|^2\right]\right)$ as $|c - \mathbb{E}[c]| \to 0$.

Theorem 1 shows that risk and ambiguity have separate negative effects on expected utility, while the expected consumption has a positive effect. In particular, the higher the risk, $\mathbb{V}\mathrm{ar}[c]$, or the aversion to risk, γ^i, the lower the certainty equivalent utility. Similarly, the higher the ambiguity, $\mho^2[c]$, or the aversion to ambiguity, η^i, the lower the certainty equivalent utility. The remainder of order $o\left(\mathbb{E}\left[|c - \mathbb{E}[c]|^2\right]\right)$ means that the third and higher absolute central moments of the uncertain consumption, c, are of a strictly smaller order than the second absolute central moment of c, and are

[22] Considering constant absolute risk aversion, Ljungqvist and Sargent (2004) show that the mean-variance preference representation can be extracted using a Taylor approximation.

therefore negligible. Thus, henceforth, the remainder is ignored (Ljungqvist and Sargent, 2004) and the equal sign is used instead of the approximation sign for simplicity. The use of the approximate expected utility, rather than the expected utility, has no impact on the conclusions of the model.

Theorem 1 implies that, in order to maximize her expected utility, every investor i maximizes

$$\mathrm{F}^i\left(\mathbb{E}\left[c\right], \mathbb{S}\mathrm{td}\left[c\right], \mathbb{S}\mathrm{td}\left[c\right]\mho\left[c\right]\right) \quad = \quad \mathbb{E}\left[c\right] - \gamma^i \frac{1}{2}\mathbb{V}\mathrm{ar}\left[c\right] - \eta^i \frac{1}{2\mathbb{E}\left[c\right]}\mathbb{V}\mathrm{ar}\left[c\right]\mho^2\left[c\right], \tag{9}$$

subject to her budget constraint, where $\mho\left[c\right] = \sqrt{\int \mathrm{E}\left[\varphi\left(c\right)\right]\mathrm{Var}\left[\varphi\left(c\right)\right]dc}$ and $\mathbb{S}\mathrm{td}\left[c\right] = \sqrt{\mathbb{V}\mathrm{ar}\left[c\right]}$. With this representation in place, the mean-variance-ambiguity preference representation is immediately obtained.

Theorem 2. *The preferences of a risk- and ambiguity-averse investor can be formed in the mean-variance-ambiguity space by*

$$\mathrm{F}^i\left(\mathbb{E}\left[c\right], \mathbb{S}\mathrm{td}\left[c\right], \mathbb{S}\mathrm{td}\left[c\right]\mho\left[c\right]\right),$$

where

$$\frac{\partial \mathrm{F}^i}{\partial \mathbb{E}\left[c\right]} > 0; \qquad \frac{\partial \mathrm{F}^i}{\partial \mathbb{S}\mathrm{td}\left[c\right]} \leq 0; \qquad \textit{and} \qquad \frac{\partial \mathrm{F}^i}{\partial \mho\left[c\right]} \leq 0.$$

Theorem 2 establishes a representation of preferences for risk and ambiguity as mean-variance-ambiguity preferences in the mean-variance-ambiguity space. The mean-variance-ambiguity space extends the mean-variance space by adding ambiguity as a third dimension.[23] In this three-dimensional space, to scale ambiguity to the units of consumption, as the other two dimensions, $\mathbb{E}\left[c\right]$ and $\mathbb{S}\mathrm{td}\left[c\right]$, the third dimension is represented by $\mathbb{S}\mathrm{td}\left[c\right]\mho\left[c\right]$. An alternative representation of this third dimension by $\mho\left[c\right]$ would not alter the conclusions of the model.

Since the only source of consumption available to investor i is the payoff of her asset portfolio $\mathbf{x}^i$, investor i's maximization problem in Equation (6) can be reframed as

$$\max_{\mathbf{x}^i \in \mathcal{B}^i} \quad \mathrm{F}^i\left(\mathbb{E}\left[\mathbf{x}^{i\prime}\mathbf{y}\right], \mathbb{S}\mathrm{td}\left[\mathbf{x}^{i\prime}\mathbf{y}\right], \mathbb{S}\mathrm{td}\left[\mathbf{x}^{i\prime}\mathbf{y}\right]\mho\left[\mathbf{x}^{i\prime}\mathbf{y}\right]\right). \tag{10}$$

The solution to this optimization problem, subject to the market clearing conditions in Equation (7), provides the optimal portfolio holding of each investor and thereby the general equilibrium prices and allocations.

Every portfolio with a non-zero cost has a return $r_{\mathbf{x}} = \dfrac{\mathbf{x}'\mathbf{y}}{\mathbf{x}'\mathbf{p}} - 1$, where $\mathbf{x}'\mathbf{p}$ is the cost (value) of the portfolio and $\mathbf{x}'\mathbf{y}$ is its payoff. Thus, the equivalent representation of the maximization problem

[23] Other extensions of the mean-variance space to $\mathbb{R}^3$ have been proposed. For example, Kraus and Litzenberger (1976) extend it to mean-variance-skewness space.

in Equation (10) is

$$\max_{\mathbf{x}^i\in\mathcal{B}^i} \quad \mathrm{F}^i\Big(\mathbb{E}\left[\mathbf{x}^{i\prime}\left(\mathbf{1}+\mathbf{r}\right)\right], \mathbb{S}\mathrm{td}\left[\mathbf{x}^{i\prime}\left(\mathbf{1}+\mathbf{r}\right)\right], \mathbb{S}\mathrm{td}\left[\mathbf{x}^{i\prime}\left(\mathbf{1}+\mathbf{r}\right)\right]\mho\left[\mathbf{x}^{i\prime}\left(\mathbf{1}+\mathbf{r}\right)\right]\Big), \tag{11}$$

where

$$\mathcal{B}^i \quad = \quad \left\{\mathbf{x}^i\in\mathbb{R}^{n+1} \mid \left(\bar{\mathbf{x}}^i-\mathbf{x}^i\right)'\mathbf{1}=0\right\}; \tag{12}$$

$\mathbf{x}\in\mathbb{R}^{n+1}$ is the values in terms of consumption units allocated to each asset instead of asset units; $\mathbf{r}\in\mathbb{R}^{n+1}$ is the vector of returns of the assets in the economy; and $\mathbf{1}\in\mathbb{R}^{n+1}$ is a vector of 1s.

Modern portfolio theory asserts that, in an efficient market, a rational investor holds an asset portfolio that maximizes the expected return for a given level of risk. The maximization problem in Equation (11) generalizes this concept to ambiguity: a rational (risk- and ambiguity-averse) investor holds an asset portfolio that maximizes the expected return for a given level of risk and a given level of ambiguity.[24] With this notion, a portfolio $\mathbf{x}$ is *efficient* if there is no other portfolio with the same risk, the same ambiguity, and a strictly higher expected return.

The preferences for risk and ambiguity define sets of portfolios over which the investor is indifferent. Every such indifference set represents a specific level of expected utility. A rational investor chooses from among all feasible portfolios the one placing her on the indifference set that represents the highest level of expected utility.

3 Equilibrium prices and asset allocations

To characterize the mean-variance preferences and to measure risk by the variance of returns, it is commonly assumed that returns are normally distributed (Lintner, 1965; Merton, 1973; Acharya and Pedersen, 2005), so that the probability distributions are completely characterized by the mean and the variance of returns. To maintain our settings as closely as possible to those of the standard CAPM, it is assumed that returns are normally distributed; however, the parameters governing the distributions, mean and variance, are uncertain. This assumption allows for a closed-form formalization of the effect a change in a portfolio composition has on its degree of ambiguity. Since the EUUP model and its derived ambiguity measure are not restricted to a special class of probability distributions, the model that the current paper introduces can be generalized to other classes of probability distributions, including elliptically distributed returns.[25] In addition, all other assumptions of the standard CAPM

[24]In particular, given two portfolios with identical risk and ambiguity, a *rational* (risk- and ambiguity-averse) investor would prefer the portfolio with the higher expected return; given two portfolios with identical expected return and risk, she would prefer the portfolio with the lower ambiguity; given two portfolios with identical expected return and ambiguity, she would prefer the portfolio with the lower risk.

[25]The normal probability distribution is a subclass of elliptical distributions, which are fully characterized by the first two moments, mean and variance (Owen and Rabinovitch, 1983; Zhou, 1993).

are maintained.[26]

Formally, $\mathbf{r} \sim \mathrm{N}\left(\boldsymbol{\mu}_{\mathbf{r}}, \boldsymbol{\Sigma}_{\mathbf{rr}}\right)$, where $\boldsymbol{\Sigma}_{\mathbf{rr}}$ is a symmetric, positive definite matrix of rank $n+1$. Since ambiguity is present, $\boldsymbol{\mu}_{\mathbf{r}}$ and $\boldsymbol{\Sigma}_{\mathbf{rr}}$ are uncertain, and determined by a (joint) prior $\mathrm{P} \in \mathcal{P}$, with probability ξ. Each $\mathrm{P} \in \mathcal{P}$ is associated with a (joint) normal probability density function $\phi(\cdot)$ according to which $\mathbf{r}$ may be distributed. The second-order probability distribution ξ, determining which $\mathrm{P} \in \mathcal{P}$ is realized, is assumed to induce a symmetric distribution over each entry of $\boldsymbol{\mu}_{\mathbf{r}}$. The assumption that the set of probability distributions $\mathcal{P}$ consists of parametric (normal) probability distributions implies that a change in the parameters of the distributions may cause a change in the risk, $\mathbb{V}\mathrm{ar}[\mathbf{r}]$, as well as in the ambiguity, $\mho^2[\mathbf{r}]$. Risk may be changed directly by the change in the parameters. In the particular case of the normal distribution, an increase in the parameter $\boldsymbol{\Sigma}_{\mathbf{rr}}$ is an increase in risk. Ambiguity may be changed because the change in the parameters alters the priors within the set of priors. Further, in the class of continuous parametric probability distributions, a change in the parameters of the distribution alters the partition of the state space (Papoulis and Pillai, 2002); thereby, changes the degree of ambiguity. By Lemma 3 in Appendix A.1, in the particular case of normal distributions, an increase in the parameter $\boldsymbol{\Sigma}_{\mathbf{rr}}$ decreases the degree of ambiguity.

3.1 Optimal portfolio selection

To solve for investor i's optimal portfolio, the maximization problem in Equation (11) can be written explicitly as[27]

$$\max_{\mathbf{x}^i \in \mathcal{B}^i} \quad \mathbb{E}\left[\mathbf{x}^{i\prime}(\mathbf{1}+\mathbf{r})\right] - \gamma^i \tfrac{1}{2}\mathbb{V}\mathrm{ar}\left[\mathbf{x}^{i\prime}\mathbf{r}\right] - \eta^i \tfrac{1}{2\mathbb{E}\left[\mathbf{x}^{i\prime}(\mathbf{1}+\mathbf{r})\right]}\mathbb{V}\mathrm{ar}\left[\mathbf{x}^{i\prime}\mathbf{r}\right]\mho^2\left[\mathbf{x}^{i\prime}\mathbf{r}\right]. \tag{13}$$

The Lagrangian of this maximization problem can then be written as

$$\begin{aligned} \mathcal{L}\left(\mathbf{x}^i, \theta\right) \quad = \quad & \mathbb{E}\left[\mathbf{x}^{i\prime}(\mathbf{1}+\mathbf{r})\right] - \gamma^i \frac{1}{2}\mathbb{V}\mathrm{ar}\left[\mathbf{x}^{i\prime}\mathbf{r}\right] \\ & -\eta^i \frac{1}{2\mathbb{E}\left[\mathbf{x}^{i\prime}(\mathbf{1}+\mathbf{r})\right]}\mathbb{V}\mathrm{ar}\left[\mathbf{x}^{i\prime}\mathbf{r}\right]\mho^2\left[\mathbf{x}^{i\prime}\mathbf{r}\right] - \theta\left(\bar{\mathbf{x}}^i - \mathbf{x}^i\right)'\mathbf{1}. \end{aligned} \tag{14}$$

Using the Lagrangian, the next theorem identifies the optimal portfolio. To this end, the next theorem defines $\mathbf{r} \in \mathbb{R}^n$ as the return vector of the risky and ambiguous assets, and $r_f \in \mathbb{R}$ as the return of the risk-free asset.[28] Accordingly, henceforth, $\mathbf{x}^{*i} \in \mathbb{R}^n$ is investor i's *optimal* portfolio of risky and ambiguous assets.

---

[26] That is, markets are efficient in the sense that all information is available to all investors, who behave competitively. All of them have equal access to all assets in a market with no taxes and no commissions, and they can short any asset and hold any fraction of any asset.

[27] Note that the variance is invariant to uniform linear shifts in outcomes, and the ambiguity it invariant to uniform linear shifts in distributions (Lemma 3 in A.1).

[28] In Sharpe (1964), investors can borrow or lend unlimited quantities at a constant risk-free rate of return which is exogenous. Here, following Mossin's (1966) approach, the quantity of the risk-free assets is limited and the risk-free rate is endogenously determined in general equilibrium.

Theorem 3. *Investor i's optimal portfolio of risky and ambiguous assets, $\mathbf{x}^{*i} \in \mathbb{R}^n$, satisfies*

$$\mathbf{x}^{*i} = \left(\left(\gamma^i + \eta^i \frac{\mho^2 \left[\mathbf{x}^{*i\prime} \mathbf{r} \right]}{\mathbb{E}\left[x_f^{*i} (1 + r_f) + \mathbf{x}^{*i\prime} (\mathbf{1} + \mathbf{r}) \right]} \right) (\mathrm{E}\left[\mathbf{\Sigma}_{\mathbf{rr}}\right] + \mathbf{\Sigma}_{\boldsymbol{\mu}_\mathbf{r}\boldsymbol{\mu}_\mathbf{r}}) + \eta^i \frac{\mathbb{V}\mathrm{ar}\left[\mathbf{x}^{*i\prime} \mathbf{r} \right]}{\mathbb{E}\left[x_f^{*i} (1 + r_f) + \mathbf{x}^{*i\prime} (\mathbf{1} + \mathbf{r}) \right]} \Theta \right)^{-1}$$
$$\times \left(\mathbb{E}\left[\mathbf{r}\right] - \mathbf{1} r_f \right) \left(1 + \eta^i \frac{1}{2} \frac{\mathbb{V}\mathrm{ar}\left[\mathbf{x}^{*i\prime} \mathbf{r} \right]}{\mathbb{E}^2\left[x_f^{*i} (1 + r_f) + \mathbf{x}^{*i\prime} (\mathbf{1} + \mathbf{r}) \right]} \mho^2 \left[\mathbf{x}^{*i\prime} \mathbf{r} \right] \right), \tag{15}$$

where $\mathbf{r} \in \mathbb{R}^n$ is the vector of returns of the risky and ambiguous assets; $\mathbf{1} \in \mathbb{R}^n$ is a vector of 1s; $x_f^{*i} \in \mathbb{R}$ is the investor's optimal allocation to the risk-free asset; r_f is the risk-free rate of return; and Θ satisfies $\frac{\partial \mho^2 \left[\mathbf{x}^{*i\prime} \mathbf{r} \right]}{\partial \mathbf{x}^{*i}} = \Theta \mathbf{x}^{*i} 2$, as detailed in Lemma 6 in Appendix A.1.

