Universität Trier

Quantitative Finance and Risk Analysis

Working Paper Series Nº 19 - 6:

Optimal Investment for Retail Investors with Floored and Capped Costs

Christoph Belak

Lukas Mich

Frank T. Seifried

Optimal Investment for Retail Investors with Floored and Capped Costs

Christoph Belak* Lukas Mich† Frank T. Seifried‡

September 3, 2019

We study optimal portfolio decisions for a retail investor that faces proportional costs which are floored and capped at some minimal and maximal cost levels, respectively, in a classical Black-Scholes market. We provide a construction of optimal trading strategies and characterize the value function as the unique viscosity solution of the associated quasi-variational inequalities. Moreover, we numerically investigate the optimal trading regions and find a distinct structure: The no-trading region is vVv-shaped, and all optimal trades for small (large) levels of wealth incur the floored (capped) cost; proportional cost trades occur only in a narrow intermediate wealth regime.

Mathematics Subject Classification (2010): 93E20, 91G80, 49L25.

JEL Classification: G11, C61

Keywords: Portfolio Optimization, Transaction Costs, Retail Investor.

1 Introduction

Classical transaction cost models typically assume that costs are affine functions of the trading volume,[1] i.e. either costs which are proportional to the trading volume, e.g. [12, 13, 18, 25], purely fixed costs, e.g. [1, 2, 15, 20], or a mix of the two, e.g. [2, 7, 14, 19, 20, 22]. In real-world markets, however, retail investors face a different transaction cost structure: Typically, costs

*Technische Universität Berlin, Institute of Mathematics, Straße des 17. Juni 136, 10623 Berlin, Germany, e-mail: belak@math.tu-berlin.de.

†University of Trier, Department IV – Mathematics, Universitätsring 19, 54296 Trier, Germany, e-mail: mich@uni-trier.de.

‡University of Trier, Department IV – Mathematics, Universitätsring 19, 54296 Trier, Germany, e-mail: seifried@uni-trier.de.

[1] Here, we think of transaction costs as brokerage fees and do not consider implicit costs caused by frictions such as price impact as, e.g., in [16]. Costs that are proportional to the investor's wealth have also been studied in the literature, see, e.g., [21].

are proportional to the trading volume, but additionally floored and capped at certain minimal and maximal cost levels. Thus the transaction cost as a function of the trading volume is

$$\mathrm{C} : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \qquad \Delta \mapsto \mathrm{C}(\Delta) \triangleq \min\{\max\{\mathrm{C}_{\min}, \mathrm{c}|\Delta|\}, \mathrm{C}_{\max}\}, \tag{1.1}$$

where $0 < \mathrm{C}_{\min} < \mathrm{C}_{\max}$ and $\mathrm{c} \in (0, 1)$. Here, $\mathrm{C}_{\min}$ represents the minimal cost due per trade, c is the proportional cost per unit trading volume, and $\mathrm{C}_{\max}$ is the maximal trading cost. We wish to stress that this is not the same as a fixed-plus-proportional cost with a cap, as (1.1) features a regime where costs are exactly proportional to the trading volume.

In the literature, rather little is known about optimal investment decisions in the presence of transaction costs such as (1.1), except for general results that are agnostic toward the specific cost structure. Thus in [9] a risk-sensitive growth rate criterion is considered for general cost functions, and the authors provide a verification theorem for the value function which allows to construct an optimal investment strategy under the assumption of the existence of a sufficiently smooth solution of the associated Bellman equation.[2] Moreover, in [23], an iterated optimal stopping approach is employed to construct optimal trading strategies for a lifetime consumption-portfolio problem with general cost functions, in which consumption is only allowed to take place at trading dates. While the focus of [23] is on the theoretical study of existence of optimal strategies in their general setting, this paper provides a detailed investigation, including a qualitative analysis of optimal strategies, for the specific cost structure (1.1).

Thus in this paper we consider a retail investor in a Black-Scholes market that faces transaction costs of the form (1.1) and who wishes to maximize expected utility from terminal wealth. Being a retail investor, it is natural to assume that short sales and leverage, i.e. short positions in either the money market account or the stock, are prohibited.[3] In this setting, we use arguments based on the stochastic Perron's method (see [4, 5, 6] for early developments) to characterize the value function as the unique viscosity solution of the associated Bellman equation, which in this setting is represented by a system of quasi-variational inequalities (QVIs). Given this characterization of the value function, we employ the superharmonic function technique introduced in [10] and further refined in [7, 8] to provide an explicit construction of optimal investment strategies in terms of a trading region and post-trade target positions. On the basis of our theoretical results, we provide a detailed numerical investigation of the structure and shape of the trading regions and post-trade target positions leading to the optimal investment strategy.

Our numerical results exhibit a distinct structure of optimal trading strategies: First, we find vVv-shaped, rather than classical V-shaped, no-trading regions. In particular, there exist portfolio positions for which it is not optimal to trade, even though there are both portfolios closer to and further away from the frictionless optimizer for which it *is* optimal to make a transaction. Second, we identify some novel boundary effects for short time horizons. Third and most

[2] Note, however, that since the cost in (1.1) is bounded above, one expects this to lead to a degenerate solution for a growth rate criterion. In particular, it seems to be difficult to verify the assumptions of the verification theorem in [9] for the costs in (1.1).

[3] Borrowing from their cash accounts to attain leverage on their stock positions is difficult for small retail investors; while it is possible for retail investors that are able to pledge sufficient additional assets, even in that case the borrowing rate is typically significantly higher than the rate earned on cash deposits.

importantly, we find that optimal transactions can be characterized via three distinct regimes: For moderate amounts of wealth, the retail investor optimally trades only at the floored costs, with target portfolios distinct from the frictionless optimizer. By contrast, investors with large levels of wealth trade at the capped costs and onto the frictionless optimal position, thus effectively facing fixed transaction costs. For intermediate levels of wealth, proportional costs also occur at the optimum; for a real-world parametrization, this regime obtains in a narrow range of cash holdings between 100,000\$ and 135,000\$ for sell orders and between 145,000\$ and 190,000\$ for buy orders.

The remainder of this article is structured as follows. In Section 2 we introduce the model and provide the mathematical formulation of the retail investor's portfolio optimization problem. In Section 3 we state the main mathematical results of this paper. Section 4 contains our main qualitative findings via a detailed analysis of optimal trading strategies. Finally, Section 5 provides the proof of the viscosity characterization and the construction of optimal strategies.

2 Retail Investor Portfolio Problem

In all that follows, we fix a filtered probability space $(\Omega, \mathfrak{A}, \mathfrak{F}, \mathbb{P})$ where $\mathfrak{F} = \{\mathfrak{F}_t\}_{t\in[0,T]}$ that supports a standard $\mathfrak{F}$-Wiener process $W = \{W_t\}_{t\in[0,T]}$ and satisfies the usual conditions.

Financial Market and Transaction Costs. We consider a retail investor that has access to a classical Black-Scholes market $P = (P^0, P^1)$ consisting of a money market account $P^0 = \{P^0_t\}_{t\in[0,T]}$ with risk-free rate $r \in \mathbb{R}$ and a stock (or stock index) $P^1 = \{P^1_t\}_{t\in[0,T]}$ with drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$. The dynamics of P^0 and P^1 are thus given by

$$\mathrm{d}P^0_t = rP^0_t\,\mathrm{d}t \qquad \text{and} \qquad \mathrm{d}P^1_t = \mu P^1_t\,\mathrm{d}t + \sigma P^1_t\,\mathrm{d}W_t, \qquad t \in [0,T].$$

The investor faces transaction costs that are proportional to the volume traded, with both a floor (i.e., a minimum cost charged per trade) and a cap (i.e., a maximal cost amount). More precisely, the transaction cost incurred by a transaction of size Δ is given by the cost function[4]

$$\mathrm{C} : \mathbb{R} \to \mathbb{R}_+, \qquad \Delta \mapsto \mathrm{C}(\Delta) \triangleq \min\{\max\{\mathrm{C}_{\min}, \mathrm{c}|\Delta|\}, \mathrm{C}_{\max}\} \tag{2.1}$$

where $0 < \mathrm{C}_{\min} < \mathrm{C}_{\max}$ denote the minimal and maximal transaction costs and $\mathrm{c} \in (0,1)$ represents the proportional cost factor. Portfolio positions are specified as vectors $x = (x_0, x_1) \in \mathbb{R}^2$ where x_0 and x_1 represent the dollar amounts invested in the money market account and in the stock, respectively. The sets of portfolios without leveraged or short positions in the stock (equivalently, without short positions in either the money market account or the stock) and the set of non-zero portfolios without leverage or shorting are defined by

$$\overline{\mathcal{S}} \triangleq \mathbb{R}^2_+ \qquad \text{and} \qquad \mathcal{S} \triangleq \mathbb{R}^2_+ \setminus \{0\}.$$

[4]Note that $\mathrm{C}(0) = \mathrm{C}_{\min}$, i.e. a degenerate transaction of size zero leads to a strictly positive cost. We shall see below that such degenerate transactions are strictly suboptimal for the retail investor's portfolio problem. However, they are admissible in the real world, and it is convenient to include them mathematically to ensure compactness of the set of feasible transactions.

Trading Strategies and Portfolio Dynamics. It is well-known that, in the presence of transaction costs that are bounded from below, trading with infinite activity leads to immediate bankruptcy. Hence a *trading strategy* is specified by a sequence $\Lambda = \{(\tau_k, \Delta_k)\}_{k\in\mathbb{N}}$, where $\{\tau_k\}_{k\in\mathbb{N}}$ is an increasing sequence of $\mathfrak{F}$-stopping times representing the trading dates, and each Δ_k, $k \in \mathbb{N}$, is an $\mathbb{R}$-valued $\mathfrak{F}_{\tau_k}$-measurable random variable specifying the volume of the k^{th} trade. Starting from an initial portfolio position $x \in \overline{\mathcal{S}}$ at time $t \in [0, T]$, the dynamics of the retail investor's portfolio $X = X^{t,x,\Lambda} = \{X_s^{t,x,\Lambda}\}_{s\in[t,T]}$ are given by

$$X_s^0 = x_0 + \int_t^s rX_u^0 \,\mathrm{d}u - \sum_{k=1}^{\infty} \big[\Delta_k + \mathrm{C}(\Delta_k)\big] \mathbb{1}_{\{\tau_k \le s\}}, \qquad s \in [t, T],$$

$$X_s^1 = x_1 + \int_t^s \mu X_u^1 \,\mathrm{d}u + \int_t^s \sigma X_u^1 \,\mathrm{d}W_u + \sum_{k=1}^{\infty} \Delta_k \mathbb{1}_{\{\tau_k \le s\}}, \qquad s \in [t, T].$$

We furthermore set $X_{t-}^{t,x,\Lambda} \triangleq x$ to account for the possibility of a trade at time t. A trading strategy Λ is called *admissible* for the initial portfolio position (t, x) if it does not involve leverage or borrowing, i.e.

$$\tau_1 \ge t \qquad \text{and} \qquad X_s^{t,x,\Lambda} \in \overline{\mathcal{S}}, \qquad s \in [t, T].$$

The set of all trading strategies that are admissible for the initial position (t, x) is denoted by $\mathcal{A}(t, x)$.

Remark. Since transaction costs are bounded from below, admissibility implies in particular that the investor trades only finitely many times a.s., i.e. we have

$$\mathbb{P}\Big[\lim_{k\to\infty} \tau_k > T\Big] = 1 \qquad \text{for all } \Lambda = \{(\tau_k, \Delta_k)\}_{k\in\mathbb{N}} \in \mathcal{A}(t, x); \tag{2.2}$$

see [7, Lemma A.4] for a formal argument. Moreover, since leveraged positions are ruled out, a standard moments estimate for SDEs, see [7, Lemma A.5], yields a constant $M > 0$ such that

$$\sup_{\Lambda\in\mathcal{A}(t,x)} \mathbb{E}\Big[\sup_{s\in[t,T]} \big|X_s^{t,x,\Lambda}\big|^2\Big] \le M\big(1 + |x|^2\big), \qquad (t, x) \in [0, T] \times \overline{\mathcal{S}}. \tag{2.3}$$

Portfolio Problem. The retail investor aims to maximize expected utility from liquid wealth at terminal time T. Her risk preferences are captured by a power utility function with relative risk aversion parameter $1 - p$ where $p \in (0, 1)$, so the investor's utility function for liquid wealth is given by

$$U : \mathbb{R}_+ \to \mathbb{R}_+, \qquad \ell \mapsto U(\ell) \triangleq \tfrac{1}{p}\ell^p. \tag{2.4}$$

We denote by $\mathrm{L}(x)$ the liquidation value of a portfolio $x \in \overline{\mathcal{S}}$, where

$$\mathrm{L} : \overline{\mathcal{S}} \to \mathbb{R}_+, \quad x \mapsto \mathrm{L}(x) \triangleq x_0 + \big(x_1 - \mathrm{C}(-x_1)\big)^+.$$

This definition of L guarantees that the investor liquidates her stock position only in case this does not induce a net loss, i.e., the revenue from selling is at least as big as the trading cost; note that this is the case if and only if the position in the stock exceeds the minimal cost amount. Conversely, stocks being limited liability securities, the investor cannot be forced to sell them, and she will thus not do so if she were to incur a loss in case she did. Setting $U_{\mathrm{L}} \triangleq U \circ \mathrm{L}$, the retail investor's portfolio problem reads

$$\mathcal{V}(t,x) \triangleq \sup_{\Lambda \in \mathcal{A}(t,x)} \mathbb{E}\Big[U_{\mathrm{L}}\big(X_T^{t,x,\Lambda}\big)\Big], \qquad (t,x) \in [0,T] \times \overline{\mathcal{S}}. \tag{2.5}$$

3 Mathematical Results

In this section, we state and discuss the main mathematical results of this article; their proofs are deferred to Section 5 below.