In Theorem 3, $\mathbf{x}^{*i}$ describes the optimal values that investor i allocates to each of the risky and ambiguous assets. The difference between the value of her total initial endowment and the total value allocated to the risky and ambiguous assets defines the optimal allocation to the risk-free asset, which can be positive (lending) or negative (borrowing). The total value allocated to the risky and ambiguous assets can also be positive (long position) or negative (short position). However, in general equilibrium, the investor either has positive holdings in *every* risky and ambiguous asset, or negative holdings in *every* risky and ambiguous asset, as shown in Section 3.3.

Investor i's optimal allocation is a function of her aversion to risk, γ^i, and aversion to ambiguity, η^i. When she is neutral to ambiguity ($\eta^i = 0$), her optimal allocation is

$$\mathbf{x}^{*i} = \frac{1}{\gamma^i} \left(\mathrm{E}\left[\mathbf{\Sigma}_{\mathbf{rr}}\right] + \mathbf{\Sigma}_{\boldsymbol{\mu}_\mathbf{r}\boldsymbol{\mu}_\mathbf{r}} \right)^{-1} \left(\mathbb{E}\left[\mathbf{r}\right] - \mathbf{1} r_f \right),$$

and, in the absence of ambiguity, her optimal allocation is

$$\mathbf{x}^{*i} = \frac{1}{\gamma^i} \mathbf{\Sigma}_{\mathbf{rr}}^{-1} \left(\mathbb{E}\left[\mathbf{r}\right] - \mathbf{1} r_f \right),$$

similar to standard asset pricing models (Sharpe, 1964; Treynor, 1965).

An important insight delivered by Theorem 3 is that the optimal allocation (optimal asset portfolio) in the presence of ambiguity is different from the optimal allocation in the absence of ambiguity. Since the allocation to each asset determines its equilibrium price, the prices of assets in the presence of ambiguity would also be different from those in the absence of ambiguity. Theorem 3 implies that accounting for ambiguity has the potential to provide an explanation for observable asset allocations and prices, thereby helps resolve some asset pricing anomalies (Brennan and Xia, 2001; Fama and French, 2008).[29] Extant literature highlights the discrepancy between the observed asset allocations

[29] A few studies use different approaches to analyze the effect of ambiguity on optimal asset allocations. Pflug and Wozabal (2007) consider an optimal portfolio problem with a confidence set of probability distributions. Garlappi et al. (2007) and Boyle et al. (2012) consider a similar problem with a confidence interval for the estimated mean returns, and

and the predicted optimal ones (Canner et al., 1997). The asset allocations, suggested in Theorem 3, may be empirically compelling to the extent that they are consistent with asset allocation puzzles.

3.2 Two-fund separation

The identification of the optimal portfolio allocation in Theorem 3 delivers an important property: a description of the optimal relative *proportion* of the value allocated to each risky and ambiguous asset.

Theorem 4. *In general equilibrium, the relative proportional allocation of any two risky and ambiguous assets j and h satisfies*

$$\frac{x_j^{*i}}{x_h^{*i}} = \frac{\mathbb{E}\left[r_j\right] - r_f}{\mathbb{E}\left[r_h\right] - r_f}, \tag{16}$$

for each investor i.

Theorem 4 suggests an ambiguity-adjusted version of the optimal allocation in a risk-only economy (Sharpe, 1964; Treynor, 1965). In the presence of ambiguity, the relative proportions of risky and ambiguous assets might be different from those in the absence of ambiguity due to the ambiguity premium that affects asset expected returns (Theorem 1). The description of the optimal allocation in Theorem 4 delivers an important insight: the two-fund separation theorem holds true also in the presence of ambiguity.

Theorem 5. *Suppose n risky and ambiguous assets, whose returns are normally distributed with uncertain means and uncertain variances, and a risk-free (and ambiguity-free) asset.*

(i) There exists a unique pair of efficient portfolios (mutual funds): one containing only the risk-free asset and the other containing only the risky and ambiguous assets, such that independent of preferences (attitudes toward risk and ambiguity) or wealth, all investors are indifferent between choosing portfolios from the original $n+1$ assets or from these two funds.

(ii) The return of the risky and ambiguous fund is normally distributed with an uncertain mean and an uncertain variance.

(iii) The relative proportion of an investor's initial wealth invested in the j^{th} risky and ambiguous asset is the same for any investor i, independent of her preferences for risk and ambiguity.

Theorem 5 is an ambiguity-adjusted version of the Markowitz-Tobin separation theorem (Merton, 1973, Theorem 1). Tobin's (1958) separation theorem asserts that, in equilibrium, any investor holds the risk-free asset and a unique optimal portfolio of risky assets, called the *market portfolio*. Theorem 5

Maenhout (2004) explores the effect of model uncertainty, which may be interpreted as ambiguity.

shows that Tobin's insight holds true also in the presence of ambiguity. This implies that investment decisions can be broken into two separate phases: the first phase considers the choice of a unique optimal risky and ambiguous asset portfolio (the market portfolio); the second phase considers the allocation of funds between the risk-free asset and the risky and ambiguous asset portfolio (the market portfolio).

Investors may have different intensities of aversion to risk and to ambiguity. Nevertheless, in their investment decisions, they are different only in their decisions regarding the proportions of funds allocated to the risk-free asset and to the risky and ambiguous portfolio (the market portfolio). Thus, in equilibrium, every investor holds risky and ambiguous assets in the same relative proportions as the assets in the market portfolio, which means the same relative proportions as represented by the market value of assets (Theorem 3). The nature of the market portfolio in the presence of ambiguity, however, is different from Tobin's market portfolio.[30] Whereas Tobin's market portfolio demonstrates minimum risk for a given expected return, in the presence of ambiguity, the market portfolio has minimum consolidated risk and ambiguity for a given expected return, but not necessarily a minimum risk. The reason being the tradeoff between risk and ambiguity.[31]

Since investors are different in their intensities of aversion to risk and to ambiguity, the proportions of the risk-free asset and the market portfolio they choose to hold may be different. More conservative investors, for example, choose to allocate a larger proportion of their initial wealth to the risk-free asset. More aggressive investors may even decide to borrow money, i.e., to make a negative allocation to the risk-free asset, in order to invest more than their initial wealth in the market portfolio.

3.3 Equilibrium

To extract the optimal allocation, Theorem 3 constitutes n equations for each investor, describing her demand for the n risky and ambiguous assets in the economy. These equations can also be used to identify the demand for the risk-free asset. To determine a general equilibrium, the equality between demand and supply must be satisfied for each asset. That is, the market clearing condition in Equation (7) must be satisfied. Since the budget constraint in Equation (12) holds true for every investor i at their optimum, summing the budget constraint equations over all investors delivers the required market clearing condition in Equation (7), which completes the conditions describing the general equilibrium.

[30] In a model of investors with neutral preferences for parameter uncertainty, Klein and Bawa (1976) and Brown (1979) show that the market portfolio in the presence of parameter uncertainty is different from the market portfolio in the absence of parameter uncertainty, due to the Bayesian approach used.

[31] A few recent studies investigate the negative relation between risk and ambiguity (e.g., Brenner and Izhakian, 2018; Augustin and Izhakian, 2020).

As in Mossin (1966), the equilibrium allocation represents a Pareto optimum. That is, due to the property of a competitive equilibrium, in which preferences are concave, it would be impossible to increase one investor's expected utility by a new allocation without reducing the expected utility of at least one other investor. It is important to note that, in equilibrium, the problem of negative asset holdings is ruled out. By Theorems 4 and 5, in equilibrium, the relative proportion invested in any risky and ambiguous asset is the same for every investor. Thus, if one investor has a negative relative allocation to a given asset, then all other investors have a negative relative allocation to that asset, implying a violation of the market clearing condition. Therefore, the equilibrium (relative) allocation is positive for all risky and ambiguous assets.

Corollary 1. *In equilibrium, every risky and ambiguous asset j has a strictly positive proportion in the market portfolio and, therefore, a strictly positive capital market value.*

Furthermore, the market portfolio is unique and, therefore, so is the equilibrium.

Corollary 2. *In equilibrium, the proportions of assets in the market portfolio are unique. Therefore, the market portfolio and the equilibrium are unique.*

To recognize the uniqueness of the market portfolio, note that since the market is in equilibrium, which is governed by supply and demand, the proportion of each asset in the market portfolio is determined by its capital market value divided by the capital value of the whole market. The capital market value of an asset (the total worth of its shares) is unique, which implies that the proportion of each asset in the market portfolio is unique. Therefore, the market portfolio is unique.

3.4 Fund allocation decision

Suppose that the capital asset market is in equilibrium. The total resources available to investor i are $w^i = \bar{\mathbf{x}}^{i\prime}\mathbf{1}$, where $\bar{\mathbf{x}}^i$ is the value in terms of consumption units of all the assets that investor i brings to the market (including the risk-free asset). Based on the equilibrium prices and her attitudes toward risk and ambiguity, each investor allocates these resources into an optimal portfolio with a particular expected return, a particular level of risk, and a particular level of ambiguity. By Theorem 5, this portfolio consists of two funds: one containing only the risk-free asset and the other containing all the risky and ambiguous assets—the market portfolio, denoted $\mathbf{m}$.

To maximize her consumption, c^i, conditional on her preferences for risk and ambiguity, investor i chooses a proportion α of her resources, w^i, to invest in the market portfolio and a proportion $1 - \alpha$

to invest in the risk-free asset. Thus, investor i's maximization problem can be simplified to

$$\max_{\alpha} \quad \mathbb{E}\left[c^i\right] - \gamma^i \tfrac{1}{2} \mathbb{V}\mathrm{ar}\left[c^i\right] - \eta^i \tfrac{1}{2\mathbb{E}[c^i]} \mathbb{V}\mathrm{ar}\left[c^i\right] \mho^2\left[c^i\right], \tag{17}$$

where

$$c^i \quad = \quad w^i\big((1-\alpha)(1+r_f) + \alpha(1+r_{\mathbf{m}})\big),$$

and $\mathbb{E}\left[c^i\right] > 0$. That is, α is the investor's decision in equilibrium. This means that all investors solve the same optimization problem to maximize expected return, conditional on the degrees of risk and ambiguity. Since all investors have the same investment opportunities to choose from, the same information, and the same decision procedure, every portfolio selected by a rational investor is in the set of efficient portfolios, i.e., the set of portfolios that maximize the expected return for a given level of risk and a given level of ambiguity.

4 The capital market line

The set of feasible efficient portfolios defines the capital market line (CML). In the mean-variance-ambiguity space, a rational investor maximizes the expected return for given degrees of risk and ambiguity. Therefore, all portfolios lying on the CML are efficient in the sense that they attain the maximum expected return for a given degree of consolidated risk and ambiguity. The CML identifies the reward (in terms of expected return) of efficient portfolios per unit of consolidated risk and ambiguity borne; i.e., the price of risk and ambiguity. Therefore, in equilibrium, subject to the investor's aversion to risk and to ambiguity, every investment decision is made on the CML. The CML is described as follows.

Definition 1. *The capital market line is defined by*

$$\mathbb{E}[r] \quad = \quad r_f + \mathbb{S}\mathrm{td}[r]\sqrt{1+\mho^2[r]}\frac{\mathbb{E}[r_{\mathbf{m}}] - r_f}{\mathbb{S}\mathrm{td}[r_{\mathbf{m}}]\sqrt{1+\mho^2[r_{\mathbf{m}}]}}. \tag{18}$$

Definition 1 is obtained geometrically by the piecewise line segment originating from the risk-free rate, r_f, and passing through the market portfolio's expected return, $\mathbb{E}[r_{\mathbf{m}}]$, in the three-dimensional mean-variance-ambiguity space. The CML in Definition 1 implies that the ratio between the expected excess return, $\mathbb{E}[r] - r_f$, and the consolidated risk and ambiguity borne, $\mathbb{S}\mathrm{td}[r]\sqrt{1+\mho^2[r]}$, is the same for every investor, regardless of her intensity of aversion to risk or to ambiguity. This ratio is also the same for every asset, including the market portfolio.

The expected return of an optimal portfolio rewards for three elements: the time value of money, the risk borne, and the ambiguity borne. By Definition 1, the reward for the time value of money is equal to the risk-free rate, r_f. The reward for risk is a premium, proportional to the amount of risk borne, which is proportional to the holdings in the market portfolio. The reward for ambiguity consists of two elements: a fixed participation premium and a premium proportional to the amount of ambiguity borne, which is proportional to the holdings in the market portfolio. To observe this, let r_0 be the return of a portfolio with infinitely small proportional holdings in the market portfolio; i.e., $r_0 = (1-\alpha) r_f + \alpha r_{\mathbf{m}} \mid_{\alpha \to 0}$. In this case the risk associated with this portfolio is $\mathbb{S}\mathrm{td}[r_0] = 0$. However, once an investor holds any portion of the market portfolio, she bears a fixed amount of consolidated risk and ambiguity, $\mathbb{S}\mathrm{td}[r_0]\sqrt{1+\mho^2[r_0]}$, for which she is rewarded by a positive *participation premium* of magnitude $\mathbb{E}[r_0] - r_f$. The next corollary elicits this consolidated risk and ambiguity and its related premium.

Corollary 3. *The expected rate of return of a portfolio with infinitely small proportional holdings in the market portfolio is*

$$\mathbb{E}[r_0] \quad = \quad r_f + (\mathbb{E}[r_{\mathbf{m}}] - r_f)\sqrt{\frac{\mho^2[r_{\mathbf{m}}]}{1+\mho^2[r_{\mathbf{m}}]}}, \tag{19}$$

rewarding for its consolidated risk and ambiguity $\mathbb{S}\mathrm{td}[r_0]\sqrt{1+\mho^2[r_0]} = \mathbb{S}\mathrm{td}[r_{\mathbf{m}}]\,\mho[r_{\mathbf{m}}]$.

Corollary 3 implies that any non-zero holding of risky and ambiguous assets exposes the investor to the discrete inherited market ambiguity.[32] Thus, in the presence of ambiguity, all the risky and ambiguous portfolios with $\mathbb{S}\mathrm{td}[r]\,\mho[r] \in (0, \mathbb{S}\mathrm{td}[r_{\mathbf{m}}]\,\mho[r_{\mathbf{m}}]]$ are unfeasible, which implies that the CML has a segment of unfeasible portfolios. In this respect, Bossaerts et al. (2010) find that investors who are sufficiently averse to ambiguity have open sets of prices for which they refuse to hold ambiguous portfolios. Figure 2 depicts the two-dimensional section of the mean-variance-ambiguity space that contains the CML.

The portfolio with expected return $\mathbb{E}[r_0]$ is referred to as the *zero-beta portfolio* (Merton, 1973). In the absence of ambiguity, the reward per unit of risk is equal to $\frac{\mathbb{E}[r_{\mathbf{m}}]-r_f}{\mathbb{S}\mathrm{td}[r_{\mathbf{m}}]}$. In this case, Definition 1 reduces to the standard CML (Sharpe, 1964; Lintner, 1965; Mossin, 1966) in which the rate of substitution between a unit of expected excess return and a unit of risk is constant. Analogously, in the presence of ambiguity, the CML is a straight line, which means that the rate of substitution between a unit of expected excess return and a unit of uncertainty (consolidated risk and ambiguity) is constant. However, the consolidated risk and ambiguity, $\mathbb{S}\mathrm{td}[r]\sqrt{1+\mho^2[r]}$, is not linear in the proportion allocated to the market portfolio. In particular, by Lemma 3 in Appendix A.1, $\mathbb{S}\mathrm{td}[\alpha r_{\mathbf{m}}]\sqrt{1+\mho^2[\alpha r_{\mathbf{m}}]} = \mathbb{S}\mathrm{td}[r_{\mathbf{m}}]\sqrt{\alpha^2+\mho^2[r_{\mathbf{m}}]}$.

In general equilibrium, by Corollary 3, the expected return of the market portfolio is at least as

[32] Note that, since the risk associated with $\mathbb{E}[r_0]$ is zero, the entire consolidated risk and ambiguity is attributed to the normalized ambiguity $\mathbb{S}\mathrm{td}[r_{\mathbf{m}}]\,\mho[r_{\mathbf{m}}]$.

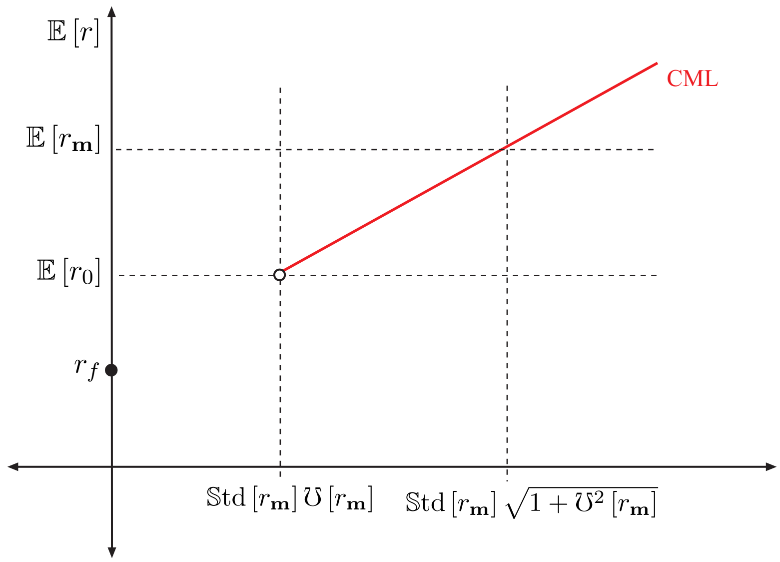

Figure 2: **The capital market line**

high as the expected return of the zero-beta portfolio, which is at least as high as the risk-free rate. The risk-free rate is lower than the expected return of the portfolio with the minimum possible risk and ambiguity, i.e., the *global minimum* risk and ambiguity portfolio; otherwise, all investors with mean-variance-ambiguity preferences would attempt to short this portfolio, a situation that cannot represent an equilibrium.[33]

By Definition 1 and Corollary 3, regardless of their aversion to risk and ambiguity, in the mean-variance-ambiguity space $(\mathbb{E}[r], \mathbb{S}\mathrm{td}[r], \mathbb{S}\mathrm{td}[r]\,\mho[r])$ all investors share the same goal: to maximize the expected excess return for a given level of consolidated risk and ambiguity (uncertainty). Therefore, in equilibrium, each investor can be thought of as solving the following maximization problem:

$$\max_{\mathbf{x}\in\mathcal{B}} \quad \frac{\mathbb{E}[\mathbf{x}'\mathbf{r}]-\mathbb{E}[r_0]}{\mathbb{S}\mathrm{td}[\mathbf{x}'\mathbf{r}]\sqrt{1+\mho^2[\mathbf{x}'\mathbf{r}]}-\mathbb{S}\mathrm{td}[r_0]\sqrt{1+\mho^2[r_0]}}, \tag{20}$$

where

$$\mathcal{B} \quad = \quad \left\{\mathbf{x}\in\mathbb{R}^n \mid \mathbf{x}'\mathbf{1}=1\right\};$$

and $\mathbf{x}$ is the proportional capital value of her assets relative to the capital value of her total assets. Therefore, excluding the risk-free asset, $\mathbf{x}$ can also be viewed as the relative proportion of each asset in the market portfolio, which is determined by its capital market value divided by the capital value of the market portfolio.

[33] For the same reason, in an economy with no ambiguity, the risk-free rate is lower than the expected return of the portfolio with the minimum possible risk (Cochrane, 2005).

5 Capital asset pricing

The simplified maximization problem in Equation (20) can be utilized to extract the expected return of assets. To this end, a calculus of variations-type argument can be applied to extend the classical CAPM to account for ambiguity. In the obtained closed-form pricing model, labeled the *Capital Asset Pricing Model under Ambiguity* (ACAPM), the expected return of an asset corresponds to its ambiguity and risk, relative to the market ambiguity and risk, rather than to its own ambiguity and risk. The next theorem, which is the central result of the current paper, introduces the ACAPM.

Theorem 6. *Suppose that the rates of return are normally distributed with uncertain means, μ, and uncertain variances, σ^2. The expected return of asset j is then*

$$\mathbb{E}\left[r_j\right] = r_f + \underbrace{\zeta_j^{\mathtt{P}}\left(\mathbb{E}\left[r_{\mathbf{m}}\right] - r_f\right)}_{\textit{Participation Premium}} + \underbrace{\beta_j^{\mathtt{R}}\left(1-\zeta_j^{\mathtt{P}}\right)\left(\mathbb{E}\left[r_{\mathbf{m}}\right] - r_f\right)}_{\textit{Risk Premium}} + \underbrace{\beta_j^{\mathtt{A}}\left(1-\zeta_j^{\mathtt{P}}\right)\left(\mathbb{E}\left[r_{\mathbf{m}}\right] - r_f\right)}_{\textit{Ambiguity Premium}}. \tag{21}$$

Zeta participation is

$$\zeta_j^{\mathtt{P}} = \sqrt{\frac{\mho^2\left[r_{\mathbf{m}}\right]}{1+\mho^2\left[r_{\mathbf{m}}\right]}}\mathrm{I}_{\{j\neq f\}}, \tag{22}$$

where the indicator function $\mathrm{I}_{\{j\neq f\}}$ takes the value one for non risk-free assets, and zero otherwise. Beta risk is

$$\beta_j^{\mathtt{R}} = \frac{\mathbb{C}\mathrm{ov}\left[r_{\mathbf{m}}, r_j\right]}{\mathbb{V}\mathrm{ar}\left[r_{\mathbf{m}}\right]}\frac{1+\mho^2\left[r_{\mathbf{m}}\right]}{1+\mho^2\left[r_{\mathbf{m}}\right]+\Lambda\left[r_{\mathbf{m}}, r_{\mathbf{m}}\right]}. \tag{23}$$

Beta ambiguity is

$$\beta_j^{\mathtt{A}} = \frac{\Lambda\left[r_{\mathbf{m}}, r_j\right]}{1+\mho^2\left[r_{\mathbf{m}}\right]+\Lambda\left[r_{\mathbf{m}}, r_{\mathbf{m}}\right]}, \tag{24}$$

where

$$\Lambda\left[r_{\mathbf{m}}, r_j\right] = \int \mathrm{E}\left[\phi\left(r\,|\,\mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\right]\mathrm{Cov}\left[\phi\left(r\,|\,\mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right), \phi\left(r\,|\,\mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\lambda\left(r\,|\,\mu_{\mathbf{m}}, \mu_j, \sigma_{\mathbf{m}}^2, \sigma_{\mathbf{m},j}\right)\right] dr \tag{25}$$

and

$$\lambda\left(r\,|\,\mu_{\mathbf{m}}, \mu_j, \sigma_{\mathbf{m}}^2, \sigma_{\mathbf{m},j}\right) = \frac{r-\mu_{\mathbf{m}}}{\sigma_{\mathbf{m}}^2}\left(\frac{\sigma_{\mathbf{m},j}}{\sigma_{\mathbf{m}}^2}\left(r-\mu_{\mathbf{m}}\right)+\mu_j\right) - \frac{\sigma_{\mathbf{m},j}}{\sigma_{\mathbf{m}}^2}. \tag{26}$$

Theorem 6 decomposes the price of an asset, in terms of expected return, into four components: the price of time, the price of risk, the price of ambiguity, and the price of market participation. The price of time, r_f, is the pure risk-free rate of return, rewarding for the time value of money. The price of risk, $\beta_j^{\mathtt{R}}(1-\zeta_j^{\mathtt{P}})\left(\mathbb{E}\left[r_{\mathbf{m}}\right] - r_f\right)$, is an additional expected return, rewarding for the systematic risk borne, referred to as the *risk premium.* The price of ambiguity, $\beta_j^{\mathtt{A}}(1-\zeta_j^{\mathtt{P}})\left(\mathbb{E}\left[r_{\mathbf{m}}\right] - r_f\right)$, is a second additional expected return, rewarding for the systematic ambiguity borne, referred to as the *ambiguity* *premium.* The participation price, $\zeta_j^{\mathtt{P}}\left(\mathbb{E}\left[r_{\mathbf{m}}\right]-r_f\right)$, is a third discrete additional fixed expected return, rewarding for the exposure to the fundamental ambiguity in the market, referred to as the *participation premium.* All three premia are independent of investors' attitudes toward risk and ambiguity, and depend only on beliefs (information).

A marginal exposure to the market portfolio implies a proportional marginal exposure to risk. In contrast, a marginal exposure to the market portfolio exposes the investor to a discrete fixed level of ambiguity, for which the participation premium is rewarding. In other words, when moving away from pure risk-free and ambiguity-free holdings, there is a discrete change in ambiguity, which exposes the investor to the fundamental ambiguity of the market portfolio. The sum of these three premia—the *uncertainty premium*—on the market portfolio, $\mathrm{E}\left[r_{\mathbf{m}}\right]-r_f$, is the aggregate excess return, rewarding for risk and ambiguity borne by the market portfolio, $\mathbf{m}$. The participation, risk, and ambiguity premia on asset j are proportional to the uncertainty premium on $\mathbf{m}$, as determined by the coefficients $\zeta_j^{\mathtt{P}}$, $\beta_j^{\mathtt{R}}$ and $\beta_j^{\mathtt{A}}$, respectively.

In the absence of ambiguity, μ and σ are certain for all assets, and so are $\mu_{\mathbf{m}}$, $\sigma_{\mathbf{m}}$, and $\sigma_{\mathbf{m},j}$. In this case, Theorem 6 reduces to the classical CAPM, in which only the systematic risk is rewarded. This also holds true when all investors are ambiguity neutral, since then they compound probabilities linearly.

Corollary 4. *In the absence of ambiguity or in an economy with ambiguity-neutral investors, for every asset j,*

$$\zeta_j^{\mathtt{P}} \;=\; 0, \qquad \beta_j^{\mathtt{R}} \;=\; \frac{\mathbb{C}\mathrm{ov}\left[r_{\mathbf{m}}, r_j\right]}{\mathbb{V}\mathrm{ar}\left[r_{\mathbf{m}}\right]}, \qquad \textit{and} \qquad \beta_j^{\mathtt{A}} \;=\; 0.$$

In Theorem 6, beta risk, $\beta_j^{\mathtt{R}}$, measures the sensitivity of asset j's return to the market return. It corresponds to the covariation of asset return and market return, which is assessed using expected probabilities (linearly compounded first- and second-order probabilities). Since there is uncertainty about probabilities, when investors are sensitive to ambiguity, the risk premium is adjusted for this uncertainty through $\zeta_j^{\mathtt{P}}$. In Theorem 6, beta ambiguity, $\beta_j^{\mathtt{A}}$, measures the sensitivity of asset j's return probabilities to the market return probabilities. It corresponds to the covariation of asset return probabilities and market return probabilities, which is assessed using the second-order probabilities (the joint probability distribution of the uncertain parameters μ and σ). This correspondence is formulated by λ in the expression of beta ambiguity in Equation (24). By Equation (26), in λ, the component $\mu_j-\frac{\sigma_{\mathbf{m},j}}{\sigma_{\mathbf{m}}^2}\mu_{\mathbf{m}}$ reflects the uncertainty about the *location* of asset j's return distribution.[34] The addi-

[34] The difference $\mu_j-\frac{\sigma_{\mathbf{m},j}}{\sigma_{\mathbf{m}}^2}\mu_{\mathbf{m}}$ can also be interpreted as the *unexpected mean return.* In this respect, Merton (1980) argues that the mean return (the location of the distribution) is difficult to estimate precisely.

tional component $\frac{\sigma_{\mathbf{m},j}}{\sigma_{\mathbf{m}}^2}$ reflects the uncertainty about the *precision* of asset j's return distribution (the uncertainty about σ_j^2). Since return distributions are fully characterized by their mean and variance (normally distributed), the uncertainty of the location and precision jointly generate the uncertainty of the return distribution.

The greater the absolute value of $\Lambda[r_{\mathbf{m}}, r_j]$, the greater the sensitivity of asset j's return distribution to the market return distribution and, accordingly, the greater the absolute value of $\beta_j^{\mathtt{A}}$. A positive $\Lambda[r_{\mathbf{m}}, r_j]$ implies a positive relation between asset j's ambiguity and the market ambiguity, resulting in a positive $\beta_j^{\mathtt{A}}$ and a positive ambiguity premium. A negative $\Lambda[r_{\mathbf{m}}, r_j]$ implies a negative relation between asset j's ambiguity and the market ambiguity, resulting in a negative $\beta_j^{\mathtt{A}}$ and a negative ambiguity premium. The intuition for a negative $\beta_j^{\mathtt{A}}$ is that investors are willing to pay (in terms of a negative premium) for holding the asset in order to hedge against the ambiguity in the market portfolio. For an asset with a positive $\beta_j^{\mathtt{A}}$ (a positive covariation of asset return probabilities and market return probabilities), investors ask for an additional positive premium as a reward for bearing ambiguity.