Characterization of the Value Function. Our first main result characterizes the value function

$$\mathcal{V} : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}_+, \qquad (t,x) \mapsto \mathcal{V}(t,x) \triangleq \sup_{\Lambda \in \mathcal{A}(t,x)} \mathbb{E}\Big[U_{\mathrm{L}}\big(X_T^{t,x,\Lambda}\big)\Big]$$

as the unique continuous viscosity solution of the dynamic programming equation associated with the retail investor's portfolio optimization problem. In order to state this result, we need to introduce some notation. First, we denote the *infinitesimal generator* of the uncontrolled state process by

$$\mathcal{L}[\varphi](t,x) \triangleq -\frac{\partial \varphi}{\partial t}(t,x) - r x_0 \frac{\partial \varphi}{\partial x_0}(t,x) - \mu x_1 \frac{\partial \varphi}{\partial x_1}(t,x) - \frac{1}{2}\sigma^2 x_1^2 \frac{\partial^2 \varphi}{\partial x_1^2}(t,x)$$

for all $(t,x) \in [0,T] \times \overline{\mathcal{S}}$ and every sufficiently smooth function $\varphi : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$. Second, given the cost function (2.1), the transaction $\Delta \in \mathbb{R}$ shifts a portfolio $x = (x_0, x_1) \in \overline{\mathcal{S}}$ to the new position $\Gamma(x, \Delta)$, where the *rebalancing function* Γ is given by

$$\Gamma : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2, \qquad (x, \Delta) \mapsto \Gamma(x,\Delta) \triangleq \big(x_0 - \Delta - \mathrm{C}(\Delta), x_1 + \Delta\big).$$

A transaction Δ is called *feasible* for the portfolio $x \in \overline{\mathcal{S}}$ if it does not result in a short position in either asset, and we denote the set of all feasible transactions by[5]

$$\mathcal{D}(x) \triangleq \big\{\Delta \in \mathbb{R} : \Gamma(x,\Delta) \in \overline{\mathcal{S}}\big\}.$$

[5]Since C is continuous, it is clear that $\mathcal{D}(x)$ is compact. Note, however, that $\mathcal{D}(x)$ may be empty if the position x is not sufficiently valuable; more precisely, it is easily seen that if any transaction is feasible, then liquidating the stock position is also feasible, i.e.

$$\mathcal{D}(x) \neq \emptyset \qquad \text{if and only if} \qquad -x_1 \in \mathcal{D}(x) \qquad \text{if and only if} \qquad x_0 + x_1 \geq \mathrm{C}_{\min}.$$

Moreover, $\mathcal{S}_\emptyset$ denotes the set of portfolio positions for which no feasible transaction exists, i.e.

$$\mathcal{S}_\emptyset \triangleq \big\{x \in \overline{\mathcal{S}} : \mathcal{D}(x) = \emptyset\big\} = \big\{x \in \overline{\mathcal{S}} : x_0 + x_1 < \mathrm{C_{min}}\big\}$$

and $\overline{\mathcal{S}}_\emptyset$ and $\partial\mathcal{S}_\emptyset$ denote the closure and the $\overline{\mathcal{S}}$-relative boundary of $\mathcal{S}_\emptyset$, respectively, i.e.

$$\overline{\mathcal{S}}_\emptyset \triangleq \big\{x \in \overline{\mathcal{S}} : x_0 + x_1 \leq \mathrm{C_{min}}\big\} \quad \text{and} \quad \partial\mathcal{S}_\emptyset \triangleq \overline{\mathcal{S}}_\emptyset \setminus \mathcal{S}_\emptyset = \big\{x \in \overline{\mathcal{S}} : x_0 + x_1 = \mathrm{C_{min}}\big\}.$$

Figure 1 illustrates $\mathcal{S}_\emptyset$ and the set $\{\Gamma(x,\Delta) : \Delta \in \mathcal{D}(x)\}$ of portfolios which can be reached by a transaction from x.

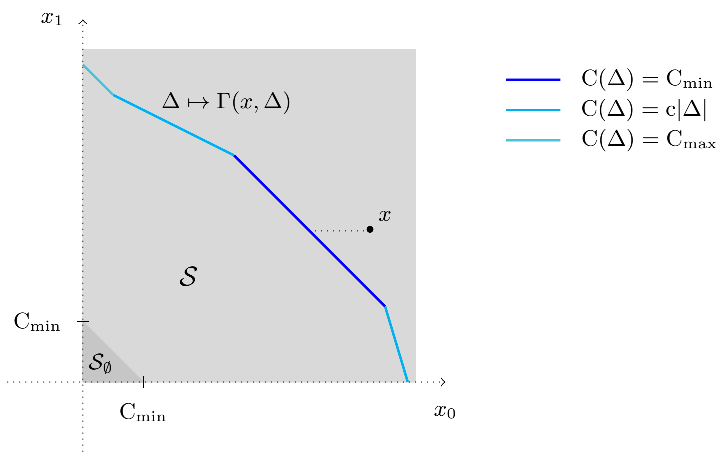

Figure 1 Illustration of the solvency region $\mathcal{S}$, the set $\mathcal{S}_\emptyset$ of portfolios for which no feasible transactions exist, and the set $\{\Gamma(x,\Delta) : \Delta \in \mathcal{D}(x)\}$ of portfolios which can be reached by a feasible transaction from x.

Finally, for every locally bounded function $\varphi : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ we define[6]

$$\mathcal{M}[\varphi](t,x) \triangleq \begin{cases} \sup\limits_{\Delta \in \mathcal{D}(x)} \varphi\big(t, \Gamma(x,\Delta)\big) & \text{if } \mathcal{D}(x) \neq \emptyset \\ \inf\limits_{(\bar{t},\bar{x}) \in [0,T] \times \overline{\mathcal{S}}_\emptyset} \big[\varphi(\bar{t},\bar{x}) - 1\big] & \text{if } \mathcal{D}(x) = \emptyset \end{cases}$$

for all $(t,x) \in [0,T] \times \overline{\mathcal{S}}$. With this notation in place, the first main result of this article can be stated as follows.

Main Result 1 (Viscosity Characterization)**.** *The value function $\mathcal{V}$ defined in* (2.5) *is a continuous viscosity solution of the quasi-variational inequalities (QVIs)*

$$\min\big\{\mathcal{L}[\mathcal{V}](t,x), \mathcal{V}(t,x) - \mathcal{M}[\mathcal{V}](t,x)\big\} = 0, \qquad (t,x) \in [0,T) \times \mathcal{S}. \tag{3.1}$$

[6]The definition of $\mathcal{M}[\varphi](t,x)$ in the case $\mathcal{D}(x) = \emptyset$ is mainly a technical convention. It is chosen to guarantee that $\varphi(t,x) > \mathcal{M}[\varphi](t,x)$ on $\mathcal{S}_\emptyset$ and that $\mathcal{M}$ preserves upper semicontinuity; see Lemma 5.2 below.

Moreover, $\mathcal{V}$ is unique in the class of functions satisfying the growth condition

$$0 \leq \mathcal{V}(t,x) \leq K\big(1+|x|^p\big), \qquad (t,x) \in [0,T] \times \overline{\mathcal{S}}$$

for $p \in (0,1)$ from (2.4) *and some $K > 0$ and the boundary/terminal conditions*

$$\mathcal{V}(t,x) = U_{\mathrm{L}}(x), \qquad (t,x) \in \big([0,T] \times \{0\}\big) \cup \big(\{T\} \times \overline{\mathcal{S}}\big).$$

Proof. The result is a direct consequence of Theorem 5.5, Theorem 5.12 and Theorem 5.13 in Section 5. □

Main Result 1 not only provides a characterization of the value function $\mathcal{V}$ for the retail investor's portfolio problem, but simultaneously demonstrates that $\mathcal{V}$ is continuous. This is the key ingredient required to explicitly construct optimal trading strategies; we elaborate on this in the following.

Construction of Optimal Trading Strategies. We first define a candidate optimal strategy in terms of the continuation region $\mathcal{C}$ and the intervention region $\mathcal{I}$ induced by the value function $\mathcal{V}$, i.e.[7]

$$\mathcal{C} \triangleq \big\{(t,x) \in [0,T] \times \overline{\mathcal{S}} : \mathcal{V}(t,x) > \mathcal{M}[\mathcal{V}](t,x)\big\},$$
$$\mathcal{I} \triangleq \big\{(t,x) \in [0,T] \times \overline{\mathcal{S}} : \mathcal{V}(t,x) = \mathcal{M}[\mathcal{V}](t,x)\big\}.$$

Main Result 1 guarantees that $\mathcal{V} \geq \mathcal{M}[\mathcal{V}]$ and hence the sets $\mathcal{C}$ and $\mathcal{I}$ partition the state space. The candidate optimal strategy is intuitively described as follows: Do nothing as long as the portfolio remains inside the continuation region $\mathcal{C}$; if the intervention region $\mathcal{I}$ is hit, trade a volume that corresponds to a maximizer of $\mathcal{M}[\mathcal{V}]$.
To make this precise, note that since $\mathcal{V}$ is continuous and each of the sets $\mathcal{D}(x)$ is compact, the measurable selection result in [24] implies that there exists a Borel measurable function

$$\delta : [0,T] \times \big(\overline{\mathcal{S}} \setminus \mathcal{S}_{\emptyset}\big) \to \mathbb{R}, \qquad (t,x) \mapsto \delta(t,x),$$

that satisfies

$$\delta(t,x) \in \mathcal{D}(x) \quad \text{and} \quad \mathcal{M}[\mathcal{V}](t,x) = \mathcal{V}\big(t, \Gamma\big(x, \delta(t,x)\big)\big), \quad (t,x) \in [0,T] \times \big(\overline{\mathcal{S}} \setminus \mathcal{S}_{\emptyset}\big).$$

Given an initial time $t \in [0,T]$ and an initial portfolio $x \in \overline{\mathcal{S}}$, we set $\tau_0^* \triangleq t$ and define the candidate optimal trading strategy $\Lambda^* = \{(\tau_k^*, \Delta_k^*)\}_{k \in \mathbb{N}}$ iteratively by setting

$$\tau_k^* \triangleq \inf\big\{u \in (\tau_{k-1}^*, T] : (u, X_{u-}^{t,x,\Lambda^*}) \in \mathcal{I}\big\} \quad \text{and} \quad \Delta_k^* \triangleq \delta\big(\tau_k^*, X_{\tau_k^*-}^{t,x,\Lambda^*}\big) \mathbb{1}_{\{\tau_k^* \leq T\}} \tag{3.2}$$

[7] In the proofs in Section 5, it is mathematically more convenient to use a slightly different line of argument: We first construct a viscosity solution $\mathbb{V}$ of the QVIs (Theorem 5.12) and define the candidate optimal strategy in terms of $\mathbb{V}$; then we establish a verification theorem (Theorem 5.13) and apply it to show simultaneously that (i) $\mathbb{V} = \mathcal{V}$, i.e. $\mathbb{V}$ coincides with the value function; and (ii) the candidate strategy is optimal. The conclusions stated in Main Results 1 and 2 below are, of course, the same.

for each $k \in \mathbb{N}$. Our second main result demonstrates that this iteration is well-defined, and that Λ^* is optimal for the retail investor's portfolio problem.

Main Result 2 (Optimal Strategy)**.** *Let* $(t,x) \in [0,T] \times \overline{\mathcal{S}}$. *Then* $\Lambda^* = \{(\tau_k^*, \Delta_k^*)\}_{k\in\mathbb{N}}$ *in* (3.2) *is well-defined and optimal for the retail investor's portfolio problem, i.e.*

$$\Lambda^* \in \mathcal{A}(t,x) \qquad \textit{and} \qquad \mathcal{V}(t,x) = \mathbb{E}\Big[U_{\mathrm{L}}\big(X_T^{t,x,\Lambda^*}\big)\Big].$$

Proof. This follows immediately from Theorem 5.12 and Theorem 5.13 in Section 5. □

Together, Main Results 1 and 2 provide a complete solution of the retail investor's portfolio problem. In particular, the retail investor's optimal trading strategy is fully described by the no-trading region $\mathcal{C}$ and the target positions on its boundary; these can be identified numerically by solving the QVIs (3.1).

4 Analysis of Optimal Trading Strategies

In this section, we analyze, illustrate and discuss the structure of optimal trading strategies for the retail investor's portfolio problem in detail. Unless stated otherwise, quantitative results are based on the model parameters in Table 1. Our numerical results are obtained by solving the QVIs (3.1) using a finite difference scheme based on penalization of the non-local term, followed by a policy iteration. The scheme is implemented in C++ using the QuantPDE library.[8] Finally, we denote by $\tau \triangleq T - t$ the remaining investment horizon.

r	μ	σ	p	T	$\mathrm{C}_{\mathrm{min}}$	$\mathrm{C}_{\mathrm{max}}$	c
3.0%	10.2%	40.0%	0.1	5	8.90$	58.90$	0.25%

Table 1 Model parameters for numerical simulations.

The market coefficients in Table 1 are such that, in the absence of transaction costs, the optimal fraction of wealth invested in the stock is given by

$$\pi^* \triangleq \frac{\mu - r}{(1-p)\sigma^2} = \frac{1}{2},$$

i.e., the investor optimally holds equal amounts of money in the money market account and the stock at all times. In all subsequent plots, these frictionless optimal positions are indicated by a solid black line, which we refer to as the *Merton line*. *Pre-trade portfolio positions*, i.e. portfolios in the intervention region $\mathcal{I} = \{\mathcal{V} = \mathcal{M}\mathcal{V}\}$, are colored in blue; the associated *target positions*, i.e. the portfolio positions resulting after optimal trades, are colored in red.

Moreover, using light, medium and dark shades of blue and red, we visually distinguish three regions: We use light shades for the *floored cost region*, where optimal trades incur the minimal

[8] See http://github.com/parsiad/QuantPDE and [3].

transaction cost $\mathrm{C}_{\min}$; medium shades for the *proportional cost region*, where optimal trades incur transaction costs in the interval $(\mathrm{C}_{\min}, \mathrm{C}_{\max})$; and dark shades for the *capped cost region* with optimal transaction cost $\mathrm{C}_{\max}$. Note that, given the model parameters in Table 1, we have

$$
\begin{aligned}
\mathrm{C}(\Delta) &= \mathrm{C}_{\min} && \text{for} && |\Delta| \leq 3{,}560, \\
\mathrm{C}(\Delta) &= \mathrm{c}|\Delta| && \text{for} && 3{,}560 \leq |\Delta| \leq 23{,}560, \\
\mathrm{C}(\Delta) &= \mathrm{C}_{\max} && \text{for} && 23{,}560 \leq |\Delta|.
\end{aligned}
$$

4.1 Optimal Trading Regions and Target Portfolios

Figure 2 depicts the optimal trading regions for $\tau = 1$; for larger investment time horizons, the optimal trading regions become stationary and hardly differ from the trading regions displayed in Figure 2. For illustration, Figure 3 displays optimal trading regions and target portfolios for an investment horizon $\tau = 5$. Thus the following discussion applies as long as the outstanding investment horizon is not too small; boundary effects as terminal time approaches are discussed separately in Subsection 4.2 below. In general, as expected, the investor trades whenever the portfolio is sufficiently far away from the Merton line; optimal transactions always move the portfolio position towards it; and target positions are in the no-trading region.

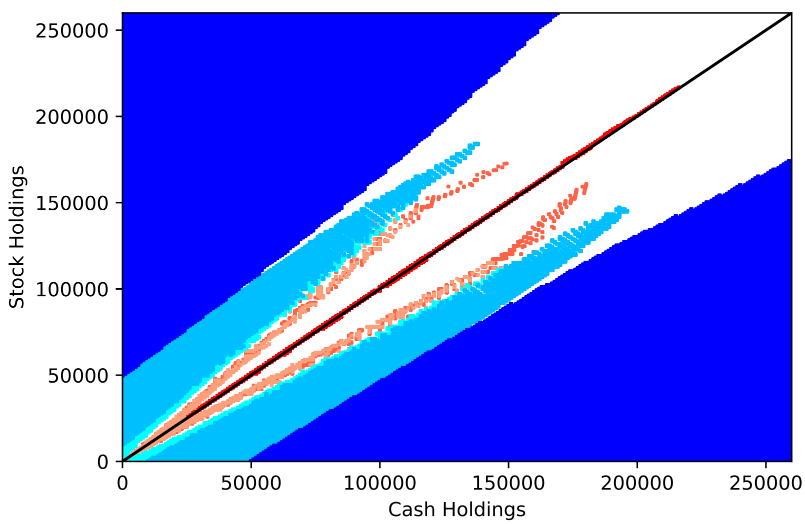

Figure 2 Trading regions for time to maturity $\tau = 1$.

A surprising feature of the trading regions in Figure 2 is the emergence of the two white v-shaped areas splitting the intervention region with proportional cost trades (medium blue) from the intervention region with capped cost trades (dark blue), resulting in a vVv-shaped no-trading region. Note that portfolios inside the two outer v-shaped wedges are no-trade portfolios, i.e. it is optimal for the investor to leave her portfolio unchanged. If the portfolio

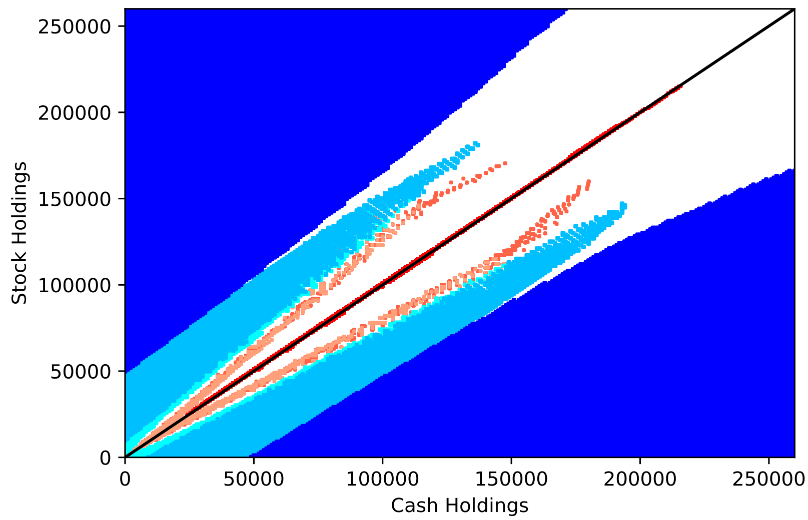

Figure 3 Trading regions for time to maturity $\tau = 5$.

moves sufficiently far away from the Merton line, a capped cost trade onto the Merton line is performed, whereas, if her portfolio moves closer to the Merton line, a proportional cost trade onto the medium red wedge is performed. Note that, once the optimal portfolio position is shifted outside, it never returns into the outer two v-shaped wedges, so this can occur only for the first transaction. On the other hand, for initial positions sufficiently far away from the frictionless optimizer, it is optimal to perform an immediate trade onto the Merton line.

The possibly most important insight from Figure 2 is that we are able to identify three distinct regimes of optimal transactions (disregarding the first trade, i.e. disregarding the two outer v-shaped areas of the no-trading region, see above): In the *moderate wealth regime* (cash holdings below $100{,}000\$$ in our parametrization), all optimal trades incur the floored cost[9] and feature a transaction size $\mathrm{C}_{\min}/c$; note that this is the largest volume tradable at the floored cost. In the *large wealth regime* (cash holdings above $190{,}000\$$) all optimal transactions involve the capped cost[10] and the target portfolios are on the Merton line. In particular, a retail investor with a large amount of wealth acts exactly as though she faced fixed transaction costs of size $\mathrm{C}_{\max}$. Between these two there is an *intermediate regime* (cash holdings between $100{,}000\$$ and $190{,}000\$$) where also proportional cost trades occur (more precisely, sell orders with proportional costs occur for cash holdings between $100{,}000\$$ and $135{,}000\$$ and buy orders with proportional costs between $145{,}000\$$ and $190{,}000\$$); the kinks in the wedge of medium red target portfolios indicate the transitions between the floored cost regime and the intermediate regime. Finally, we observe that there are no optimal transactions with volumes below $\mathrm{C}_{\min}/c$ in either of the regimes.

[9] Equivalently, the optimal portfolio exits the no-trading region only in the light blue areas, located on the boundary of the medium blue area.

[10] Equivalently, the optimal portfolio exits the no-trading region only in the dark blue areas.

4.2 Short Investment Time Horizons

Horizons $\tau = 0.25$ and $\tau = 0.15$. Figures 4 and 5 display the optimal trading regions for time to maturity $\tau = 0.25$ and $\tau = 0.15$, respectively. In both cases, the qualitative structure is analogous to the case $\tau = 1$.

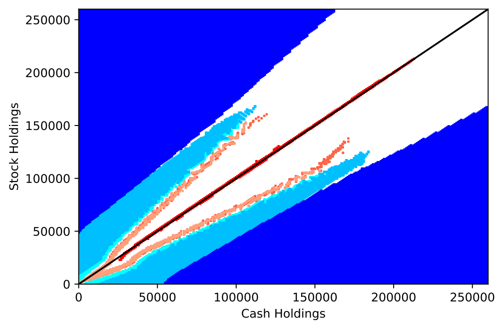

Figure 4 Trading regions for $\tau = 0.25$.

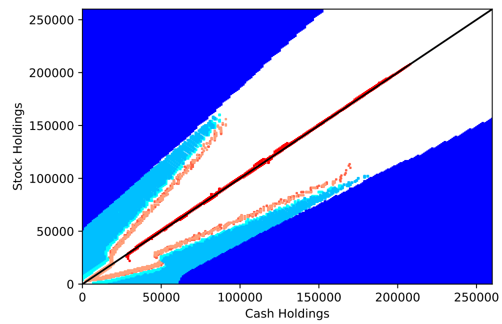

Figure 5 Trading regions for $\tau = 0.15$.

The main difference emerges in the moderate wealth regime (cash and stock holdings below $100{,}000$\$). The target portfolios no longer form a wedge around the Merton line, but develop a kink on both the selling side (above the Merton line) and buying side (below the Merton line). This is due to the fact that the investor anticipates the end of the investment period, where the entire risky position is to be liquidated. Notice that both kinks of the restarting positions are in a vicinity of stock holdings of around $23{,}560$\$, which is exactly the threshold between proportional and capped cost trades. On the buying side, it becomes less and less attractive to trade towards this level, as any risky assets bought would have to be liquidated within a short time frame, incurring transaction costs twice. By contrast, for larger stock holdings, the liquidation at terminal time is expected to be in the capped cost region, thus bounding the liquidation cost. This causes the continuation region to widen faster for moderate wealth levels than for large wealth levels, producing the kink on the buying side.

The kink on the selling side emerges for a similar reason: As noted above, in the capped cost region it becomes less attractive to sell shares shortly before the end of the investment horizon. With proportional costs, however, this makes almost no difference, as selling a part of the stock holdings before maturity and liquidating the rest at terminal time incurs approximately the same cost as liquidating the entire position at once. For this reason, the continuation region widens in the capped cost region (it is preferrable to keep the portfolio until the end), whereas it shrinks in the proportional cost region (the investor begins to liquidate the portfolio early, with the added benefit of taking it closer to the Merton line), thus causing the appearance of the kink on the selling side.

Finally, for $\tau = 0.15$, we observe that the target positions associated with capped cost trades for moderate wealth levels are located below the Merton line. This may be explained by the fact that these positions are further away from the intervention region than the Merton line, hence increasing the probability that no further trade is necessary before terminal time.

Horizons $\tau = 0.10$, $\tau = 0.06$, $\tau = 0.03$ and $\tau = 0.01$. Figures 6 to 9 illustrate the evolution of the optimal trading regions as the investment horizon tends to zero.

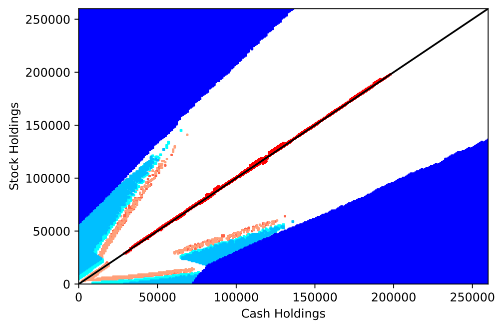

Figure 6 Trading regions for $\tau = 0.10$.