The risk-free asset bears no risk and no ambiguity; accordingly, all three premia are identically zero.

Corollary 5. *For the risk-free asset,*

$$\zeta_f^{\mathtt{P}} = 0, \qquad \beta_f^{\mathtt{R}} = 0, \qquad \textit{and} \qquad \beta_f^{\mathtt{A}} = 0.$$

It is possible for asset j to be characterized by $\beta_j^{\mathtt{R}} \neq 0$ and $\beta_j^{\mathtt{A}} = 0$. This may happen when the market return probabilities and asset j's return probabilities are perfectly known.[35] Notice that, when the market return probabilities are uncertain, an asset with no ambiguity (with perfectly-known probabilities) may still have a non-zero beta ambiguity if the correlation between the asset return and the market return is uncertain. In this case, zeta participation would be positive. It is also possible for an asset to be characterized by $\beta_j^{\mathtt{R}} = 0$ and $\beta_j^{\mathtt{A}} \neq 0$. This may happen when the covariance between r_j and $r_{\mathbf{m}}$, assessed using the expected probabilities, is zero; e.g., when $\mathbb{E}[\sigma_{r_{\mathbf{m}},j}] = 0$ and $\text{Cov}[\mu_{\mathbf{m}}, \mu_j] = 0$ (Lemma 1 in Appendix A.1). A special case is the zero-beta portfolio (or asset), for which $\beta_0^{\mathtt{R}} = 0$ and $\beta_0^{\mathtt{A}} = 0$.

Corollary 6. *For the zero-beta portfolio,*

$$\zeta_0^{\mathtt{P}} = \sqrt{\frac{\mho^2[r_{\mathbf{m}}]}{1 + \mho^2[r_{\mathbf{m}}]}}, \qquad \beta_0^{\mathtt{R}} = 0, \qquad \textit{and} \qquad \beta_0^{\mathtt{A}} = 0.$$

[35] For example, when μ_j and σ_j are both certain (i.e., a risk-only asset) or the ambiguity of the market portfolio is perfectly diversified in equilibrium, then $\Lambda[r_{\mathbf{m}}, r_j] = 0$ and $\beta_j^{\mathtt{A}} = 0$.

Corollary 6 suggests that, in the presence of ambiguity, the excess return of the zero-beta portfolio is not identically zero. Merton (1973) shows that the expected return of a risky asset may differ from the risk-free rate, even for an asset with no systematic risk. He attributes this difference to shifts in the investment opportunity set that are correlated with a zero-beta portfolio. In the ACAPM, this difference is attributed to the zeta participation, which is non-zero even when the beta risk (systematic risk) and the beta ambiguity (systematic ambiguity) are identically zero. Moreover, in the ACAPM, the additional hedging portfolio, implied by Merton's three-fund theorem, is not required.[36]

A special case considers the market portfolio, as defined in the next corollary.

Corollary 7. *For the market portfolio,*

$$\zeta_{\mathbf{m}}^{\mathtt{P}} = \sqrt{\frac{\mho^2\left[r_{\mathbf{m}}\right]}{1+\mho^2\left[r_{\mathbf{m}}\right]}},$$

$$\beta_{\mathbf{m}}^{\mathtt{R}} = \frac{1+\mho^2\left[r_{\mathbf{m}}\right]}{1+\mho^2\left[r_{\mathbf{m}}\right]+\Lambda\left[r_{\mathbf{m}},r_{\mathbf{m}}\right]} \qquad \textit{and} \qquad \beta_{\mathbf{m}}^{\mathtt{A}} = \frac{\Lambda\left[r_{\mathbf{m}},r_{\mathbf{m}}\right]}{1+\mho^2\left[r_{\mathbf{m}}\right]+\Lambda\left[r_{\mathbf{m}},r_{\mathbf{m}}\right]}.$$

Corollary 7 implies the following.

Corollary 8. *For the market portfolio,*

$$\zeta_{\mathbf{m}}^{\mathtt{P}} + \beta_{\mathbf{m}}^{\mathtt{R}}\left(1-\zeta_{\mathbf{m}}^{\mathtt{P}}\right) + \beta_{\mathbf{m}}^{\mathtt{A}}\left(1-\zeta_{\mathbf{m}}^{\mathtt{P}}\right) = 1.$$

Thus,

$$\beta_{\mathbf{m}}^{\mathtt{R}} + \beta_{\mathbf{m}}^{\mathtt{A}} = 1.$$

Corollary 8 implies that $\beta_{\mathbf{m}}^{\mathtt{R}} + \beta_{\mathbf{m}}^{\mathtt{A}} = 1$, while in the classical CAPM $\beta_{\mathbf{m}}^{\mathtt{R}} = 1$. In this respect, the ACAPM can be viewed as decomposing the observable beta of the market portfolio into three components: the first is derived from the systematic risk, the second from the systematic ambiguity, and the third from the participation in the ambiguous market.

An important property of beta risk and beta ambiguity is stressed in the next proposition.

Proposition 1. *Beta risk and beta ambiguity are both additive. That is,*

$$\beta_{\mathbf{x}}^{\mathtt{R}} = \mathbf{x}'\boldsymbol{\beta}^{\mathtt{R}} \qquad \textit{and} \qquad \beta_{\mathbf{x}}^{\mathtt{A}} = \mathbf{x}'\boldsymbol{\beta}^{\mathtt{A}},$$

where $\boldsymbol{\beta}$ *is a vector of the assets' betas, and* $\mathbf{x}$ *is a vector of the proportions of the assets in the portfolio.*

Proposition 1 suggests that the beta ambiguity (risk) of an asset portfolio is the value-weighted

[36] In Merton (1973), the betas are formally defined by the correlations between the state variables dominating the instantaneous investment opportunities and a non-tangency hedging portfolio.

average of the beta ambiguity (risk) of the individual assets comprising the portfolio. It implies that, similarly to the classical CAPM, the ACAPM is a linear beta pricing model. Consider a portfolio consisting of n risky and ambiguous assets with proportions $\mathbf{x} = (x_1, \ldots, x_n)'$. The expected excess return of portfolio $\mathbf{x}$ can then be expressed as

$$\mathbb{E}[r_{\mathbf{x}}] - r_f \;=\; \mathbf{x}'\mathbb{E}[\mathbf{r}] - r_f \;=\; \left(\mathbf{x}'\boldsymbol{\zeta}^{\mathtt{P}} + \mathbf{x}'\boldsymbol{\beta}^{\mathtt{R}}\left(1 - \zeta^{\mathtt{P}}\right) + \mathbf{x}'\boldsymbol{\beta}^{\mathtt{A}}\left(1 - \zeta^{\mathtt{P}}\right)\right)\left(\mathbb{E}[r_{\mathbf{m}}] - r_f\right),$$

where $\boldsymbol{\zeta}^{\mathtt{P}}$ is a vector of the assets' zeta participation.

In the ACAPM, an optimal portfolio has the maximal expected return for a given level of consolidated risk and ambiguity. Since risk and ambiguity diversification do not necessarily coincide, systematic risk and systematic ambiguity are *optimal* but not necessarily minimal. Similar to the classical CAPM, investors are rewarded via a higher rate of return for the systematic risk and ambiguity borne, while the idiosyncratic risk and ambiguity are not rewarded. However, idiosyncratic risk in the ACAPM is different than in the standard CAPM, since in the latter optimal portfolios have minimal risk. This difference in measuring idiosyncratic risk may help explain the idiosyncratic volatility anomaly (Ang et al., 2006; Liu et al., 2018).

The ACAPM suggests that there may be a tradeoff between risk diversification and ambiguity diversification. Theoretically, Uppal and Wang (2003) and Boyle et al. (2012) show that the presence of ambiguity leads to a strong bias in portfolio holdings (under diversification), such that full risk diversification is not optimal. Empirically, risk and ambiguity can be inversely related, such that risk reduction incurs higher ambiguity (Izhakian and Yermack, 2017; Brenner and Izhakian, 2018; Augustin and Izhakian, 2020). This inverse relation implies that, in the presence of ambiguity, a full risk diversification may not be optimal. Recall that, in the presence of ambiguity, the equilibrium asset proportions comprising the market portfolio may be different from those in the absence of ambiguity. By identifying the equilibrium prices, Theorem 6 characterizes the optimal amounts of risk and ambiguity, accounting for the tradeoff between the two.

Theorem 6 generalizes the classical CAPM and shows that the ambiguity premium on an asset is proportional to the part of its ambiguity that is derived from the market ambiguity. In earlier models, the ambiguity premium is attributed to the entire ambiguity of the market (e.g., Izhakian and Benninga, 2011; Ui, 2011) or of the asset (e.g., Chen and Epstein, 2002; Epstein and Ji, 2013). Theorem 6 adds to this literature the insight that the ambiguity premium corresponds to the covariation of asset ambiguity with market ambiguity.

6 The security market line

In the ACAPM, the security market line (SML) characterizes the linear relation between systematic risk and ambiguity (captured by beta risk and beta ambiguity) and expected return. By Theorem 6, the SML of the classical CAPM can be generalized to accommodate ambiguity and be defined as follows.

Definition 2. *The security market line is defined by*

$$\mathbb{E}[r_j] \quad = \quad \underbrace{r_f + \zeta_j^{\mathtt{P}} \left(\mathbb{E}[r_{\mathbf{m}}] - r_f\right)}_{\textit{Intercept}} + \left(\beta_j^{\mathtt{R}} + \beta_j^{\mathtt{A}}\right) \underbrace{\left(1 - \zeta_j^{\mathtt{P}}\right) \left(\mathbb{E}[r_{\mathbf{m}}] - r_f\right)}_{\textit{Slope}}. \tag{27}$$

In Definition 2, the intercept of the SML, $\mathbb{E}[r_0] = r_f + \zeta_j^{\mathtt{P}} (\mathbb{E}[r_{\mathbf{m}}] - r_f)$, captures the time value of money and the participation premium. The slope of the SML, $(1 - \zeta_j^{\mathtt{P}}) (\mathbb{E}[r_{\mathbf{m}}] - r_f)$, is the adjusted risk and ambiguity premium on the market portfolio. The coefficient $\beta_j^{\mathtt{R}} + \beta_j^{\mathtt{A}}$ corresponds to asset j's level of systematic risk and ambiguity.

Figure 1 provides a graphical representation of the SML in the presence and the absence of ambiguity. The x-axis depicts the magnitude of $\beta_j^{\mathtt{R}} + \beta_j^{\mathtt{A}}$, and the y-axis depicts the expected rate of return. The sloped dashed line describes the SML in the absence of ambiguity, and the solid sloped line describes it in the presence of ambiguity. The SML slope in the presence of ambiguity is flatter than in the absence of ambiguity. Specifically, in the presence of ambiguity, the slope is $(1 - \zeta_j^{\mathtt{P}}) (\mathbb{E}[r_{\mathbf{m}}] - r_f)$, while in the absence of ambiguity, it is $\mathbb{E}[r_{\mathbf{m}}] - r_f$. This implies that, in the presence of ambiguity, assets with relatively low standard beta risk have a greater excess return (equity premium) than in the classical CAPM; assets with relatively high standard beta risk have a smaller excess return than in the classical CAPM.

All possible portfolios, efficient and inefficient, lie on the SML, where the risk-free asset is a point of discontinuity on the SML. Every non-zero (even very small) holding of a non-risk-free asset bears an exposure to the market fundamental ambiguity, which is rewarded by a fixed discrete participation premium of size $\mathbb{E}[r_0] - r_f$. In other words, the SML reflects a fixed premium, attributed to the ambiguity borne by stock market participation. Since market values (prices) also reflect both ambiguity and the participation premia, in the presence of ambiguity, the equilibrium asset proportions comprising the market portfolio may be different from those in the absence of ambiguity.

The theoretical SML delivered by the ACAPM in Definition 2 offers a possible explanation for the inconsistency between the empirical SML and the SML predicted by the classical CAPM. The SML delivered by the ACAPM might be more consistent with the empirical findings than the SML delivered by the classical CAPM, and may explain some well-known related anomalies, including the zero-beta anomaly—the expected return being higher than the risk-free rate, even for assets having no systematic risk (Black et al., 1972; Merton, 1973); the beta anomaly—the empirical SML being too flat to be explained by the theoretical prediction of the classical CAPM (Black et al., 1972; Frazzini and Pedersen, 2014); the idiosyncratic volatility anomaly—the idiosyncratic volatility being (negatively) priced in contrast to the prediction of the classical CAPM (Ang et al., 2006; Liu et al., 2018)[37]; and the size and value anomalies—the additional positive premia associated with firms with small market capitalization and high book-to-market ratio (Fama and French, 1992).

Other extensions to the SML have been proposed in the literature. For example, Merton (1973) extends the standard CAPM to hedging portfolios using three funds; Kraus and Litzenberger (1976) extend the standard CAPM to accommodate return skewness, also using three different funds; and Acharya and Pedersen (2005) extend the standard CAPM to accommodate liquidity risk, consisting of a constant liquidation cost premium. In contrast to other models, to draw the SML in the ACAPM, only two funds are required: the risk-free asset and the market portfolio.

7 Performance measures

A natural application of the mean-variance-ambiguity preferences would be to measure portfolios' performance relative to their associated risk and ambiguity. A broadly used performance measure is the Sharpe (1966) ratio, which measures the reward in terms of excess return per unit of the total (systematic and idiosyncratic) risk borne. This ratio can be extended to account for ambiguity using the CML in the mean-variance-ambiguity space.[38] Definition 1 implies that the risk and ambiguity premium per unit of the total consolidated risk and ambiguity borne can be measured by

$$\frac{\mathbb{E}\left[r_j\right]-r_f}{\mathbb{S}\mathrm{td}\left[r_j\right]\sqrt{1+\mho^2\left[r_j\right]}}.$$

A second broadly used performance measure is the Treynor (1965) ratio, which measures the reward in terms of excess return per unit of systematic risk borne. This ratio can be extended to account for systematic ambiguity using the SML delivered by the ACAPM in Definition 2.[39] Definition 2 implies that the risk and ambiguity premium per unit of the systematic risk and ambiguity borne can be measured by

$$\frac{\mathbb{E}\left[r_j\right]-r_f}{\zeta_j^{\mathtt{P}}+\left(1-\zeta_j^{\mathtt{P}}\right)\left(\beta_j^{\mathtt{R}}+\beta_j^{\mathtt{A}}\right)}.$$

A third broadly used performance measure is the Jensen (1968) alpha, which measures the abnor-

[37]Notice that under the ACAPM, the idiosyncratic volatility is reformulated due to the new structure of the SML, which redefines the set of the optimal portfolios.

[38]Modigliani and Modigliani (1997) adjust the Sharpe ratio for portfolio leverage.

[39]Hübner (2005) extends the Treynor ratio to multiple indices.

mal return over the theoretical expected return. The Jensen alpha can be extended to account for ambiguity, using the theoretical expected return defined by the SML in Definition 2, as follows:[40]

$$r_j - r_f - \zeta_j^{\mathtt{P}} \left(\mathbb{E}\left[r_{\mathbf{m}}\right] - r_f\right) - \left(\beta_j^{\mathtt{R}} + \beta_j^{\mathtt{A}}\right) \left(1 - \zeta_j^{\mathtt{P}}\right) \left(\mathbb{E}\left[r_{\mathbf{m}}\right] - r_f\right).$$

8 Empirical implications

The main contribution of the ACAPM is portraying a more realistic picture of uncertainty and its effects on capital asset pricing. A second main contribution of the ACAPM is providing a theoretical foundation of cross-sectional empirical tests. With the new structure of the SML introduced in Definition 2, the ACAPM can be tested empirically, paving the way for further understanding of the effect of ambiguity on capital asset pricing. In this regard, it is important to note that the slope coefficient of a linear regression test of assets' excess return on the market's excess return does not capture the effect of ambiguity, since linear regression tests assume a known unique covariance matrix. Therefore, beta risk, beta ambiguity, and zeta participation must be computed directly, as formulated in Theorem 6. This can be done using the methodology to compute ambiguity presented in recent literature (e.g., Izhakian and Yermack, 2017; Brenner and Izhakian, 2018; Augustin and Izhakian, 2020). These estimates can then be used as the first-stage estimates in Fama and MacBeth's (1973) cross-sectional regression tests.

The introduction of an additional uncertainty factor—ambiguity—into capital asset pricing alters the SML and thereby the identification of overvalued and undervalued assets. Similar to the standard SML, assets above the SML are considered undervalued, since for a given degree of risk and ambiguity they yield a relatively high return, implying a relatively low price. Assets below the SML are considered overvalued, since for a given degree of risk and ambiguity they yield a relatively low return, implying a relatively high price. However, due to the different structure of the SML in Definition 2 relative to the standard SML, assets that are classified undervalued by the standard SML may be classified overvalued by the SML in the presence of ambiguity and vice versa. This insight has important implications for valuation and investment decisions.

The new structure of the theoretical SML delivered by the ACAPM in Definition 2 may address the inconsistency between the empirical evidence about the SML and the theoretical SML predicted by the classical CAPM (Fama and French, 1992, 2004). The theoretical intercept of the SML delivered by the ACAPM is higher than that in the classical CAPM (due to the participation premium), which may explain the zero-beta anomaly (Black et al., 1972; Merton, 1973). Specifically, the intercept of the SML in the ACAPM is $\mathbb{E}\left[r_0\right] = r_f + \zeta_j^{\mathtt{P}} \left(\mathbb{E}\left[r_{\mathbf{m}}\right] - r_f\right)$, while in the classical CAPM it is r_f. The

[40] Connor and Korajczyk (1986) develop multi-factor counterparts of the Jensen alpha.

theoretical slope of the SML in the ACAPM is flatter than that in the classical CAPM, which may explain the beta anomaly (Black et al., 1972; Frazzini and Pedersen, 2014). Specifically, the slope of the SML in the ACAPM is $(1 - \zeta_j^{\mathrm{P}})\,(\mathbb{E}\,[r_{\mathbf{m}}] - r_f)$, while in the classical CAPM it is $(\mathbb{E}\,[r_{\mathbf{m}}] - r_f)$. Since the SML delivered by the ACAPM can be estimated from the data, it may help explain these well-known asset pricing anomalies.

The SML in the ACAPM defines the set of optimal portfolios in the presence of ambiguity, and is different from the SML in the classical CAPM. Therefore, it also redefines the idiosyncratic risk and the idiosyncratic uncertainty (the consolidated idiosyncratic risk and ambiguity). This difference in defining and measuring idiosyncratic risk may help explain the idiosyncratic volatility anomaly (Ang et al., 2006; Liu et al., 2018). The size and value anomalies (Fama and French, 1992) may also be explained by the ACAPM, since firms with high book-to-market ratios are characterized by highly ambiguous investment opportunities (Herron and Izhakian, 2018, 2019).

9 Conclusion

This paper introduces a new capital asset pricing model that accounts for ambiguity—the uncertainty of probabilities—a real-world situation in which probabilities of outcomes are not uniquely assigned. It relaxes the main assumption of modern portfolio theory, according to which the probabilities of returns are known, and instead assumes that probabilities are uncertain. In this view, the mean-variance paradigm is generalized to a *mean-variance-ambiguity* paradigm, in which preferences are characterized. The three-way tradeoff between risk, ambiguity, and expected return sheds new light on capital asset pricing and optimal portfolio selection.

In general equilibrium, the mean-variance-ambiguity preferences deliver an important fundamental result: the Tobin two-fund separation theorem holds true in the presence of ambiguity. That is, optimally, every investor holds only two funds: the risk-free asset and the market portfolio (a unique optimal portfolio of risky and ambiguous assets). The proportions allocated to these two funds are determined by the investor's aversions to risk and to ambiguity. Asset proportions, comprising the market portfolio, may be different from those in the absence of ambiguity, since in the presence of ambiguity, market values (prices) also reflect ambiguity and participation premia.

The mean-variance-ambiguity preferences provide the theoretical underpinning for the extension of the classical capital asset pricing model (CAPM) to the Capital Asset Pricing Model under Ambiguity (ACAPM). In this extended model, a closed-form representation of *beta ambiguity*, in addition to the ambiguity-adjusted *beta risk*, is obtained. Asset prices in this model correspond to their systematic risk and systematic ambiguity borne. In addition, asset prices consist of an added fixed participation premium, generated by the market fundamental ambiguity. A natural application of the proposed model is the generalization of the Treynor (1965) and Sharpe (1966) ratios to account for ambiguity, allowing for the measurement of portfolios' performance relative to their consolidated risk and ambiguity borne. A generalization of the Jensen (1968) alpha is demonstrated as well. These measures are applicable in capital-budgeting estimations and in evaluating professionally managed portfolios.

The predictions of the classical CAPM are inconsistent with existing empirical findings, suggesting that the slope of the empirical SML is flatter, and the intercept is higher than predicted by the traditional theory; inconsistencies that generate multiple anomalies. The model that the current paper introduces may help explain these empirical inconsistencies and the related anomalies, including the zero-beta anomaly (Black et al., 1972; Merton, 1973); the beta-anomaly (Black et al., 1972; Frazzini and Pedersen, 2014); the idiosyncratic volatility anomaly (Ang et al., 2006; Liu et al., 2018); and the size and value anomalies (Fama and French, 1992).

The novel theoretical model, the ACAPM, introduced in this paper, provides important insights that pave the way for further research into the three-way risk-ambiguity-return relation. This model provides a theoretical foundation for empirical cross-sectional tests of the three-way tradeoff between risk, ambiguity, and expected return. A notable merit of the model is that it is trackable, applicable, and can be utilized in empirical studies, improving our understanding of capital asset pricing in the financial markets. The model can also be used in other applications, including portfolio selection and value at risk.

For more than half a century, the standard CAPM has been criticized for not portraying a realistic picture reflecting the empirical evidence regarding the risk-return relation. While advancing the literature toward addressing this puzzle, the model introduced in this paper may be further developed to support broader settings. The concepts introduced in this paper may also stimulate further thinking that will advance the literature toward a better understanding of the implication of ambiguity.

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A Appendix

A.1 Lemmata

Lemma 1. *Suppose that y and z are distributed with uncertain means, μ_y and μ_z, and uncertain variances, σ_y^2 and σ_z^2. Their covariance, computed using the expected probabilities, is then*

$$\mathbb{C}\text{ov}\left[y, z\right] = \text{E}\left[\sigma_{yz}\right] + \text{Cov}\left[\mu_y, \mu_z\right].$$

Lemma 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be some function. The volatility of the probabilities of x, $\text{Var}\left[\varphi\left(x\right)\right]$, is then uncorrelated with $f(x)$. That is, $\mathfrak{Cov}\left[f(x), \text{Var}\left[\varphi\left(x\right)\right]\right] = 0$, implying that*

$$\mathfrak{E}\left[f(x)\text{Var}\left[\varphi\left(x\right)\right]\right] = \mathfrak{E}\left[f(x)\right]\ \mathfrak{E}\left[\text{Var}\left[\varphi\left(x\right)\right]\right],$$

where $\mathfrak{E}$ is the expectation taken using either any $\text{P} \in \mathcal{P}$ or the expected probabilities; and $\mathfrak{Cov}$ is the covariance taken using either any $\text{P} \in \mathcal{P}$ or the expected probabilities.

Lemma 3. *Let δ and α be constants. When r is normally distributed with an uncertain mean and an uncertain variance, $\mho^2\left[\delta + \alpha r\right] = \frac{1}{\alpha^2}\mho^2\left[r\right]$ for any $\alpha \neq 0$, and $\mho^2\left[\delta + \alpha r\right] = 0$ for $\alpha = 0$.*

Lemma 4. *The expression*

$$I = \int \text{E}\left[\phi\left(r \,|\, \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right)\left(\left(\frac{\left(r - \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}\right)^2}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}} - 1\right)\frac{\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}} + \frac{\left(r - \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}\right)\boldsymbol{\mu}_{\mathbf{r}}}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}\right)\right]\text{Var}\left[\phi\left(r \,|\, \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right)\right] dr$$

is identically a vector of zeros.

Lemma 5. *The expression*

$$I = \int \text{E}\left[\phi\left(r \,|\, \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\lambda\left(r \,|\, \mu_{\mathbf{m}}, \mu_j, \sigma_{\mathbf{m}}^2, \sigma_{j,\mathbf{m}}\right)\right]\text{Var}\left[\phi\left(r \,|\, \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\right] dr$$

is identically zero.

Lemma 6. *Suppose that $\mathcal{P}$ consists of only normal probability distributions. The ambiguity $\mho^2\left[\mathbf{x}'\mathbf{r}\right]$ of the return $\mathbf{x}'\mathbf{r}$ of asset portfolio $\mathbf{x}$ then satisfies*

$$\frac{\partial\mho^2\left[\mathbf{x}'\mathbf{r}\right]}{\partial\mathbf{x}} = \Theta\mathbf{x}2,$$

where

$$\Theta = \int \text{E}\left[\phi\left(r \,|\, \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right)\right]\text{Cov}\left[\phi\left(r \,|\, \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right), \phi\left(r \,|\, \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right)\left(\frac{\left(r - \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}\right)^2}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}} - 1\right)\frac{\boldsymbol{\Sigma}_{\mathbf{rr}}}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}\right] dr.$$

A.2 Proofs

Proof of Lemma 1. The covariance can be written explicitly

$$\mathbb{C}\mathrm{ov}[y,z]=\int\int \mathrm{E}[\varphi(y,z)](y-\mathbb{E}[y])(z-\mathbb{E}[z])\,dydz=\int\int \mathrm{E}[\varphi(y,z)](y-\mathrm{E}[\mu_y])(z-\mathrm{E}[\mu_z])\,dydz,$$

where $\varphi(y,z)$ stands for the joint distribution of y and z; and $\mathrm{E}[\mu_y]$ is the expectation taken using the second-order probabilities. The expectation over probabilities can be taken out to obtain

$$\mathbb{C}\mathrm{ov}[y,z] \;=\; \mathrm{E}\left[\int\int \varphi(y,z)\big((y-\mu_y)+(\mu_y-\mathrm{E}[\mu_y])\big)\big((z-\mu_z)+(\mu_z-\mathrm{E}[\mu_z])\big)dydz\right].$$

Organizing terms provides

$$\begin{aligned}\mathbb{C}\mathrm{ov}[y,z] \;=\;& \mathrm{E}\left[\int\int \varphi(y,z)(y-\mu_y)(z-\mu_z)\,dydz\right]+\mathrm{E}\left[\int\int \varphi(y,z)(y-\mu_y)(\mu_z-\mathrm{E}[\mu_z])\,dydz\right]+\\ & \mathrm{E}\left[\int\int \varphi(y,z)(\mu_y-\mathrm{E}[\mu_y])(z-\mu_z)\,dydz\right]+\mathrm{E}\left[\int\int \varphi(y,z)(\mu_y-\mathrm{E}[\mu_y])(\mu_z-\mathrm{E}[\mu_z])\,dydz\right],\end{aligned}$$

which simplifies to

$$\mathbb{C}\mathrm{ov}[y,z] \;=\; \mathrm{E}[\sigma_{yz}]+\mathrm{Cov}[\mu_y,\mu_z]\,.$$

□

Proof of Lemma 2. Let $y=\varphi(x)$, then $\mathrm{Var}[\varphi(x)]$ can be written $\mathrm{Var}[y|x]=\mathrm{E}[y^2|x]-\mathrm{E}^2[y|x]$. In turn, $\mathfrak{Cov}\big[\mathrm{Var}[\varphi(x)],f(x)\big]$ can be written explicitly

$$\begin{aligned}\mathfrak{Cov}\big[\mathrm{Var}[\varphi(x)],f(x)\big] &= \mathfrak{E}\left[\left(\mathrm{E}[y^2|x]-\mathrm{E}^2[y|x]-\mathfrak{E}\left[\mathrm{E}[y^2|x]-\mathrm{E}^2[y|x]\right]\right)\left(f(x)-\mathfrak{E}[f(x)]\right)\right]\\ &= \mathfrak{E}\Big[f(x)\big(\mathrm{E}[y^2|x]-\mathrm{E}^2[y|x]\big)\Big]-\mathfrak{E}\Big[f(x)\mathfrak{E}\big[\mathrm{E}(y^2|x)-\mathrm{E}^2[y|x]\big]\Big].\end{aligned}$$

Applying the tower property to the first term and the law of iterated expectations to the second term (e.g., Goldberger, 1991, page 47, T8), provides

$$\mathfrak{Cov}\big[\mathrm{Var}[\varphi(x)],f(x)\big] \;=\; \mathfrak{E}\Big[f(x)\big(\mathrm{E}[y^2]-\mathrm{E}^2[y]\big)\Big]-\mathfrak{E}\Big[f(x)\mathfrak{E}\big[\mathrm{E}[y^2]-\mathrm{E}^2[y]\;\mid x\big]\Big].$$