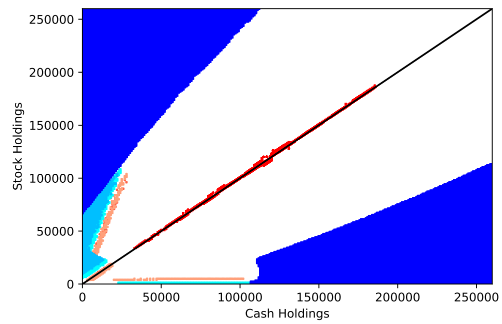

Figure 7 Trading regions for $\tau = 0.06$.

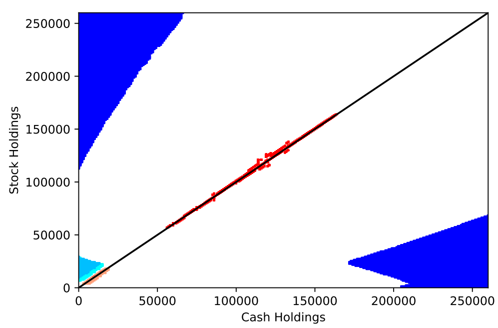

Figure 8 Trading regions for $\tau = 0.03$.

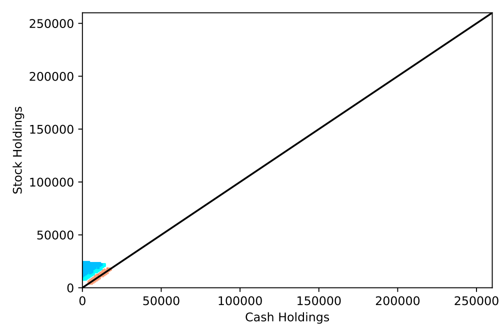

Figure 9 Trading regions for $\tau = 0.01$.

Several effects emerge, most of which are explained by the difference in speed by which the continuation region widens in the different cost regions. In Figure 6, i.e. for $\tau = 0.10$, we see that the target positions for floored/proportional cost transactions are no longer connected. On the buying side, for cash holdings below $25{,}000\$$ proportional trades disappear and are replaced by floored cost trades; in Figure 7, i.e. for $\tau = 0.06$, proportional trades on the buying side are eliminated entirely. The novel feature in Figure 8, i.e. for $\tau = 0.03$, is that the trading region is given by three connected regions instead of two, as the sell-side trade region with capped costs splits from the sell-side trade region with floored and proportional costs. Finally, in Figure 9, i.e. for $\tau = 0.01$, it is no longer optimal to make any trades unless the portfolio is on the selling side and in the proportional cost region (see the discussion above).

4.3 Negative Excess Return

Our mathematical results apply also for $\mu < r$, i.e. when the stock features a negative excess return. We briefly investigate this degenerate case in the following. In the absence of transaction costs but without short-selling, the optimal strategy is to immediately liquidate all risky positions and invest everything into the money market account. By contrast, with strictly positive transaction costs there is a trade-off between liquidation costs and negative returns. While it is clear that sufficiently large stock positions should be liquidated, it is equally clear that immediate liquidation is suboptimal if the position's liquidation value is zero, i.e. $0 < s < \mathrm{C}(-s)$. The key question is, therefore, to identify the threshold.

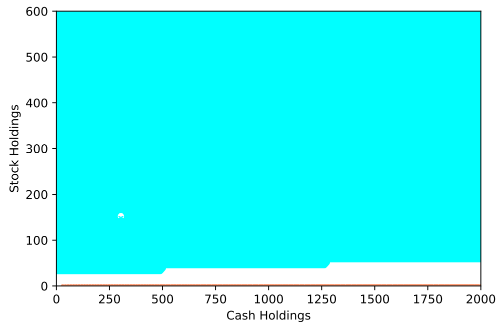

Figure 10 Trading regions for $\tau = 1.00$ ($\mu < r$).

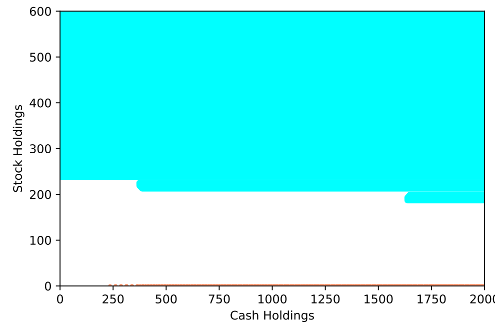

Figure 11 Trading regions for $\tau = 0.01$ ($\mu < r$).

This issue is addressed in Figure 10 for a long time horizon of $\tau = 1$ and in Figure 11 for a short time horizon of $\tau = 0.01$ with $\mu = 0.025 < 0.03 = r$ (the remaining parameters being the same as in Table 1). As expected, if the time horizon is sufficiently long, the investor optimally liquidates even relatively small stock positions immediately. In particular, for zero cash holdings, any stock positions worth more than approximately 35\$ are liquidated if $\tau = 1$ (incurring the floored cost of 8.90\$). On the other hand, for $\tau = 0.01$, there is significantly less time to benefit from the larger interest rate and the trader is willing to keep stock holdings of up to 235\$ if the current cash holdings are zero. In a typical real-world scenario, these thresholds imply that the retail investor liquidates her portfolio immediately at the optimum.

5 Viscosity Characterization and Optimal Strategies

In this section we prove the two main results announced in Section 3: The viscosity characterization of the value function and the optimality of the candidate trading strategy.

Our mathematical approach is based on the stochastic Perron's method and the superharmonic function technique, similarly as in the analysis of portfolio problems with fixed plus proportional costs in [7]. In the present setting, however, some key technical arguments can be sharpened and streamlined. Thus we directly characterize the smallest stochastic supersolution $\mathbb{V}$ as the unique viscosity solution of the Bellman equation (Theorems 5.5 and 5.12); then we define a candidate optimal strategy in terms of $\mathbb{V}$ and provide a verification theorem (Theorem 5.13) that simultaneously establishes optimality of the candidate strategy and the fact that $\mathbb{V}$ coincides with the value function.[11] The main advantage of this direct approach is that it is

[11]By contrast, in [7] the approach is to first characterize the value function as the unique viscosity solution of the

significantly easier to verify the viscosity supersolution property, as we can avoid the iterated optimal stopping approximation of the value function used in [7].

5.1 Preliminary Results

Our first important observation concerns sequential hemicontinuity of the set-valued mapping $x \mapsto \mathcal{D}(x)$.

Lemma 5.1 (Sequential Hemicontinuity of $\mathcal{D}$). *Let $\{x^k\}_{k\in\mathbb{N}}$ be a sequence in $\overline{\mathcal{S}} \setminus \mathcal{S}_\emptyset$ which converges to some $x \in \overline{\mathcal{S}} \setminus \mathcal{S}_\emptyset$.*

1. *Let $\Delta^k \in \mathcal{D}(x^k)$ for all $k \in \mathbb{N}$. Then there exists a subsequence of $\{\Delta^k\}_{k\in\mathbb{N}}$ that converges to some $\Delta \in \mathcal{D}(x)$.*
2. *Let $\Delta \in \mathcal{D}(x)$. Then there exists a sequence $\{\Delta^k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ converging to Δ such that $\Delta^k \in \mathcal{D}(x^k)$ for all $k \in \mathbb{N}$.*

In other words, $x \mapsto \mathcal{D}(x)$ is sequentially hemicontinuous on its effective domain $\overline{\mathcal{S}} \setminus \mathcal{S}_\emptyset$.

Proof. First observe that, for all $\bar{x} = (\bar{x}_0, \bar{x}_1) \in \overline{\mathcal{S}} \setminus \mathcal{S}_\emptyset$, the feasibility constraint $\Delta \in \mathcal{D}(\bar{x})$ implies that

$$-\bar{x}_1 \leq \Delta \leq \bar{x}_0 - \mathrm{C}_{\min}, \qquad \Delta \in \mathcal{D}(\bar{x}).$$

From this and boundedness of the sequence $\{x^k\}_{k\in\mathbb{N}}$, it follows that $\bigcup_{k\in\mathbb{N}} \mathcal{D}(x^k)$ is bounded.
ad 1. Suppose that $\Delta^k \in \mathcal{D}(x^k)$ for all $k \in \mathbb{N}$, which is to say that $\Gamma(x^k, \Delta^k) \in \overline{\mathcal{S}}$. Since $\bigcup_{k\in\mathbb{N}} \mathcal{D}(x^k)$ is bounded, there is a subsequence of $\{\Delta^k\}_{k\in\mathbb{N}}$ that converges to some $\Delta \in \mathbb{R}$. Since Γ is continuous and $\overline{\mathcal{S}}$ is closed, we obtain $\Gamma(x, \Delta) \in \overline{\mathcal{S}}$, i.e. $\Delta \in \mathcal{D}(x)$, thus proving the first part of the claim.
ad 2. Now fix $\Delta \in \mathcal{D}(x)$. If

$$x_0 - \Delta - \mathrm{C}(\Delta) > 0 \qquad \text{and} \qquad x_1 + \Delta > 0,$$

then the same must be true if we replace (x_0, x_1) by (x_0^k, x_1^k) for all $k \in \mathbb{N}$ sufficiently large. Thus $\Delta \in \mathcal{D}(x^k)$ eventually and we conclude. If, on the other hand, $x_1 + \Delta = 0$, i.e. $\Delta = -x_1$, then we may choose

$$\Delta^k \triangleq -x_1^k \in \mathcal{D}(x^k), \qquad k \in \mathbb{N}.$$

It is then immediate that $\Delta^k \to \Delta$ as $k \to \infty$ and we are done as well. Thus, we are left with the case where $x_1 + \Delta > 0$ and $x_0 - \Delta - \mathrm{C}(\Delta) = 0$. In this case, we set

$$\Delta^k \triangleq \Delta - \frac{(x_0 - x_0^k)^+}{1 - \mathrm{c}}, \qquad k \in \mathbb{N}.$$

Since $x^k \to x$, it follows that $\Delta^k \to \Delta$ and hence $x_1^k + \Delta^k \geq 0$ for eventually all $k \in \mathbb{N}$. Moreover, using the fact that $x_0 - \Delta - \mathrm{C}(\Delta) = 0$, we obtain

$$x_0^k - \Delta^k - \mathrm{C}(\Delta_k) = x_0^k - \Delta^k - \mathrm{C}(\Delta_k) - x_0 + \Delta + \mathrm{C}(\Delta)$$

Bellman equation and show that it coincides with the smallest stochastic supersolution; then define a candidate optimal strategy in terms of the value function, and finally verify its optimality.

$$\geq -(x_0 - x_0^k)^+ + \frac{(x_0 - x_0^k)^+}{1 - \mathrm{c}} + \mathrm{C}(\Delta) - \mathrm{C}(\Delta_k).$$

Since the cost function C is Lipschitz continuous with Lipschitz constant c, we have

$$\mathrm{C}(\Delta) - \mathrm{C}(\Delta_k) \geq -\mathrm{c}|\Delta - \Delta^k| = -\mathrm{c}\frac{(x_0 - x_0^k)^+}{1 - \mathrm{c}}$$

and hence

$$x_0^k - \Delta^k - \mathrm{C}(\Delta_k) \geq -(x_0 - x_0^k)^+ + \frac{(x_0 - x_0^k)^+}{1 - \mathrm{c}} - \mathrm{c}\frac{(x_0 - x_0^k)^+}{1 - \mathrm{c}} = 0,$$

which concludes the proof. □

We subsequently denote by LSC and USC the sets of lower and upper semicontinuous functions $h : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$, respectively. If $h : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$ is locally bounded, we denote its lower semicontinuous envelope by h_* and its upper semicontinuous envelope by h^*.

Lemma 5.2 (Semicontinuity of $\mathcal{M}$)**.** *For any function $h : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}_+$, the following holds:*

1. *If $h \in \mathrm{USC}$, then $\mathcal{M}[h]^*(t, x) = \mathcal{M}[h](t, x)$ for all $(t, x) \in [0, T] \times \overline{\mathcal{S}}$.*
2. *If $h \in \mathrm{LSC}$, then $\mathcal{M}[h]_*(t, x) = \mathcal{M}[h](t, x)$ for all $(t, x) \in [0, T] \times (\overline{\mathcal{S}} \setminus \overline{\mathcal{S}}_\emptyset)$.*

Proof. ad 1. Let $h \in \mathrm{USC}$. To show that $\mathcal{M}[h]^* = \mathcal{M}[h]$, it obviously suffices to show that $\mathcal{M}[h]$ is upper semicontinuous. For this, let $(t, x) \in [0, T] \times \overline{\mathcal{S}}$ and choose a sequence $\{(t_k, x_k)\}_{k \in \mathbb{N}} \subset [0, T] \times \overline{\mathcal{S}}$ converging to (t, x). Since $\mathcal{M}[h]$ is constant on $[0, T] \times \mathcal{S}_\emptyset$, we may assume that $x \in \overline{\mathcal{S}} \setminus \mathcal{S}_\emptyset$. Moreover, by dropping to a subsequence, we may assume that either

$$x_k \in \overline{\mathcal{S}} \setminus \mathcal{S}_\emptyset \quad \text{for all } k \in \mathbb{N} \qquad \text{or} \qquad x_k \in \mathcal{S}_\emptyset \quad \text{for all } k \in \mathbb{N}.$$

In the latter case, we have $x \in \partial \mathcal{S}_\emptyset$ and hence $\mathcal{D}(x) = \{-x_1\}$ and $\Gamma(x, -x_1) = 0$; but this and the definition of $\mathcal{M}[h]$ on $[0, T] \times \mathcal{S}_\emptyset$ imply that

$$\limsup_{k \to \infty} \mathcal{M}[h](t_k, x_k) \leq \limsup_{k \to \infty} h(t_k, 0) \leq h(t, 0) = h\big(t, \Gamma(x, -x_1)\big) = \mathcal{M}[h](t, x),$$

thus giving upper semicontinuity. Hence in the following we assume that $x_k \in \overline{\mathcal{S}} \setminus \mathcal{S}_\emptyset$ for all $k \in \mathbb{N}$. We drop to a subsequence if necessary to ensure that

$$\limsup_{k \to \infty} \mathcal{M}[h](t_k, x_k) = \lim_{k \to \infty} \mathcal{M}[h](t_k, x_k).$$

For each $k \in \mathbb{N}$, the set $\mathcal{D}(x_k)$ is non-empty and compact. By upper semicontinuity of h, we therefore find $\Delta_k \in \mathcal{D}(x_k)$ such that

$$\mathcal{M}[h](t_k, x_k) = h\big(t_k, \Gamma(x_k, \Delta_k)\big).$$

Dropping to yet another subsequence if necessary, Lemma 5.1 shows that $\{\Delta_k\}_{k \in \mathbb{N}}$ converges to some $\Delta \in \mathcal{D}(x)$. But then upper semicontinuity of h yields

$$\limsup_{k\to\infty} \mathcal{M}[h](t_k, x_k) = \lim_{k\to\infty} \mathcal{M}[h](t_k, x_k)$$
$$= \lim_{k\to\infty} h\big(t_k, \Gamma(x_k, \Delta_k)\big) \leq h\big(t, \Gamma(x, \Delta)\big) \leq \mathcal{M}[h](t, x),$$

which concludes the first part of the proof.

ad 2. Now suppose that $h \in \mathrm{LSC}$. We fix $(t, x) \in [0, T] \times (\overline{\mathcal{S}} \setminus \overline{\mathcal{S}}_\emptyset)$ and choose an arbitrary sequence $\{(t_k, x_k)\}_{k\in\mathbb{N}} \subset [0, T] \times \overline{\mathcal{S}}$ converging to (t, x). Since $x \notin \overline{\mathcal{S}}_\emptyset$, it follows that $x_k \notin \overline{\mathcal{S}}_\emptyset$ eventually and hence, without loss of generality, $\mathcal{D}(x_k) \neq \emptyset$ for all $k \in \mathbb{N}$. Now take as given some $\Delta \in \mathcal{D}(x)$. By Lemma 5.1, for each $k \in \mathbb{N}$, we find $\Delta_k \in \mathcal{D}(x_k)$ such that $\Delta_k \to \Delta$ as $k \to \infty$. But then

$$\liminf_{k\to\infty} \mathcal{M}[h](t_k, x_k) \geq \liminf_{k\to\infty} h\big(t_k, \Gamma(x_k, \Delta_k)\big) \geq h\big(t, \Gamma(x, \Delta)\big).$$

Since $\Delta \in \mathcal{D}(x)$ was chosen arbitrarily, this implies that

$$\liminf_{k\to\infty} \mathcal{M}[h](t_k, x_k) \geq \mathcal{M}[h](t, x),$$

i.e. $\mathcal{M}[h]$ is lower semicontinuous on $[0, T] \times (\overline{\mathcal{S}} \setminus \overline{\mathcal{S}}_\emptyset)$ and thus equal to $\mathcal{M}[h]_*$. □

We close this subsection by introducing a suitable notion of viscosity solutions of the QVIs (3.1). Since (3.1) are the only quasi-variational inequalities in this paper, we henceforth briefly refer to (3.1) as *the QVIs.*

Definition 5.3 (Viscosity Solutions of QVIs)**.** *Let $h : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be locally bounded.*

(i) We say that h is a viscosity subsolution *of the QVIs if, for all $(t, x) \in [0, T) \times \mathcal{S}$ and all $\varphi \in \mathrm{C}^2([0, T) \times \mathcal{S})$ with $\varphi \geq h^*$ and $\varphi(t, x) = h^*(t, x)$, we have*

$$\min\big\{\mathcal{L}[\varphi](t, x), h^*(t, x) - \mathcal{M}[h]^*(t, x)\big\} \leq 0.$$

(ii) We say that h is a viscosity supersolution *of the QVIs if, for all $(t, x) \in [0, T) \times \mathcal{S}$ and all $\varphi \in \mathrm{C}^2([0, T) \times \mathcal{S})$ with $\varphi \leq h_*$ and $\varphi(t, x) = h_*(t, x)$, we have*

$$\min\big\{\mathcal{L}[\varphi](t, x), h_*(t, x) - \mathcal{M}[h]_*(t, x)\big\} \geq 0.$$

(iii) h is called a viscosity solution *of the QVIs if it is both a viscosity sub- and supersolution.*

5.2 A Comparison Principle for the QVIs

The aim of this subsection is to establish a comparison principle that is sufficiently strong to establish uniqueness and continuity for viscosity solutions of the QVIs. The comparison principle is obtained by perturbing viscosity solutions with a (strict) classical supersolution, an idea which goes back to [17]. The supersolution we use is given by the following result.

Lemma 5.4 (Classical Supersolution). *Let $\varepsilon \in \{0,1\}$, $q \in [p,1)$, $\lambda > q\max\{r,\mu,0\}$, and $C > 0$ and define*

$$\Psi^q_\varepsilon : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}_+, \qquad (t,x) \mapsto \Psi^q_\varepsilon(t,x) \triangleq C\big(\varepsilon + x_0 + x_1\big)^q e^{\lambda(T-t)}. \tag{5.1}$$

Then there exists a continuous function $\kappa : \overline{\mathcal{S}} \to \mathbb{R}_+$ that is strictly positive on $\mathcal{S}$ such that

$$\min\big\{\mathcal{L}[\Psi^q_\varepsilon](t,x), \Psi^q_\varepsilon(t,x) - \mathcal{M}[\Psi^q_\varepsilon](t,x)\big\} \geq \kappa(x) > 0, \qquad (t,x) \in [0,T) \times \mathcal{S}.$$

Proof. Fix $(t,x) \in [0,T) \times \mathcal{S}$. An explicit computation shows that

$$\begin{aligned}\mathcal{L}[\Psi^q_\varepsilon](t,x) = Ce^{\lambda(T-t)}(\varepsilon + x_0 + x_1)^{q-1}\Big[&\lambda\varepsilon + (\lambda - qr)x_0 + (\lambda - q\mu)x_1 \\ &+ \frac{1}{2}(1-q)q\sigma^2\frac{x_1^2}{\varepsilon + x_0 + x_1}\Big] \\ \geq \big(\lambda - q\max\{r,\mu,0\}\big)&Ce^{\lambda(T-t)}(\varepsilon + x_0 + x_1)^q > 0.\end{aligned}$$

Moreover, whenever $x \notin \mathcal{S}_\emptyset$,

$$\begin{aligned}\Psi^q_\varepsilon(t,x) - \mathcal{M}[\Psi^q_\varepsilon](t,x) &= Ce^{\lambda(T-t)} \inf_{\Delta \in \mathcal{D}(x)}\Big[(\varepsilon + x_0 + x_1)^q - \big(\varepsilon + x_0 + x_1 - \mathrm{C}(\Delta)\big)^q\Big] \\ &= Ce^{\lambda(T-t)}\Big[(\varepsilon + x_0 + x_1)^q - \big(\varepsilon + x_0 + x_1 - \mathrm{C}_{\min}\big)^q\Big] > 0.\end{aligned}$$

Since $\Psi^q_\varepsilon(t,x) - \mathcal{M}[\Psi^q_\varepsilon](t,x) \geq \Psi^q_\varepsilon(t,x) - \Psi^q_\varepsilon(t,x) + 1 = 1$ if $x \in \mathcal{S}_\emptyset$, this completes the proof. $\square$

Before we state the comparison principle, we introduce some short-hand notation by defining[12]

$$F_{\mathcal{L}} : \overline{\mathcal{S}} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}^3 \to \mathbb{R}$$

via

$$F_{\mathcal{L}}(x,a,b,M) \triangleq -a - rx_0b_0 - \mu x_1 b_1 - \tfrac{1}{2}\sigma^2 x_1^2 M_{33}$$

for all $x = (x_0,x_1) \in \overline{\mathcal{S}}$, $a \in \mathbb{R}$, $b = (b_0,b_1) \in \mathbb{R}^2$, and $M = (M_{ij})_{i=1,2,3}^{j=1,2,3} \in \mathbb{S}^3$. Note that

$$\mathcal{L}[\varphi](t,x) = F_{\mathcal{L}}\Big(x, \frac{\partial\varphi}{\partial t}(t,x), \frac{\partial\varphi}{\partial x}(t,x), \frac{\partial^2\varphi}{\partial x^2}(t,x)\Big), \qquad (t,x) \in [0,T) \times \mathcal{S},$$

for every $\varphi \in \mathrm{C}^2([0,T) \times \mathcal{S})$.

Theorem 5.5 (Comparison Principle). *Let $u \in$ USC and $v \in$ LSC be a viscosity subsolution of the QVIs and a viscosity supersolution of the QVIs, respectively. Suppose that*

$$u(t,0) = 0 \quad \textit{for all } t \in [0,T] \qquad \textit{and} \qquad u(T,x) \leq v(T,x) \quad \textit{for all } x \in \overline{\mathcal{S}}, \tag{5.2}$$

[12]Here, $\mathbb{S}^3 \subset \mathbb{R}^{3\times 3}$ denotes the set of symmetric 3×3 matrices.

and that there exists a constant $K > 0$ such that

$$0 \le u(t,x), v(t,x) \le K\big(1+|x|^p\big), \qquad (t,x) \in [0,T] \times \overline{\mathcal{S}}. \tag{5.3}$$

Then v dominates u everywhere, i.e.

$$u(t,x) \le v(t,x), \qquad (t,x) \in [0,T] \times \overline{\mathcal{S}}.$$

Proof. Fix $q \in (p,1)$ and choose $C > 0$ sufficiently large such that $u, v \le \Psi_1^q$ on $[0,T] \times \overline{\mathcal{S}}$, where Ψ_1^q is given by (5.1); this is possible by (5.3). For any $\eta > 1$, we define $u_\eta \in \mathrm{USC}$ and $v_\eta \in \mathrm{LSC}$ by

$$u_\eta \triangleq \frac{\eta+1}{\eta} u - \frac{1}{\eta}\Psi_1^q \qquad \text{and} \qquad v_\eta \triangleq \frac{\eta-1}{\eta} v + \frac{1}{\eta}\Psi_1^q.$$

We proceed to show that $u_\eta \le v_\eta$ on $[0,T] \times \overline{\mathcal{S}}$, which implies the result once we send $\eta \to \infty$. We argue by contradiction and suppose that

$$u_\eta(t^*,x^*) > v_\eta(t^*,x^*) \qquad \text{for some } (t^*,x^*) \in [0,T] \times \overline{\mathcal{S}}.$$

Step 1. For each $k \in \mathbb{N}_0$, we define $\phi_k : ([0,T] \times \overline{\mathcal{S}})^2 \to \mathbb{R}$ by

$$\phi_k(t,x,\hat{t},\hat{x}) \triangleq u_\eta(t,x) - v_\eta(\hat{t},\hat{x}) - \frac{k}{2}\big[|t-\hat{t}|^2 + |x-\hat{x}|^2\big], \qquad (t,x),(\hat{t},\hat{x}) \in [0,T] \times \overline{\mathcal{S}},$$

and set

$$\Theta_k \triangleq \sup_{(t,x),(\hat{t},\hat{x}) \in [0,T]\times\overline{\mathcal{S}}} \phi_k(t,x,\hat{t},\hat{x}) \qquad \text{and} \qquad \Theta \triangleq \sup_{(t,x)\in[0,T]\times\overline{\mathcal{S}}} \phi_0(t,x,t,x).$$