By Karlin and Taylor (2012, page 8), $\mathfrak{E}\big[f(x)\mathfrak{E}[g(y)|x]\big]=\mathfrak{E}\big[f(x)g(y)\big]$. Therefore,

$$\mathfrak{E}\Big[f(x)\mathfrak{E}\big[\mathrm{E}[y^2]-\mathrm{E}^2[y]\;\mid x\big]\Big] \;=\; \mathfrak{E}\Big[f(x)\big(\mathrm{E}[y^2]-\mathrm{E}^2[y]\big)\Big],$$

which completes the proof. □

Proof of Lemma 3. Since $\mathbb{V}\mathrm{ar}[\delta]=0$, $\mathbb{V}\mathrm{ar}\left[\delta+\alpha r\mid\sigma^2\right]=\alpha^2\sigma^2$. Thus, the ambiguity of $\delta+\alpha r$ can be written explicitly

$$\mho^2[\delta+\alpha r] \;=\; \int \mathrm{E}\left[\frac{1}{\sqrt{2\pi}\alpha\sigma}e^{-\frac{(r-\delta-\alpha\mu)^2}{2\alpha^2\sigma^2}}\right]\mathrm{Var}\left[\frac{1}{\sqrt{2\pi}\alpha\sigma}e^{-\frac{(r-\delta-\alpha\mu)^2}{2\alpha^2\sigma^2}}\right]dr.$$

When $\alpha \neq 0$, changing the integration variable to $r+\delta$ and then to αr, provides

$$\mho^2[\delta+\alpha r] = \int \mathrm{E}\left[\frac{1}{\sqrt{2\pi}\alpha\sigma}e^{-\frac{(r-\mu)^2}{2\sigma^2}}\right]\mathrm{Var}\left[\frac{1}{\sqrt{2\pi}\alpha\sigma}e^{-\frac{(r-\mu)^2}{2\sigma^2}}\right]\alpha dr = \frac{1}{\alpha^2}\mho^2[r].$$

When $\alpha = 0$, the probability of δ is constant and thus $\mho^2[\delta]=0$. □

Proof of Lemma 4. Changing the order of integration provides

$$I = \mathrm{E}\left[\int \phi\left(r \,|\, \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right)\left(\left(\frac{\left(r-\mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}\right)^2}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}-1\right)\frac{\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}+\frac{\left(r-\mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}\right)\boldsymbol{\mu}_{\mathbf{r}}}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}\right)\mathrm{Var}\left[\phi\left(r \,|\, \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right)\right]dr\right].$$

By Lemma 2,

$$I = \mathrm{E}\left[\begin{array}{l}\int \phi\left(r \,|\, \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right)\left(\left(\frac{\left(r-\mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}\right)^2}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}-1\right)\frac{\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}+\frac{\left(r-\mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}\right)\boldsymbol{\mu}_{\mathbf{r}}}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}\right)dr \\ \times \int \phi\left(r \,|\, \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right)\mathrm{Var}\left[\phi\left(r \,|\, \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right)\right]dr\end{array}\right],$$

which implies

$$I = \mathrm{E}\left[\begin{array}{l}\left(\begin{array}{l}\frac{\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}\int \phi\left(r \,|\, \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right)\left(\frac{\left(r-\mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}\right)^2}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}-1\right)dr \\ +\frac{\boldsymbol{\mu}_{\mathbf{r}}}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}\int \phi\left(r \,|\, \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right)\left(r-\mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}\right)dr\end{array}\right) \\ \times \int \phi\left(r \,|\, \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right)\mathrm{Var}\left[\phi\left(r \,|\, \mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right)\right]dr\end{array}\right] = 0.$$ □

Proof of Lemma 5. Writing the integral explicitly by substituting Equation (26) for $\lambda\left(r \,|\, \mu_{\mathbf{m}}, \mu_j, \sigma^2_{\mathbf{m}}, \sigma_{\mathbf{m},j}\right)$ provides

$$I = \int \mathrm{E}\left[\phi\left(r \,|\, \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\left(\frac{r-\mu_{\mathbf{m}}}{\sigma^2_{\mathbf{m}}}\left(\frac{\sigma_{\mathbf{m},j}}{\sigma^2_{\mathbf{m}}}\left(r-\mu_{\mathbf{m}}\right)+\mu_j\right)-\frac{\sigma_{\mathbf{m},j}}{\sigma^2_{\mathbf{m}}}\right)\right]\mathrm{Var}\left[\phi\left(r \,|\, \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\right]dr.$$

Changing the order of integration provides

$$I = \mathrm{E}\left[\int \phi\left(r \,|\, \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\left(\frac{r-\mu_{\mathbf{m}}}{\sigma^2_{\mathbf{m}}}\left(\frac{\sigma_{\mathbf{m},j}}{\sigma^2_{\mathbf{m}}}\left(r-\mu_{\mathbf{m}}\right)+\mu_j\right)-\frac{\sigma_{\mathbf{m},j}}{\sigma^2_{\mathbf{m}}}\right)\mathrm{Var}\left[\phi\left(r \,|\, \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\right]dr\right].$$

By Lemma 2,

$$\begin{aligned} I &= \mathrm{E}\left[\begin{array}{l}\int \phi\left(r \,|\, \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\left(\frac{r-\mu_{\mathbf{m}}}{\sigma^2_{\mathbf{m}}}\left(\frac{\sigma_{\mathbf{m},j}}{\sigma^2_{\mathbf{m}}}\left(r-\mu_{\mathbf{m}}\right)+\mu_j\right)-\frac{\sigma_{\mathbf{m},j}}{\sigma^2_{\mathbf{m}}}\right)dr \\ \times \int \phi\left(r \,|\, \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\mathrm{Var}\left[\phi\left(r \,|\, \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\right]dr\end{array}\right] \\ &= \mathrm{E}\left[\frac{\mu_j}{\sigma^2_{\mathbf{m}}}\int \phi\left(r \,|\, \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\left(r-\mu_{\mathbf{m}}\right)dr\int \phi\left(r \,|\, \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\mathrm{Var}\left[\phi\left(r \,|\, \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}\right)\right]dr\right] = 0.\end{aligned}$$ □

Proof of Lemma 6. Writing the ambiguity measure explicitly provides

$$\mho^2\left[\mathbf{x}'\mathbf{r}\right] = \int \mathrm{E}\left[\frac{1}{\sqrt{2\pi\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}}e^{-\frac{\left(r-\mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}\right)^2}{2\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}}\right]\mathrm{Var}\left[\frac{1}{\sqrt{2\pi\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}}e^{-\frac{\left(r-\mathbf{x}'\boldsymbol{\mu}_{\mathbf{r}}\right)^2}{2\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}}\right]dr.$$

Differentiating the expected probability with respect to $\mathbf{x}$ provides

$$
\begin{aligned}
I &= \frac{\partial \mho^2 [\mathbf{x}'\mathbf{r}]}{\partial \mathrm{E}[\cdot]} \frac{\partial \mathrm{E}[\cdot]}{\partial \mathbf{x}} \\
&= \int \mathrm{E}\left[\frac{e^{-\frac{(r-\mathbf{x}'\boldsymbol{\mu}_\mathbf{r})^2}{2\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}}}{\sqrt{2\pi \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}} \left(\left(\frac{(r-\mathbf{x}'\boldsymbol{\mu}_\mathbf{r})^2}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}} - 1 \right) \frac{\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}} + \frac{(r-\mathbf{x}'\boldsymbol{\mu}_\mathbf{r})\boldsymbol{\mu}_\mathbf{r}}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}} \right) \right] \mathrm{Var}\left[\frac{e^{-\frac{(r-\mathbf{x}'\boldsymbol{\mu}_\mathbf{r})^2}{2\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}}}{\sqrt{2\pi \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}} \right] dr.
\end{aligned}
$$

By Lemma 4, $I = 0$. Differentiating the variance of probabilities with respect to $\mathbf{x}$ provides

$$
\begin{aligned}
II &= \frac{\partial \mho^2 [\mathbf{x}'\mathbf{y}]}{\partial \mathrm{Var}[\cdot]} \frac{\partial \mathrm{Var}[\cdot]}{\partial \mathbf{x}} \\
&= 2\int \mathrm{E}\left[\frac{e^{-\frac{(r-\mathbf{x}'\boldsymbol{\mu}_\mathbf{r})^2}{2\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}}}{\sqrt{2\pi \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}} \right] \mathrm{Cov}\left[\frac{e^{-\frac{(r-\mathbf{x}'\boldsymbol{\mu}_\mathbf{r})^2}{2\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}}}{\sqrt{2\pi \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}}, \frac{e^{-\frac{(r-\mathbf{x}'\boldsymbol{\mu}_\mathbf{r})^2}{2\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}}}{\sqrt{2\pi \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}} \left(\left(\frac{(r-\mathbf{x}'\boldsymbol{\mu}_\mathbf{r})^2}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}} - 1 \right) \frac{\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}} + \frac{(r-\mathbf{x}'\boldsymbol{\mu}_\mathbf{r})\boldsymbol{\mu}_\mathbf{r}}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}} \right) \right] dr.
\end{aligned}
$$

Since r is symmetrically distributed around $\mathbf{x}'\boldsymbol{\mu}_\mathbf{r}$, and every entry of $\boldsymbol{\mu}_\mathbf{r}$ is symmetrically distributed,

$$
II = 2\int \mathrm{E}\left[\frac{e^{-\frac{(r-\mathbf{x}'\boldsymbol{\mu}_\mathbf{r})^2}{2\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}}}{\sqrt{2\pi \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}} \right] \mathrm{Cov}\left[\frac{e^{-\frac{(r-\mathbf{x}'\boldsymbol{\mu}_\mathbf{r})^2}{2\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}}}{\sqrt{2\pi \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}}, \frac{e^{-\frac{(r-\mathbf{x}'\boldsymbol{\mu}_\mathbf{r})^2}{2\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}}}{\sqrt{2\pi \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}} \left(\frac{(r-\mathbf{x}'\boldsymbol{\mu}_\mathbf{r})^2}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}} - 1 \right) \frac{\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}} \right] dr.
$$

Thus,

$$
\frac{\partial \mho^2 [\mathbf{x}'\mathbf{r}]}{\partial \mathbf{x}} = I + II = \Theta \mathbf{x} 2,
$$

where

$$
\Theta = \int \mathrm{E}\left[\phi\left(r \mid \mathbf{x}'\boldsymbol{\mu}_\mathbf{r}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right)\right] \mathrm{Cov}\left[\phi\left(r \mid \mathbf{x}'\boldsymbol{\mu}_\mathbf{r}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right), \phi\left(r \mid \mathbf{x}'\boldsymbol{\mu}_\mathbf{r}, \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}\right) \left(\frac{(r-\mathbf{x}'\boldsymbol{\mu}_\mathbf{r})^2}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}} - 1 \right) \frac{\boldsymbol{\Sigma}_{\mathbf{rr}}}{\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{rr}}\mathbf{x}} \right] dr.
$$

□

Proof of Theorem 1. The first-order Taylor expansion of the right hand side (RHS) of Equation (8) with respect to $\mathcal{K}$, around 0, is

$$
RHS = \mathrm{U}\left(\mathbb{E}[c] - \mathcal{K}\right) = \mathrm{U}\left(\mathbb{E}[c]\right) - \mathcal{K}\mathrm{U}'\left(\mathbb{E}[c]\right) + o\left(|c|\right).
$$

By Equations (3), the left hand side (LHS) of Equation (8) can be written

$$
\begin{aligned}
LHS &= \int \mathrm{E}\left[\varphi(c)\right] \mathrm{U}(c)\, dc + \\
&\quad \eta \int_{c \le k} \mathrm{U}(c)\, \mathrm{E}\left[\varphi(c)\right] \mathrm{Var}\left[\varphi(c)\right] dc - \eta \int_{c \ge k} \mathrm{U}(c)\, \mathrm{E}\left[\varphi(c)\right] \mathrm{Var}\left[\varphi(c)\right] dc + R_2(c).
\end{aligned} \tag{28}
$$

The second-order Taylor expansion of the first component of Equation (28) with respect to c, around

$\mathbb{E}[c]$, is

$$I = \int \mathrm{E}[\varphi(c)] \left(\mathrm{U}(\mathbb{E}[c]) + \mathrm{U}'(\mathbb{E}[c])(c - \mathbb{E}[c]) + \frac{1}{2}\mathrm{U}''(\mathbb{E}[c])(c - \mathbb{E}[c])^2 + o\left(|c - \mathbb{E}[c]|^2\right) \right) dc$$
$$= \mathrm{U}(\mathbb{E}[c]) + \frac{1}{2}\mathrm{U}''(\mathbb{E}[c])\mathbb{V}\mathrm{ar}[c] + o\left(\mathbb{E}\left[|c - \mathbb{E}[c]|^2\right]\right).$$

By Judd (2003), the first-order Taylor expansion of $\mathrm{U}\left(\sqrt{c^2}\right)$ with respect to c^2, around $\mathbb{E}^2[c]$, can be written[41]

$$\mathrm{U}(c) = \begin{cases} \mathrm{U}(\mathbb{E}[c]) - \mathrm{U}'(\mathbb{E}[c]) \dfrac{1}{2\mathbb{E}[c]}\left(c^2 - \mathbb{E}^2[c]\right) + o\left(|c - \mathbb{E}[c]|^2\right), & c < k \\ \mathrm{U}(\mathbb{E}[c]) + \mathrm{U}'(\mathbb{E}[c]) \dfrac{1}{2\mathbb{E}[c]}\left(c^2 - \mathbb{E}^2[c]\right) + o\left(|c - \mathbb{E}[c]|^2\right), & c \geq k. \end{cases}$$

Since $\mathbb{E}[c]$ is relatively close to k and $\mathrm{U}(k) = 0$, then $\mathrm{U}(\mathbb{E}[c]) \approx 0$. Therefore, accounting for the sign switch of $\mathrm{E}[\varphi(x)]$ when moving from a negative to a positive utility across k (Wakker and Tversky, 1993),[42] the last two terms in Equation (28) can be written[43]

$$II = -\eta \int \mathrm{E}[\varphi(c)] \mathrm{Var}[\varphi(c)] \left(\mathrm{U}'(\mathbb{E}[c]) \frac{1}{2\mathbb{E}[c]}\left(c^2 - \mathbb{E}^2[c]\right) \right) dc + R_{II,2}(c),$$

where $R_{II,2} = o\left(\int \mathrm{E}\left[|\varphi(c) - \mathrm{E}[\varphi(c)]|^3\right] cdc\right)$ as $\int |\varphi(c) - \mathrm{E}[\varphi(c)]| \, dc \to 0$ (see Izhakian, 2020, Theorem 2). Since, by Lemma 2, $\mathrm{Var}[\varphi(c)]$ and $\left(c^2 - \mathbb{E}^2[c]\right)$ are uncorrelated,

$$II = -\eta \mathrm{U}'(\mathbb{E}[c]) \frac{1}{2\mathbb{E}[c]} \int \mathrm{E}[\varphi(c)]\left(c^2 - \mathbb{E}^2[c]\right) dc \int \mathrm{E}[\varphi(c)] \mathrm{Var}[\varphi(c)] \, dc + R_{II,2}(c).$$