It is immediately seen that

$$0 < u_\eta(t^*,x^*) - v_\eta(t^*,x^*) \le \Theta \le \Theta_{k+1} \le \Theta_k \le \Theta_0, \qquad k \in \mathbb{N}.$$

This implies that every maximizing sequence for some Θ_k, $k \in \mathbb{N}_0$, must eventually be contained in the set

$$F \triangleq \Big\{(t,x,\hat{t},\hat{x}) \in \big([0,T]\times\overline{\mathcal{S}}\big)^2 : u_\eta(t,x) - v_\eta(\hat{t},\hat{x}) \ge 0\Big\}.$$

Since u_η and $-v_\eta$ are upper semicontinuous, F is closed. Moreover, by (5.3) and the fact that $q > p$, F is bounded and hence compact. But then, for all $k \in \mathbb{N}_0$,

$$\Theta_k = \phi_k(t_k,x_k,\hat{t}_k,\hat{x}_k) < \infty \qquad \text{for some } (t_k,x_k,\hat{t}_k,\hat{x}_k) \in F,$$

and after dropping to a subsequence we may assume that $\{(t_k,x_k,\hat{t}_k,\hat{x}_k)\}_{k\in\mathbb{N}}$ is convergent. Since $\Theta_k > 0$ and $u_\eta, -v_\eta \in \mathrm{USC}$, we have

$$\frac{k}{2}\big[|t_k-\hat{t}_k|^2 + |x_k-\hat{x}_k|^2\big] \le \sup_{(t,x,\hat{t},\hat{x})\in F}\Big[u_\eta(t,x) - v_\eta(\hat{t},\hat{x})\Big] < \infty,$$

so we must have

$$(\bar{t},\bar{x}) \triangleq \lim_{k\to\infty}(t_k,x_k) = \lim_{k\to\infty}(\hat{t}_k,\hat{x}_k).$$

But then, since $\Theta \le \Theta_k$ and $u_\eta, -v_\eta \in \mathrm{USC}$, we obtain

$$\begin{aligned} 0 \le \limsup_{k\to\infty} \frac{k}{2}\left[|t_k-\hat{t}_k|^2+|x_k-\hat{x}_k|^2\right] \\ = \limsup_{k\to\infty}\Big[u_\eta(t_k,x_k)-v_\eta(\hat{t}_k,\hat{x}_k)-\Theta_k\Big] \le u_\eta(\bar{t},\bar{x})-v_\eta(\bar{t},\bar{x})-\Theta \le 0. \end{aligned}$$

We have thus shown that

$$(\bar{t},\bar{x}) = \lim_{k\to\infty}(t_k,x_k) = \lim_{k\to\infty}(\hat{t}_k,\hat{x}_k) \quad \text{and} \quad \lim_{k\to\infty}\frac{k}{2}\left[|t_k-\hat{t}_k|^2+|x_k-\hat{x}_k|^2\right]=0, \tag{5.4}$$

$$\lim_{k\to\infty} u_\eta(t_k,x_k) = u_\eta(\bar{t},\bar{x}) \quad \text{and} \quad \lim_{k\to\infty} v_\eta(\hat{t}_k,\hat{x}_k) = v_\eta(\bar{t},\bar{x}), \tag{5.5}$$

and that

$$\lim_{k\to\infty}\Theta_k = \Theta = \phi_0(\bar{t},\bar{x},\bar{t},\bar{x}) = u_\eta(\bar{t},\bar{x})-v_\eta(\bar{t},\bar{x}). \tag{5.6}$$

Note that (5.6) implies in particular that $\bar{t} < T$ and hence, without loss of generality, $t_k, \hat{t}_k < T$ for every $k\in\mathbb{N}$. Indeed, if this were not the case, (5.2) and the estimate $u, v \le \Psi_1^q$ would yield the contradiction

$$\Theta = u_\eta(\bar{t},\bar{x})-v_\eta(\bar{t},\bar{x}) = u(T,\bar{x})-v(T,\bar{x})+\frac{1}{\eta}\Big[u(T,\bar{x})+v(T,\bar{x})-2\Psi_1^q(T,\bar{x})\Big] \le 0.$$

Similarly, we cannot have $\bar{x}=0$ since otherwise (5.2) and non-negativity of v_η and Ψ_1^q imply

$$\Theta = u_\eta(\bar{t},\bar{x})-v_\eta(\bar{t},\bar{x}) \le u_\eta(\bar{t},0) = -\frac{1}{\eta}\Psi_1^q(\bar{t},0) \le 0.$$

We may therefore also assume that $x_k, \hat{x}_k \in \mathcal{S}$ for every $k\in\mathbb{N}$.

Step 2. Since $(t_k,x_k), (\hat{t}_k,\hat{x}_k) \in [0,T)\times\mathcal{S}$ for all $k\in\mathbb{N}$ by Step 1, we can apply Ishii's lemma, see [11, Theorem 3.2], to obtain $M_k, N_k \in \mathbb{S}^3$ with[13]

$$\begin{pmatrix} M_k & 0 \\ 0 & -N_k \end{pmatrix} \le 3k \begin{pmatrix} \mathrm{I} & -\mathrm{I} \\ -\mathrm{I} & \mathrm{I} \end{pmatrix} \tag{5.7}$$

such that[14]

$$\begin{aligned} \Big(\big(k(t_k-\hat{t}_k), k(x_k-\hat{x}_k)\big)^\top, M_k\Big) &\in \overline{\mathcal{J}}^{2,+}u_\eta(t_k,x_k), \\ \Big(\big(k(t_k-\hat{t}_k), k(x_k-\hat{x}_k)\big)^\top, N_k\Big) &\in \overline{\mathcal{J}}^{2,-}v_\eta(\hat{t}_k,\hat{x}_k). \end{aligned}$$

[13]Here, I denotes the identity in $\mathbb{S}^3$.

[14]Here, $\overline{\mathcal{J}}^{2,+}u_\eta(t_k,x_k)$ and $\overline{\mathcal{J}}^{2,-}v_\eta(\hat{t}_k,\hat{x}_k)$ denote the closures of the second-order super- and subjets of u_η and v_η at (t_k,x_k) and $(\hat{t}_k,\hat{x}_k)$, respectively.

Since u and v are, respectively, viscosity sub- and supersolutions and Ψ_1^q is a strict classical supersolution, the same argument as in [7, Proposition 4.2] shows that

$$-\frac{\bar{\kappa}}{\eta} \geq \min\Big\{F_{\mathcal{L}}\big(x_k, k(t_k - \hat{t}_k), k(x_k - \hat{x}_k), M_k\big), u_\eta(t_k, x_k) - \mathcal{M}[u_\eta]^*(t_k, x_k)\Big\}, \tag{5.8}$$

$$\frac{\bar{\kappa}}{\eta} \leq \min\Big\{F_{\mathcal{L}}\big(\hat{x}_k, k(t_k - \hat{t}_k), k(x_k - \hat{x}_k), N_k\big), v_\eta(t_k, x_k) - \mathcal{M}[v_\eta]_*(t_k, x_k)\Big\} \tag{5.9}$$

$$\leq F_{\mathcal{L}}\big(\hat{x}_k, k(t_k - \hat{t}_k), k(x_k - \hat{x}_k), N_k\big), \tag{5.10}$$

where $\bar{\kappa} \triangleq \inf_{(t,x,\hat{t},\hat{x}) \in F} \min\{\kappa(x), \kappa(\hat{x})\} > 0$ and κ is the continuous function provided by Lemma 5.4.

Step 3. Let us now argue that in (5.8), after dropping to a subsequence, we may assume that

$$F_{\mathcal{L}}\big(x_k, k(t_k - \hat{t}_k), k(x_k - \hat{x}_k), M_k\big) \leq -\frac{\bar{\kappa}}{\eta}, \qquad k \in \mathbb{N}. \tag{5.11}$$

We argue by contradiction and assume that this is not the case, i.e. that the latter inequality is only valid for at most finitely many $k \in \mathbb{N}$. By (5.8), this means that there exists $K \in \mathbb{N}$ with

$$u_\eta(t_k, x_k) \leq \mathcal{M}[u_\eta]^*(t_k, x_k) - \frac{\bar{\kappa}}{\eta}, \qquad k \geq K. \tag{5.12}$$

Note that this is only possible if $x_k \notin \mathcal{S}_\emptyset$ for all $k \geq K$, and hence we see that $\bar{x} \notin \mathcal{S}_\emptyset$. Upon making K larger, using (5.6) and the convergence in (5.5), we furthermore find that

$$\Theta = u_\eta(\bar{t}, \bar{x}) - v_\eta(\bar{t}, \bar{x}) \leq u_\eta(t_k, x_k) - v_\eta(\hat{t}_k, \hat{x}_k) + \frac{\bar{\kappa}}{4\eta}, \qquad k \geq K. \tag{5.13}$$

Similarly, making K even larger if necessary, upper semicontinuity of $\mathcal{M}[u_\eta]^*$ yields

$$\mathcal{M}[u_\eta]^*(t_k, x_k) \leq \mathcal{M}[u_\eta]^*(\bar{t}, \bar{x}) + \frac{\bar{\kappa}}{4\eta}, \qquad k \geq K. \tag{5.14}$$

Since u_η is upper semicontinuous, we have $\mathcal{M}[u_\eta]^* = \mathcal{M}[u_\eta]$ by Lemma 5.2. But then due to compactness of $\mathcal{D}(\bar{x})$ and upper semicontinuity of u_η there exists $\Delta \in \mathcal{D}(\bar{x})$ such that

$$\mathcal{M}[u_\eta]^*(\bar{t}, \bar{x}) = \mathcal{M}[u_\eta](\bar{t}, \bar{x}) = u_\eta\big(\bar{t}, \Gamma(\bar{x}, \Delta)\big). \tag{5.15}$$

If we now successively plug (5.12), (5.14), and then (5.15) into (5.13), we arrive at

$$\Theta \leq u_\eta\big(\bar{t}, \Gamma(\bar{x}, \Delta)\big) - v_\eta(\hat{t}_k, \hat{x}_k) - \frac{\bar{\kappa}}{2\eta}, \qquad k \geq K. \tag{5.16}$$

If $\Gamma(\bar{x}, \Delta) = 0$, then (5.2) gives $u_\eta(\bar{t}, \Gamma(\bar{x}, \Delta)) \leq 0$ and hence we obtain the contradiction

$$\Theta \leq -v_\eta(\hat{t}_k, \hat{x}_k) - \frac{\bar{\kappa}}{2\eta} < 0.$$

We must therefore have $\Gamma(\bar{x}, \Delta) \neq 0$; but since $\mathcal{D}(x) = \{-x_1\}$ for all $x \in \partial\mathcal{S}_\emptyset$, this is only possible if $\bar{x} \notin \overline{\mathcal{S}}_\emptyset$. Now (5.9) gives

$$v_\eta(\hat{t}_k, \hat{x}_k) \geq \mathcal{M}[v_\eta]_*(\hat{t}_k, \hat{x}_k) + \frac{\bar{\kappa}}{\eta}, \qquad k \in \mathbb{N},$$

and by lower semicontinuity of $\mathcal{M}[v_\eta]_*$ we can assume that

$$\mathcal{M}[v_\eta]_*(\hat{t}_k, \hat{x}_k) \geq \mathcal{M}[v_\eta]_*(\bar{t}, \bar{x}) - \frac{\bar{\kappa}}{2\eta}, \qquad k \geq K.$$

Since $\bar{x} \notin \overline{\mathcal{S}}_\emptyset$ and v_η is lower semicontinuous, Lemma 5.2 and $\Delta \in \mathcal{D}(\bar{x})$ imply that

$$\mathcal{M}[v_\eta]_*(\bar{t}, \bar{x}) = \mathcal{M}[v_\eta](\bar{t}, \bar{x}) \geq v_\eta\big(\bar{t}, \Gamma(\bar{x}, \Delta)\big).$$

Plugging the latter three inequalities into (5.16) thus gives

$$\Theta \leq u_\eta\big(\bar{t}, \Gamma(\bar{x}, \Delta)\big) - v_\eta\big(\bar{t}, \Gamma(\bar{x}, \Delta)\big) - \frac{\bar{\kappa}}{\eta} \leq \Theta - \frac{\bar{\kappa}}{\eta}.$$

Since this is a contradiction, it follows that we may assume that (5.11) holds.

Step 4. Combining (5.10) from Step 2 and (5.11) from Step 3 shows that, for all $k \in \mathbb{N}$,

$$\frac{2\bar{\kappa}}{\eta} \leq F_\mathcal{L}\big(\hat{x}_k, k(t_k - \hat{t}_k), k(x_k - \hat{x}_k), N_k\big) - F_\mathcal{L}\big(x_k, k(t_k - \hat{t}_k), k(x_k - \hat{x}_k), M_k\big).$$

Using (5.7), it is readily confirmed that there exists a constant $L > 0$ such that

$$\frac{2\bar{\kappa}}{\eta} \leq F_\mathcal{L}\big(\hat{x}_k, k(t_k - \hat{t}_k), k(x_k - \hat{x}_k), N_k\big) - F_\mathcal{L}\big(x_k, k(t_k - \hat{t}_k), k(x_k - \hat{x}_k), M_k\big) \leq kL|x_k - \hat{x}_k|^2$$

for all $k \in \mathbb{N}$. Now send $k \to \infty$ and use (5.4) to obtain the final contradiction $2\bar{\kappa}/\eta \leq 0$. □

5.3 Stochastic Supersolutions and the Viscosity Property

We next demonstrate that there exists a viscosity solution $\mathbb{V}$ of the QVIs. We use a variant of the stochastic Perron's method, in which it is shown that $\mathbb{V}$ can be constructed as the pointwise minimum of the set of stochastic supersolutions of the QVIs.

Definition 5.6 (Stochastic Supersolutions)**.** *We denote by* $\mathbb{H}$ *the set of* stochastic supersolutions *of the QVIs, i.e. the set of all functions* $h : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$ *such that*

$(\mathbb{H}_1)$ h *is upper semicontinuous;*

$(\mathbb{H}_2)$ *There exists a constant* $K > 0$ *such that*

$$h(t, x) \leq K\big(1 + |x|^p\big), \qquad (t, x) \in [0, T] \times \overline{\mathcal{S}};$$

($\mathbb{H}_3$) *h satisfies the terminal condition*

$$h(T,x) \geq U_{\mathrm{L}}(x), \qquad x \in \overline{\mathcal{S}};$$

($\mathbb{H}_4$) *h is decreasing in the direction of transactions, i.e.*

$$h(t,x) \geq \mathcal{M}[h](t,x), \qquad (t,x) \in [0,T] \times \overline{\mathcal{S}};$$

($\mathbb{H}_5$) *For any pair of $\mathfrak{F}$-stopping times θ, ρ with $0 \leq \theta \leq \rho \leq T$ and any $\mathfrak{F}_\theta$-measurable random vector $\xi = (\xi_0, \xi_1)$ taking values in $\overline{\mathcal{S}}$ with $\mathbb{E}[|\xi|^2] < \infty$, we have*

$$h(\theta,\xi) \geq \mathbb{E}\Big[h\big(\rho, \bar{X}^{\theta,\xi}_\rho\big)\Big|\mathfrak{F}_\theta\Big],$$

where $\bar{X}^{\theta,\xi} = \{\bar{X}^{\theta,\xi}_t\}_{t\in[\theta,T]}$ denotes the uncontrolled portfolio process with $\bar{X}^{\theta,\xi}_\theta = \xi$.

Let us first argue that the set of stochastic supersolutions is not empty.

Lemma 5.7 (Stochastic Supersolution)**.** *Provided that $C > 1/p$, the function Ψ^p_0 defined in* (5.1) *is a stochastic supersolution of the QVIs, i.e. $\Psi^p_0 \in \mathbb{H}$.*

Proof. Being continuous, Ψ^p_0 evidently satisfies ($\mathbb{H}_1$). The growth condition ($\mathbb{H}_2$) is immediate from the definition of Ψ^p_0, and the terminal condition ($\mathbb{H}_3$) follows from the fact that

$$\Psi(T,x) = C(x_0+x_1)^p \geq \tfrac{1}{p}\big(\mathrm{L}(x)\big)^p = U_{\mathrm{L}}(x), \qquad x \in \overline{\mathcal{S}}.$$

The property ($\mathbb{H}_4$), i.e. $\Psi^p_0 - \mathcal{M}[\Psi^p_0] \geq 0$, has already been established in Lemma 5.4. Regarding ($\mathbb{H}_5$), we fix two $\mathfrak{F}$-stopping times θ, ρ with $0 \leq \theta \leq \rho \leq T$ and an $\mathfrak{F}_\theta$-measurable and $\overline{\mathcal{S}}$-valued random vector ξ with $\mathbb{E}[|\xi|^2] < \infty$. Denote by $\{\rho_k\}_{k\in\mathbb{N}}$ a localizing sequence of the local martingale

$$\int_\theta^\cdot \sigma \bar{X}^{\theta,\xi}_u \frac{\partial \Psi^q_0}{\partial x_1}\big(u, \bar{X}^{\theta,\xi}_u\big)\,\mathrm{d}W_u.$$

Then Itō's formula, the supersolution property of Ψ^p_0 established in Lemma 5.4, and Fatou's lemma show that

$$\Psi^p_0(\theta,\xi) \geq \liminf_{k\to\infty} \mathbb{E}\Big[\Psi^p_0\big(\rho_k\wedge\rho, \bar{X}^{\theta,\xi}_{\rho_k\wedge\rho}\big)\Big|\mathfrak{F}_\theta\Big] \geq \mathbb{E}\Big[\Psi^p_0\big(\rho, \bar{X}^{\theta,\xi}_\rho\big)\Big|\mathfrak{F}_\theta\Big].$$

Thus Ψ^p_0 satisfies ($\mathbb{H}_5$), and the proof is complete. □

For each $h \in \mathbb{H}$, we note that Fatou's lemma, (2.2) and (2.3) imply that

$$h\big(\,\cdot\,, X^{t,x,\Lambda}_\cdot\big) \text{ is a strong supermartingale for all } \Lambda \in \mathcal{A}(t,x) \text{ and } (t,x) \in [0,T] \times \overline{\mathcal{S}};$$

see, e.g., [8, Lemma 3.4] or [7, Lemma 5.2] for a detailed argument. Using ($\mathbb{H}_3$), it follows in particular that

$$h(t,x) \geq \mathbb{E}\Big[h\big(T, X^{t,x,\Lambda}_T\big)\Big] \geq \mathbb{E}\Big[U_{\mathrm{L}}\big(X^{t,x,\Lambda}_T\big)\Big] \quad \text{for all } \Lambda \in \mathcal{A}(t,x) \text{ and } (t,x) \in [0,T] \times \overline{\mathcal{S}}$$

and thus $h \geq \mathcal{V} \geq 0$, where $\mathcal{V}$ is the value function of the retail investor's portfolio problem, see (2.5). Thus we have $\mathbb{V} \geq \mathcal{V} \geq 0$ where the function $\mathbb{V}$ is defined by

$$\mathbb{V} : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}_+, \qquad (t,x) \mapsto \mathbb{V}(t,x) \triangleq \inf_{h \in \mathbb{H}} h(t,x). \tag{5.17}$$

By [4, Proposition 4.1], the infimum in (5.17) can be restricted to a countable subset of $\mathbb{H}$, which implies that $\mathbb{V} \in \mathbb{H}$. As a consequence, $\mathbb{V}$ is the pointwise minimum of the members of $\mathbb{H}$.

In the following, we demonstrate that $\mathbb{V}$ is a viscosity solution of the QVIs. We begin with the subsolution property.

Proposition 5.8 (Viscosity Subsolution)**.** *The function* $\mathbb{V}$ *defined in* (5.17) *is a viscosity subsolution of the QVIs.*

Proof. Being a member of $\mathbb{H}$, the function $\mathbb{V}$ is upper semicontinuous; hence we have $\mathbb{V} = \mathbb{V}^*$ and $\mathcal{M}[\mathbb{V}] = \mathcal{M}[\mathbb{V}]^*$ by Lemma 5.2. Assume by contradiction that there exist $(t^*, x^*) \in [0,T) \times \mathcal{S}$ and a test function $\varphi \in \mathrm{C}^2([0,T) \times \mathcal{S})$ with $\varphi \geq \mathbb{V}$, $\varphi(t^*, x^*) = \mathbb{V}(t^*, x^*)$, and

$$\min\big\{\mathcal{L}[\varphi](t^*, x^*), \mathbb{V}(t^*, x^*) - \mathcal{M}[\mathbb{V}](t^*, x^*)\big\} = 2\kappa > 0 \tag{5.18}$$

for some $\kappa > 0$. We can assume without loss that the maximum of $\mathbb{V} - \varphi$ at (t^*, x^*) is global (as only the behavior of φ in a neighborhood of (t^*, x^*) is relevant) and strict (consider $\bar{\varphi}(t,x) \triangleq \varphi(t,x) + |(t,x) - (t^*, x^*)|^4$ instead). Using $\varphi(t^*, x^*) = \mathbb{V}(t^*, x^*)$ in (5.18), continuity of φ and $\mathcal{L}[\varphi]$, and lower semicontinuity of $-\mathcal{M}[\mathbb{V}]$ it follows that there exists $\varepsilon > 0$ such that

$$\min\big\{\mathcal{L}[\varphi](t,x), \varphi(t,x) - \mathcal{M}[\mathbb{V}](t,x)\big\} \geq \kappa > 0, \qquad (t,x) \in \overline{\mathcal{B}}_\varepsilon(t^*, x^*), \tag{5.19}$$

where we set

$$\begin{aligned}
\overline{\mathcal{B}}_\varepsilon(t^*, x^*) &\triangleq \big\{(t,x) \in [0,T] \times \overline{\mathcal{S}} : |(t,x) - (t^*, x^*)| \leq \varepsilon\big\}, \\
\mathcal{B}_\varepsilon(t^*, x^*) &\triangleq \big\{(t,x) \in [0,T] \times \overline{\mathcal{S}} : |(t,x) - (t^*, x^*)| < \varepsilon\big\}.
\end{aligned}$$

Upon making ε smaller if necessary, we may in addition assume that

$$\overline{\mathcal{B}}_\varepsilon(t^*, x^*) \cap \big([0,T] \times \{0\}\big) = \emptyset = \overline{\mathcal{B}}_\varepsilon(t^*, x^*) \cap \big(\{T\} \times \overline{\mathcal{S}}\big). \tag{5.20}$$

Now define

$$\mathcal{D} \triangleq \overline{\mathcal{B}}_\varepsilon(t^*, x^*) \setminus \mathcal{B}_{\varepsilon/2}(t^*, x^*).$$

Since $\mathcal{D}$ is compact and the global maximum of $\mathbb{V} - \varphi \in \mathrm{USC}$ at (t^*, x^*) is strict, there exists some $\delta \in (0, \kappa)$ such that

$$\mathbb{V}(t,x) + \delta \leq \varphi(t,x), \qquad (t,x) \in \mathcal{D}. \tag{5.21}$$

Fixing $\eta \in (0, \delta)$, we define

$$\varphi^\eta : [0,T) \times \mathcal{S} \to \mathbb{R}, \qquad (t,x) \mapsto \varphi^\eta(t,x) \triangleq \varphi(t,x) - \eta,$$

and

$$h^\eta : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}, \quad (t,x) \mapsto h^\eta(t,x) \triangleq \begin{cases} \min\{\mathbb{V}(t,x), \varphi^\eta(t,x)\} & \text{if } (t,x) \in \overline{\mathcal{B}}_\varepsilon(t^*,x^*), \\ \mathbb{V}(t,x) & \text{otherwise.} \end{cases}$$

Since the partial derivatives of φ^η and φ coincide, it follows from (5.19) that

$$\mathcal{L}[\varphi^\eta](t,x) = \mathcal{L}[\varphi](t,x) \geq \kappa > 0, \qquad (t,x) \in \overline{\mathcal{B}}_\varepsilon(t^*,x^*). \tag{5.22}$$

Moreover, we clearly have $\varphi^\eta(t^*,x^*) = \varphi(t^*,x^*) - \eta = \mathbb{V}(t^*,x^*) - \eta < \mathbb{V}(t^*,x^*)$ and thus

$$h^\eta(t^*,x^*) = \varphi^\eta(t^*,x^*) < \mathbb{V}(t^*,x^*). \tag{5.23}$$

By (5.19), (5.20), (5.21), and (5.22) and a standard argument as in [8, Theorem 4.1], it follows that $h^\eta \in \mathbb{H}$. But in view of (5.23) this is incompatible with the definition of $\mathbb{V}$ in (5.17), and we conclude that $\mathbb{V}$ is a viscosity subsolution of the QVIs. □

The following two results characterize the behavior of $\mathbb{V}$ on the boundary of the state space, i.e. on the sets $\{T\} \times \mathcal{S}$ and $[0,T] \times \{0\}$.