Combining the LHS and the RHS (I and II), and substituting Equation (1) for $\mathrm{U}(\mathbb{E}[c])$ provides

$$\mathcal{K} = \gamma \tfrac{1}{2} \mathbb{V}\mathrm{ar}[c] + \eta \tfrac{1}{2} \tfrac{1}{\mathbb{E}[c]} \mathbb{V}\mathrm{ar}[c] \mho^2[c] + R_2(c).$$

By I and II, $R_2(c) = o\left(\mathbb{E}\left[|c - \mathbb{E}[c]|^2\right]\right) + o\left(\int \mathrm{E}\left[|\varphi(c) - \mathrm{E}[\varphi(c)]|^3\right] cdc\right)$. Since, $o\left(\int \mathrm{E}\left[|\varphi(c) - \mathrm{E}[\varphi(c)]|^3\right] cdc\right)$ is equivalent to $o\left(\mathbb{E}\left[|c - \mathbb{E}[c]|^3\right]\right)$, then $R_2(c) = o\left(\mathbb{E}\left[|c - \mathbb{E}[c]|^2\right]\right)$ as $|c - \mathbb{E}[c]| \to 0$. □

Proof of Theorem 2. To simplify notations, the superscript i, denoting an investor, is omitted. By Equation (9), $\dfrac{\partial \mathrm{F}}{\partial \mathbb{E}[c]} = 1 + \eta \dfrac{1}{2} \dfrac{\mathbb{V}\mathrm{ar}[c]}{\mathbb{E}^2[c]} \mho^2[c] > 0$, $\dfrac{\partial \mathrm{F}}{\partial \mathbb{S}\mathrm{td}[c]} = -\gamma \mathbb{S}\mathrm{td}[c] - \eta \dfrac{\mathbb{S}\mathrm{td}[c]}{\mathbb{E}[c]} \mho^2[c] \leq 0$, and $\dfrac{\partial \mathrm{F}}{\partial \mho[c]} = -\eta \dfrac{\mathbb{V}\mathrm{ar}[c]}{\mathbb{E}[c]} \mho[c] \leq 0$. □

Proof of Theorem 3. To simplify notations, the superscript i, denoting an investor, is omitted.

[41] Judd (2003) shows that the Taylor expansion of $f(x)$ can be improved by the change of variable $x = h(y)$, i.e., writing x as a non-linear transformation of y, to obtain h-linearization, and expanding $f(h(y))$ with respect to y. Here, the linearization is applied by c^2.

[42] By Wakker and Tversky (1993), the sign switch is determined by a linear shift, which ensures that capacities (perceived probabilities) are nonnegative. This can also be viewed through the Choquet integration over negative functions, which takes the form $\int f dQ = \int (f + c) \, dQ - c$, where $c > 0$ such that $f + c > 0$.

[43] Note that II is of the order of cubic probabilities. Thus, it is smaller by two orders of magnitude than the probabilities, and therefore smaller by two orders of magnitude than I.

Let $\mathbf{x} \in \mathbb{R}^n$ be the portfolio consisting of the risky and ambiguous assets, and $\mathbf{r} \in \mathbb{R}^n$ be the vector of returns on these assets. The Lagrangian in Equation (14) can then be written explicitly

$$\begin{aligned} \mathcal{L}(x_f, \mathbf{x}, \theta) &= \mathbb{E}\left[x_f(1+r_f) + \mathbf{x}'(\mathbf{1}+\mathbf{r})\right] - \gamma\frac{1}{2}\mathbb{V}\mathrm{ar}\left[\mathbf{x}'\mathbf{r}\right] \\ &\quad -\eta\frac{1}{2}\frac{\mathbb{V}\mathrm{ar}\left[\mathbf{x}'\mathbf{r}\right]}{\mathbb{E}\left[x_f(1+r_f) + \mathbf{x}'(\mathbf{1}+\mathbf{r})\right]}\mho^2\left[\mathbf{x}'\mathbf{r}\right] - \theta\left((\bar{x}_f - x_f) + (\bar{\mathbf{x}} - \mathbf{x})'\mathbf{1}\right). \end{aligned} \tag{29}$$

By Lemma 1, the variance of returns can be written $\mathbb{V}\mathrm{ar}\left[\mathbf{x}'\mathbf{r}\right] = \mathbf{x}'\mathrm{E}\left[\boldsymbol{\Sigma}_{\mathbf{rr}}\right]\mathbf{x} + \mathbf{x}'\boldsymbol{\Sigma}_{\boldsymbol{\mu}_\mathbf{r}\boldsymbol{\mu}_\mathbf{r}}\mathbf{x}$. The first order condition of the Lagrangian is, therefore,

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\mathbf{x}} &= \mathbb{E}\left[\mathbf{1}+\mathbf{r}\right] - \left(\mathrm{E}\left[\boldsymbol{\Sigma}_{\mathbf{rr}}\right] + \mathrm{E}\left[\boldsymbol{\Sigma}'_{\mathbf{rr}}\right] + \boldsymbol{\Sigma}_{\boldsymbol{\mu}_\mathbf{r}\boldsymbol{\mu}_\mathbf{r}} + \boldsymbol{\Sigma}'_{\boldsymbol{\mu}_\mathbf{r}\boldsymbol{\mu}_\mathbf{r}}\right)\mathbf{x}\gamma\frac{1}{2} \\ &\quad - \left(\mathrm{E}\left[\boldsymbol{\Sigma}_{\mathbf{rr}}\right] + \mathrm{E}\left[\boldsymbol{\Sigma}'_{\mathbf{rr}}\right] + \boldsymbol{\Sigma}_{\boldsymbol{\mu}_\mathbf{r}\boldsymbol{\mu}_\mathbf{r}} + \boldsymbol{\Sigma}'_{\boldsymbol{\mu}_\mathbf{r}\boldsymbol{\mu}_\mathbf{r}}\right)\mathbf{x}\eta\frac{1}{2}\frac{\mho^2\left[\mathbf{x}'\mathbf{r}\right]}{\mathbb{E}\left[x_f(1+r_f) + \mathbf{x}'(\mathbf{1}+\mathbf{r})\right]} \\ &\quad + \mathbb{E}\left[\mathbf{1}+\mathbf{r}\right]\eta\frac{1}{2}\frac{\mathbb{V}\mathrm{ar}\left[\mathbf{x}'\mathbf{r}\right]}{\mathbb{E}^2\left[x_f(1+r_f) + \mathbf{x}'(\mathbf{1}+\mathbf{r})\right]}\mho^2\left[\mathbf{x}'\mathbf{r}\right] \\ &\quad - \frac{\partial\mho^2\left[\mathbf{x}'\mathbf{r}\right]}{\partial(\mathbf{x})}\eta\frac{1}{2}\frac{\mathbb{V}\mathrm{ar}\left[\mathbf{x}'\mathbf{r}\right]}{\mathbb{E}\left[x_f(1+r_f) + \mathbf{x}'(\mathbf{1}+\mathbf{r})\right]} + \mathbf{1}\theta = 0. \end{aligned} \tag{30}$$

Since all covariance matrices are symmetric, and by Lemma 6,

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\mathbf{x}} &= \mathbb{E}\left[\mathbf{1}+\mathbf{r}\right]\left(1 + \eta\frac{1}{2}\frac{\mathbb{V}\mathrm{ar}\left[\mathbf{x}'\mathbf{r}\right]}{\mathbb{E}^2\left[x_f(1+r_f) + \mathbf{x}'(\mathbf{1}+\mathbf{r})\right]}\mho^2\left[\mathbf{x}'\mathbf{r}\right]\right) \\ &\quad - \left(\mathrm{E}\left[\boldsymbol{\Sigma}_{\mathbf{rr}}\right] + \boldsymbol{\Sigma}_{\boldsymbol{\mu}_\mathbf{r}\boldsymbol{\mu}_\mathbf{r}}\right)\mathbf{x}\left(\gamma + \eta\frac{\mho^2\left[\mathbf{x}'\mathbf{r}\right]}{\mathbb{E}\left[x_f(1+r_f) + \mathbf{x}'(\mathbf{1}+\mathbf{r})\right]}\right) \\ &\quad - \Theta\mathbf{x}\eta\frac{\mathbb{V}\mathrm{ar}\left[\mathbf{x}'\mathbf{r}\right]}{\mathbb{E}\left[x_f(1+r_f) + \mathbf{x}'(\mathbf{1}+\mathbf{r})\right]} + \mathbf{1}\theta = 0. \end{aligned} \tag{31}$$

The additional conditions are

$$\frac{\partial\mathcal{L}}{\partial x_f} = 1 + r_f + (1+r_f)\eta\frac{1}{2}\frac{\mathbb{V}\mathrm{ar}\left[\mathbf{x}'\mathbf{r}\right]}{\mathbb{E}^2\left[x_f(1+r_f) + \mathbf{x}'(\mathbf{1}+\mathbf{r})\right]}\mho^2\left[\mathbf{x}'\mathbf{r}\right] + \theta = 0 \tag{32}$$

and

$$\frac{\partial\mathcal{L}}{\partial\theta} = (\bar{x}_f - x_f) + (\bar{\mathbf{x}} - \mathbf{x})'\mathbf{1} = 0. \tag{33}$$

By Equation (32),

$$\theta = -(1+r_f)\left(1 + \eta\frac{1}{2}\frac{\mathbb{V}\mathrm{ar}\left[\mathbf{x}'\mathbf{r}\right]}{\mathbb{E}^2\left[x_f(1+r_f) + \mathbf{x}'(\mathbf{1}+\mathbf{r})\right]}\mho^2\left[\mathbf{x}'\mathbf{r}\right]\right). \tag{34}$$

Substituting for θ into Equation (31), provides

$$\begin{aligned} 0 &= \left(\mathbb{E}\left[\mathbf{r}\right] - \mathbf{1}r_f\right)\left(1 + \eta\frac{1}{2}\frac{\mathbb{V}\mathrm{ar}\left[\mathbf{x}'\mathbf{r}\right]}{\mathbb{E}^2\left[x_f(1+r_f) + \mathbf{x}'(\mathbf{1}+\mathbf{r})\right]}\mho^2\left[\mathbf{x}'\mathbf{r}\right]\right) \\ &\quad - \left(\mathrm{E}\left[\boldsymbol{\Sigma}_{\mathbf{rr}}\right] + \boldsymbol{\Sigma}_{\boldsymbol{\mu}_\mathbf{r}\boldsymbol{\mu}_\mathbf{r}}\right)\mathbf{x}\left(\gamma + \eta\frac{\mho^2\left[\mathbf{x}'\mathbf{r}\right]}{\mathbb{E}\left[x_f(1+r_f) + \mathbf{x}'(\mathbf{1}+\mathbf{r})\right]}\right) - \Theta\mathbf{x}\eta\frac{\mathbb{V}\mathrm{ar}\left[\mathbf{x}'\mathbf{r}\right]}{\mathbb{E}\left[x_f(1+r_f) + \mathbf{x}'(\mathbf{1}+\mathbf{r})\right]}. \end{aligned} \tag{35}$$

Organizing terms completes the proof. □

Proof of Theorem 4. Immediately obtained from Equation (15). □

Proof of Theorem 5. (*i*) By Theorem 4, the relative proportion of any two risky and ambiguous assets is the same for any investor, independent of their preferences or wealth. Therefore, for any investor, the holding of n risky and ambiguous assets is equivalent to holding a portion of the fund containing the risky and ambiguous assets.

(*ii*) Since all risky and ambiguous assets comprising the fund are normally distributed, the return of the fund is normally distributed (e.g. Papoulis and Pillai, 2002), with an uncertain mean and an uncertain variance.

(*iii*) Immediately by Theorem 4. □

Proof of Theorem 6. Suppose that the investor decides to allocate $1-\alpha-\varepsilon$ of her wealth to the risk-free asset, $\alpha > 0$ to the market portfolio, and ε to some asset j. The maximization problem in Equation (20) can then be written

$$\max_{\alpha,\varepsilon} \frac{\mathbb{E}\left[(1-\alpha-\varepsilon)\, r_0 + \alpha r_{\mathbf{m}} + \varepsilon r_j\right] - \mathbb{E}\left[r_0\right]}{\mathbb{S}\mathrm{td}\left[1+(1-\alpha-\varepsilon)\, r_f + \alpha r_{\mathbf{m}} + \varepsilon r_j\right]\sqrt{1+\mho^2\left[1+(1-\alpha-\varepsilon)\, r_f + \alpha r_{\mathbf{m}} + \varepsilon r_j\right]} - \mathbb{S}\mathrm{td}\left[r_0\right]\sqrt{1+\mho^2\left[r_0\right]}}.$$

Since $\mathbb{S}\mathrm{td}\left[\cdot\right]$ and, by Lemma 3, $\mho^2\left[\cdot\right]$ are invariant to linear constant shifts in returns, the maximization problem reduces to

$$\max_{\alpha,\varepsilon} \quad \frac{\mathbb{E}\left[(1-\alpha-\varepsilon)\, r_0 + \alpha r_{\mathbf{m}} + \varepsilon r_j\right] - \mathrm{E}\left[r_0\right]}{\mathbb{S}\mathrm{td}\left[\alpha r_{\mathbf{m}} + \varepsilon r_j\right]\sqrt{1+\mho^2\left[\alpha r_{\mathbf{m}} + \varepsilon r_j\right]} - \mathbb{S}\mathrm{td}\left[r_0\right]\sqrt{1+\mho^2\left[r_0\right]}}.$$

The first order condition with respect to α, evaluated at $\alpha = 1$ and $\varepsilon = 0$, is

$$\begin{aligned} 0 \;=\; & \frac{\mathbb{E}\left[r_{\mathbf{m}} - r_0\right]}{\mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right]\sqrt{1+\mho^2\left[r_{\mathbf{m}}\right]} - \mathbb{S}\mathrm{td}\left[r_0\right]\sqrt{1+\mho^2\left[r_0\right]}} - \\ & \frac{\mathbb{E}\left[r_{\mathbf{m}} - r_0\right]}{\left(\mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right]\sqrt{1+\mho^2\left[r_{\mathbf{m}}\right]} - \mathbb{S}\mathrm{td}\left[r_0\right]\sqrt{1+\mho^2\left[r_0\right]}\right)^2} \frac{\sqrt{1+\mho^2\left[r_{\mathbf{m}}\right]}}{\mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right]} \mathbb{V}\mathrm{ar}\left[r_{\mathbf{m}}\right] - \\ & \frac{\mathbb{E}\left[r_{\mathbf{m}} - r_0\right]}{\left(\mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right]\sqrt{1+\mho^2\left[r_{\mathbf{m}}\right]} - \mathbb{S}\mathrm{td}\left[r_0\right]\sqrt{1+\mho^2\left[r_0\right]}\right)^2} \frac{\mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right]}{\sqrt{1+\mho^2\left[r_{\mathbf{m}}\right]}} \frac{1}{2} \left.\frac{\partial \mho^2\left[\alpha r_{\mathbf{m}} + \varepsilon r_j\right]}{\partial \alpha}\right|_{\alpha=1,\varepsilon=0}. \end{aligned} \tag{36}$$

Let

$$\begin{aligned} \Lambda\left[r_{\mathbf{m}}, r_{\mathbf{m}}\right] \;=\; & \frac{1}{2} \left.\frac{\partial \mho^2\left[\alpha r_{\mathbf{m}} + \varepsilon r_j\right]}{\partial \alpha}\right|_{\alpha=1,\varepsilon=0} \\ =\; & \frac{1}{2} \int \mathrm{E}\left[\frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}} \lambda\left(r \,|\, \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}^2\right)\right] \mathrm{Var}\left[\frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}}\right] dr \,+ \\ & \int \mathrm{E}\left[\frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}}\right] \mathrm{Cov}\left[\frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}}, \frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}} \lambda\left(r \,|\, \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}^2\right)\right] dr, \end{aligned}$$

where

$$\lambda\left(r \,|\, \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}^2\right) \;=\; \frac{r\left(r-\mu_{\mathbf{m}}\right)}{\sigma_{\mathbf{m}}^2} - 1.$$

By Lemma 5,

$$\Lambda\left[r_{\mathbf{m}}, r_{\mathbf{m}}\right] = \int \mathrm{E}\left[\frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}}\right] \mathrm{Cov}\left[\frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}}, \frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}} \lambda\left(r \mid \mu_{\mathbf{m}}, \sigma_{\mathbf{m}}^2\right)\right] dr. \tag{37}$$

Thus, by Equation (36),

$$\mathbb{E}\left[r_{\mathbf{m}} - r_0\right] = \mathbb{E}\left[r_{\mathbf{m}} - r_0\right]\left(1 + \frac{\Lambda\left[r_{\mathbf{m}}, r_{\mathbf{m}}\right]}{1 + \mho^2\left[r_{\mathbf{m}}\right]}\right) \frac{\mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right]\sqrt{1 + \mho^2\left[r_{\mathbf{m}}\right]}}{\mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right]\sqrt{1 + \mho^2\left[r_{\mathbf{m}}\right]} - \mathbb{S}\mathrm{td}\left[r_0\right]\sqrt{1 + \mho^2\left[r_0\right]}}. \tag{38}$$

The first order condition with respect to ε, evaluated at $\alpha = 1$ and $\varepsilon = 0$, is

$$\begin{aligned} 0 = {} & \frac{\mathbb{E}\left[r_j - r_0\right]}{\mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right]\sqrt{1 + \mho^2\left[r_{\mathbf{m}}\right]} - \mathbb{S}\mathrm{td}\left[r_0\right]\sqrt{1 + \mho^2\left[r_0\right]}} - \\ & \frac{\mathbb{E}\left[r_{\mathbf{m}} - r_0\right]}{\left(\mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right]\sqrt{1 + \mho^2\left[r_{\mathbf{m}}\right]} - \mathbb{S}\mathrm{td}\left[r_0\right]\sqrt{1 + \mho^2\left[r_0\right]}\right)^2} \frac{\sqrt{1 + \mho^2\left[r_{\mathbf{m}}\right]}}{\mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right]} \mathbb{C}\mathrm{ov}\left[r_{\mathbf{m}}, r_j\right] - \\ & \frac{\mathbb{E}\left[r_{\mathbf{m}} - r_0\right]}{\left(\mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right]\sqrt{1 + \mho^2\left[r_{\mathbf{m}}\right]} - \mathbb{S}\mathrm{td}\left[r_0\right]\sqrt{1 + \mho^2\left[r_0\right]}\right)^2} \frac{\mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right]}{\sqrt{1 + \mho^2\left[r_{\mathbf{m}}\right]}} \frac{1}{2} \left.\frac{\partial \mho^2\left[\alpha r_{\mathbf{m}} + \varepsilon r_j\right]}{\partial \varepsilon}\right|_{\alpha=1,\varepsilon=0} \end{aligned} \tag{39}$$

Let

$$\begin{aligned} \Lambda\left[r_{\mathbf{m}}, r_j\right] &= \frac{1}{2} \left.\frac{\partial \mho^2\left[\alpha r_{\mathbf{m}} + \varepsilon r_j\right]}{\partial \varepsilon}\right|_{\alpha=1,\varepsilon=0} \\ &= \frac{1}{2} \int \mathrm{E}\left[\frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}} \lambda\left(r \mid \mu_{\mathbf{m}}, \mu_j, \sigma_{\mathbf{m}}^2, \sigma_{\mathbf{m},j}\right)\right] \mathrm{Var}\left[\frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}}\right] dr + \\ & \quad \int \mathrm{E}\left[\frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}}\right] \mathrm{Cov}\left[\frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}}, \frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}} \lambda\left(r \mid \mu_{\mathbf{m}}, \mu_j, \sigma_{\mathbf{m}}^2, \sigma_{\mathbf{m},j}\right)\right] dr, \end{aligned}$$

where

$$\lambda\left(r \mid \mu_{\mathbf{m}}, \mu_j, \sigma_{\mathbf{m}}^2, \sigma_{\mathbf{m},j}\right) = \frac{r - \mu_{\mathbf{m}}}{\sigma_{\mathbf{m}}^2}\left(\frac{\sigma_{\mathbf{m},j}}{\sigma_{\mathbf{m}}^2}\left(r - \mu_{\mathbf{m}}\right) + \mu_j\right) - \frac{\sigma_{\mathbf{m},j}}{\sigma_{\mathbf{m}}^2}.$$

By Lemma 5,

$$\Lambda\left[r_{\mathbf{m}}, r_j\right] = \int \mathrm{E}\left[\frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}}\right] \mathrm{Cov}\left[\frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}}, \frac{e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}}}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}} \lambda\left(r \mid \mu_{\mathbf{m}}, \mu_j, \sigma_{\mathbf{m}}^2, \sigma_{\mathbf{m},j}\right)\right] dr. \tag{40}$$

Thus, by Equation (39),

$$\mathbb{E}\left[r_j - r_0\right] = \mathbb{E}\left[r_{\mathbf{m}} - r_0\right]\left(\frac{\mathbb{C}\mathrm{ov}\left[r_{\mathbf{m}}, r_j\right]}{\mathbb{V}\mathrm{ar}\left[r_{\mathbf{m}}\right]} + \frac{\Lambda\left[r_{\mathbf{m}}, r_j\right]}{1 + \mho^2\left[r_{\mathbf{m}}\right]}\right) \frac{\mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right]\sqrt{1 + \mho^2\left[r_{\mathbf{m}}\right]}}{\mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right]\sqrt{1 + \mho^2\left[r_{\mathbf{m}}\right]} - \mathbb{S}\mathrm{td}\left[r_0\right]\sqrt{1 + \mho^2\left[r_0\right]}}. \tag{41}$$

The ratio of Equations (41) and (38) is

$$\frac{\mathbb{E}\left[r_j - r_0\right]}{\mathbb{E}\left[r_{\mathbf{m}} - r_0\right]} = \frac{\frac{\mathbb{C}\mathrm{ov}\left[r_{\mathbf{m}}, r_j\right]}{\mathbb{V}\mathrm{ar}\left[r_{\mathbf{m}}\right]} + \frac{\Lambda\left[r_{\mathbf{m}}, r_j\right]}{1 + \mho^2\left[r_{\mathbf{m}}\right]}}{1 + \frac{\Lambda\left[r_{\mathbf{m}}, r_{\mathbf{m}}\right]}{1 + \mho^2\left[r_{\mathbf{m}}\right]}},$$

which implies

$$\frac{\mathbb{E}\left[r_j - r_0\right]}{\mathbb{E}\left[r_{\mathbf{m}} - r_0\right]} = \frac{\mathbb{C}\mathrm{ov}\left[r_{\mathbf{m}}, r_j\right]}{\mathbb{V}\mathrm{ar}\left[r_{\mathbf{m}}\right]} \frac{1 + \mho^2\left[r_{\mathbf{m}}\right]}{1 + \mho^2\left[r_{\mathbf{m}}\right] + \Lambda\left[r_{\mathbf{m}}, r_{\mathbf{m}}\right]} + \frac{\Lambda\left[r_{\mathbf{m}}, r_j\right]}{1 + \mho^2\left[r_{\mathbf{m}}\right] + \Lambda\left[r_{\mathbf{m}}, r_{\mathbf{m}}\right]}.$$

By Corollary 3,

$$\mathbb{E}\left[r_0\right] = r_f + \left(\mathbb{E}\left[r_{\mathbf{m}}\right] - r_f\right) \sqrt{\frac{\mho^2\left[r_{\mathbf{m}}\right]}{1 + \mho^2\left[r_{\mathbf{m}}\right]}}.$$

Substituting $\mathbb{E}\left[r_0\right]$ and organizing terms completes the proof. □

Proof of Proposition 1. The portfolio return is the proportion-weighted average return of the assets in the portfolio. The covariance of the portfolio return and the market return is the weighted average of the covariances of the assets return and the market return. Thus, by Theorem 6,

$$\beta^{\mathtt{R}}_{\mathbf{x}} = \frac{\mathbb{C}\mathrm{ov}\left[r_{\mathbf{m}}, \mathbf{x}'\mathbf{r}\right]}{\mathbb{V}\mathrm{ar}\left[r_{\mathbf{m}}\right]} \frac{1 + \mho^2\left[r_{\mathbf{m}}\right]}{1 + \mho^2\left[r_{\mathbf{m}}\right] + \Lambda\left[r_{\mathbf{m}}, r_{\mathbf{m}}\right]} = \mathbf{x}' \frac{\mathbb{C}\mathrm{ov}\left[r_{\mathbf{m}}, \mathbf{r}\right]}{\mathbb{V}\mathrm{ar}\left[r_{\mathbf{m}}\right]} \frac{1 + \mho^2\left[r_{\mathbf{m}}\right]}{1 + \mho^2\left[r_{\mathbf{m}}\right] + \Lambda\left[r_{\mathbf{m}}, r_{\mathbf{m}}\right]} = \mathbf{x}'\boldsymbol{\beta}^{\mathtt{R}}.$$

By the same consideration, the λ of a portfolio is the proportion-weighted average λs of the assets in the portfolio. Thereby, the same holds true for the Λ and $\beta^{\mathtt{A}}_{\mathbf{x}}$ of the portfolio: $\Lambda\left[r_{\mathbf{m}}, \mathbf{x}'\mathbf{r}\right] = \mathbf{x}'\Lambda\left[r_{\mathbf{m}}, \mathbf{r}\right]$ and $\beta^{\mathtt{R}}_{\mathbf{x}} = \mathbf{x}'\boldsymbol{\beta}^{\mathtt{A}}$. □

Proof of Corollary 1. In equilibrium, the market clearing condition in Equation (7) holds true for every asset. Thus, $x_j > 0$ and the relative value of the asset is $\frac{x_j}{\mathbf{x}'\mathbf{1}} > 0$. □

Proof of Corollary 2. In equilibrium, the market value of every asset is unique (otherwise, the law one price is violated and arbitrage opportunities exist), which implies that the proportion of each asset in the portfolio is unique. Therefore, the market portfolio and the equilibrium are unique. □

Proof of Corollary 3. The expected rate of return of a portfolio, whose risk tends to zero, can be written

$$\mathbb{E}\left[r_0\right] = \lim_{\alpha \to 0} \left(1 - \alpha\right) r_f + \alpha \mathbb{E}\left[r_{\mathbf{m}}\right].$$

Thus, by Equation (18) and Lemma 3,

$$\begin{aligned}
\mathbb{E}\left[r_0\right] &= r_f + \lim_{\alpha \to 0} \alpha \mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right] \sqrt{1 + \frac{1}{\alpha^2}\mho^2\left[r_{\mathbf{m}}\right]} \frac{\mathbb{E}\left[r_{\mathbf{m}}\right] - r_f}{\mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right] \sqrt{1 + \mho^2\left[r_{\mathbf{m}}\right]}} \\
&= r_f + \left(\mathbb{E}\left[r_{\mathbf{m}}\right] - r_f\right) \sqrt{\frac{\mho^2\left[r_{\mathbf{m}}\right]}{1 + \mho^2\left[r_{\mathbf{m}}\right]}}.
\end{aligned}$$

The consolidated risk and ambiguity associated with $\mathbb{E}\left[r_0\right]$ satisfies

$$\mathbb{S}\mathrm{td}\left[r_0\right] \sqrt{1 + \mho^2\left[r_0\right]} = \lim_{\alpha \to 0} \alpha \mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right] \sqrt{1 + \frac{1}{\alpha^2}\mho^2\left[r_{\mathbf{m}}\right]} = \mathbb{S}\mathrm{td}\left[r_{\mathbf{m}}\right] \mho\left[r_{\mathbf{m}}\right].$$ □

Proof of Corollary 4. In the absence of ambiguity, $\mu_{\mathbf{m}}$, $\sigma_{\mathbf{m}}$, μ_j, σ_j and $\sigma_{\mathbf{m},j}$ are constants. Thus, $\mho^2\left[r_{\mathbf{m}}\right] = 0$ and $\Lambda\left[r_{\mathbf{m}}, r_j\right] = 0$. Equations (22), (23), and (24) then complete the proof. □

Proof of Corollary 5. Substituting r_f for r_j in Equations (22), (23) and (24) completes the proof. □

Proof of Corollary 6. Immediately by Equations (22), (23) and (24). □

Proof of Corollary 7. Immediately by Equations (22), (23) and (24). □

Proof of Corollary 8. By Equation (21), $\zeta_{\mathbf{m}}^{\mathtt{P}} + \beta_{\mathbf{m}}^{\mathtt{R}}\left(1 - \zeta_{\mathbf{m}}^{\mathtt{P}}\right) + \beta_{\mathbf{m}}^{\mathtt{A}}\left(1 - \zeta_{\mathbf{m}}^{\mathtt{P}}\right) = 1$, which implies that $\beta_{\mathbf{m}}^{\mathtt{R}} + \beta_{\mathbf{m}}^{\mathtt{A}} = 1$. □