Proposition 5.9 (Terminal Inequalities). *The function* $\mathbb{V}$ *defined in* (5.17) *satisfies*

$$\min\big\{\mathbb{V}(T,x) - U_{\mathrm{L}}(x), \mathbb{V}(T,x) - \mathcal{M}[\mathbb{V}](T,x)\big\} \leq 0, \qquad x \in \mathcal{S}.$$

Proof. We argue by contradiction and suppose that there exists $x^* \in \mathcal{S}$ with

$$\min\big\{\mathbb{V}(T,x^*) - U_{\mathrm{L}}(x^*), \mathbb{V}(T,x^*) - \mathcal{M}[\mathbb{V}](T,x^*)\big\} \triangleq \kappa > 0.$$

For each $\varepsilon, \delta > 0$, we define the sets

$$\begin{aligned} \mathcal{B}(\delta,\varepsilon) &\triangleq (T-\delta,T] \times \big\{x \in \overline{\mathcal{S}} : |x - x^*| < \varepsilon\big\}, \\ \overline{\mathcal{B}}(\delta,\varepsilon) &\triangleq [T-\delta,T] \times \big\{x \in \overline{\mathcal{S}} : |x - x^*| \leq \varepsilon\big\}, \\ \mathcal{D}(\delta,\varepsilon) &\triangleq \overline{\mathcal{B}}(\delta,\varepsilon) \setminus \mathcal{B}(\delta/2,\varepsilon/2) = [T-\delta,T-\delta/2] \times \big\{x \in \overline{\mathcal{S}} : \varepsilon/2 \leq |x - x^*| \leq \varepsilon\big\}. \end{aligned}$$

Since U_{L} is continuous and $\mathcal{M}[\mathbb{V}]$ is upper semicontinuous by Lemma 5.2, we can choose $\varepsilon \in (0,\kappa)$ such that $\varepsilon < \min\{|x^*|, T\}$ and

$$\min\big\{\mathbb{V}(T,x^*) - U_{\mathrm{L}}(x), \mathbb{V}(T,x^*) - \mathcal{M}[\mathbb{V}](t,x)\big\} \geq \varepsilon, \qquad (t,x) \in \overline{\mathcal{B}}(\varepsilon,\varepsilon). \tag{5.24}$$

Since $\mathbb{V}$ is locally bounded, there exists $\beta > 0$ sufficiently small such that

$$\mathbb{V}(T,x^*) + \frac{\varepsilon^2}{4\beta} \geq \varepsilon + \sup_{(t,x) \in \mathcal{D}(\delta,\varepsilon)} \mathbb{V}(t,x), \qquad \delta \in (0,\varepsilon]. \tag{5.25}$$

With a fixed constant $L > 0$ to be specified below, we consider the function

$$\varphi : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}, \qquad (t,x) \mapsto \varphi(t,x) \triangleq \mathbb{V}(T,x^*) + \frac{1}{\beta}|x^* - x|^2 + L(T-t).$$

Since the spatial partial derivatives of φ are independent of t and bounded on $\overline{\mathcal{B}}(\varepsilon,\varepsilon)$, and since $(\partial/\partial t)\varphi(t,x) = -L$, we can choose L sufficiently large to ensure that

$$\mathcal{L}[\varphi](t,x) \geq 0, \qquad (t,x) \in \overline{\mathcal{B}}(\varepsilon,\varepsilon). \tag{5.26}$$

Having fixed L in this way, we choose $\delta < \min\{\varepsilon/(2L), \varepsilon\}$. By (5.25) and the fact that $|x - x^*| \geq \varepsilon/2$ and $T - t \geq -\delta$ for all $(t,x) \in \mathcal{D}(\delta,\varepsilon)$, we have

$$\varphi(t,x) \geq \mathbb{V}(t,x) + \frac{\varepsilon}{2}, \qquad (t,x) \in \mathcal{D}(\delta,\varepsilon). \tag{5.27}$$

Moreover, since $T - t \leq -\delta \leq -\varepsilon/(2L)$, it follows from (5.24) that

$$\varphi(t,x) \geq \mathbb{V}(T,x^*) - L\delta \geq U_{\mathrm{L}}(x), \qquad (t,x) \in \overline{\mathcal{B}}(\delta,\varepsilon). \tag{5.28}$$

Fixing $\eta \in (0,\varepsilon/2)$, we define

$$\varphi^\eta : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}, \qquad (t,x) \mapsto \varphi^\eta(t,x) \triangleq \varphi(t,x) - \eta,$$

as well as

$$h^\eta : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}, \quad (t,x) \mapsto h^\eta(t,x) \triangleq \begin{cases} \min\{\mathbb{V}(t,x), \varphi^\eta(t,x)\} & \text{if } (t,x) \in \overline{\mathcal{B}}(\delta,\varepsilon), \\ \mathbb{V}(t,x) & \text{otherwise.} \end{cases}$$

Then $\varphi^\eta(T,x^*) = \varphi(T,x^*) - \eta = \mathbb{V}(T,x^*) - \eta < \mathbb{V}(T,x^*)$ and hence

$$h^\eta(T,x^*) = \varphi^\eta(T,x^*) < \mathbb{V}(T,x^*).$$

Using (5.26), (5.27) and (5.28), one can check as in [8, Proposition 4.2] that $h^\eta \in \mathbb{H}$, contradicting the minimality of $\mathbb{V}$. $\square$

Corollary 5.10 (Boundary Characterization)**.** *The function* $\mathbb{V}$ *defined in* (5.17) *satisfies*

$$\mathbb{V}(t,x) = U_{\mathrm{L}}(x), \qquad (t,x) \in \big([0,T] \times \{0\}\big) \cup \big(\{T\} \times \overline{\mathcal{S}}\big).$$

Proof. We first recall that $\Psi_0^p \in \mathbb{H}$ by Lemma 5.7 and hence

$$0 \leq \mathbb{V}(t,0) \leq \Psi_0^p(t,0) = 0 = U_{\mathrm{L}}(0), \qquad t \in [0,T].$$

Thus we only have to show that

$$\mathbb{V}(T,x) \leq U_{\mathrm{L}}(x), \qquad x \in \mathcal{S}, \tag{5.29}$$

as the reverse inequality follows from $\mathbb{V} \in \mathbb{H}$. The crucial observation to establish (5.29) is that

$$U_{\mathrm{L}}\big(\Gamma(x,\Delta)\big) \leq U_{\mathrm{L}}(x), \qquad \Delta \in \mathcal{D}(x),\ x \in \mathcal{S} \setminus \mathcal{S}_{\emptyset}. \tag{5.30}$$

By Proposition 5.9, we already know that

$$\min\big\{\mathbb{V}(T,x) - U_{\mathrm{L}}(x), \mathbb{V}(T,x) - \mathcal{M}[\mathbb{V}](T,x)\big\} \leq 0, \qquad x \in \mathcal{S}. \tag{5.31}$$

Suppose now that there exists $x^0 \in \mathcal{S}$ such that $\mathbb{V}(T,x^0) \leq \mathcal{M}[\mathbb{V}](T,x^0)$, which is only possible if $x^0 \notin \mathcal{S}_{\emptyset}$. We proceed to show that, in this case, we also have $\mathbb{V}(T,x^0) \leq U_{\mathrm{L}}(x^0)$. By upper semicontinuity of $\mathbb{V}$, we find that

$$\mathbb{V}(T,x^0) \leq \mathcal{M}[\mathbb{V}](T,x^0) = \mathbb{V}(T,x^1),\ \text{where } x^1 \triangleq \Gamma(x^0,\Delta_0) \text{ for some } \Delta_0 \in \mathcal{D}(x^0). \tag{5.32}$$

By (5.31), we either have $\mathbb{V}(T,x^1) \leq U_{\mathrm{L}}(x^1)$, in which case we conclude since then by (5.32) and (5.30)

$$\mathbb{V}(T,x^0) \leq \mathbb{V}(T,x^1) \leq U_{\mathrm{L}}(x^1) = U_{\mathrm{L}}\big(\Gamma(x^0,\Delta_0)\big) \leq U_{\mathrm{L}}(x^0);$$

or we must have $\mathbb{V}(T,x^1) \leq \mathcal{M}[\mathbb{V}](T,x^1)$, which is again only possible if $x^1 \notin \mathcal{S}_{\emptyset}$. But then we may repeat the same argument again with x^1 replacing x^0: There exists $\Delta_1 \in \mathcal{D}(x^1)$ such that, with $x^2 \triangleq \Gamma(x^1,\Delta_1)$,

$$\mathbb{V}(T,x^0) \leq \mathbb{V}(T,x^1) \leq \mathcal{M}[\mathbb{V}](T,x^1) = \mathbb{V}(T,x^2).$$

If $\mathbb{V}(T,x^2) \leq U_{\mathrm{L}}(x^2)$ we conclude that

$$\mathbb{V}(T,x^0) \leq \mathbb{V}(T,x^1) \leq \mathbb{V}(T,x^2) \leq U_{\mathrm{L}}(x^2) \leq U_{\mathrm{L}}(x^1) \leq U_{\mathrm{L}}(x^0),$$

or, otherwise, we must have $\mathbb{V}(T,x^2) \leq \mathcal{M}[\mathbb{V}](T,x^2)$. Since

$$x_0^2 + x_1^2 = x_0^1 + x_1^1 - \mathrm{C}(\Delta_1) = x_0^0 + x_1^0 - \mathrm{C}(\Delta_1) - \mathrm{C}(\Delta_0) \leq x_0^0 + x_1^0 - 2\mathrm{C}_{\min},$$

the above argument can only be repeated finitely many times until we find some $x^j \in \mathcal{S}_{\emptyset}$, $j \in \mathbb{N}$. But then we must necessarily have $\mathbb{V}(T,x^j) \leq U_{\mathrm{L}}(x^j)$ and thus

$$\mathbb{V}(T,x^0) \leq \ldots \leq \mathbb{V}(T,x^j) \leq U_{\mathrm{L}}(x^j) \leq \ldots \leq U_{\mathrm{L}}(x^0). \qquad \square$$

Finally, we establish the supersolution property of $\mathbb{V}$. This is simpler because it follows quite directly from the properties of the members in $\mathbb{H}$.

Proposition 5.11 (Viscosity Supersolutions)**.** *Each Borel measurable function $h : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}_+$ satisfying $(\mathbb{H}_2)$ to $(\mathbb{H}_5)$ is a viscosity supersolution of the QVIs with*

$$h_*(T,x) \geq U_{\mathrm{L}}(x), \qquad x \in \overline{\mathcal{S}}. \tag{5.33}$$

In particular, $\mathbb{V}$ is a viscosity supersolution of the QVIs.

Proof. By $(\mathbb{H}_4)$, h satisfies

$$h(t,x) \geq \mathcal{M}[h](t,x) \geq \mathcal{M}[h]_*(t,x), \qquad (t,x) \in [0,T) \times \mathcal{S}.$$

But then, since $\mathcal{M}[h]_*$ is lower semicontinuous, it must be dominated by the lower semicontinuous envelope of h, i.e.

$$h_*(t,x) - \mathcal{M}[h]_*(t,x) \geq 0, \qquad (t,x) \in [0,T) \times \mathcal{S}. \tag{5.34}$$

Now fix $(\bar{t},\bar{x}) \in [0,T) \times \mathcal{S}$ and $\varphi \in C^2([0,T) \times \mathcal{S})$ with $\varphi \leq h_*$ and $\varphi(\bar{t},\bar{x}) = h_*(\bar{t},\bar{x})$. We choose a sequence $\{(t_k,x_k)\}_{k\in\mathbb{N}} \subset [0,T) \times \mathcal{S}$ converging to $(\bar{t},\bar{x})$ such that

$$h_*(\bar{t},\bar{x}) = \lim_{k\to\infty} h(t_k,x_k).$$

Since φ is continuous and $\varphi \leq h_* \leq h$, we see that

$$0 \leq \gamma_k \triangleq h(t_k,x_k) - \varphi(t_k,x_k) \to 0 \qquad \text{as } k \to \infty.$$

Now fix a sequence $\{\delta_k\}_{k\in\mathbb{N}}$ of strictly positive real numbers with

$$\lim_{k\to\infty} \delta_k = \lim_{k\to\infty} \frac{\gamma_k}{\delta_k} = 0.$$

Moreover, let $\varepsilon > 0$ and define

$$\rho_k \triangleq \inf\big\{t \in [t_k,T] : |\bar{X}^k_u - x_k| \geq \varepsilon\big\} \wedge (t_k + \delta_k) \wedge T, \qquad k \in \mathbb{N},$$

where $\bar{X}^k \triangleq \bar{X}^{t_k,x_k}$. Using $(\mathbb{H}_5)$, the inequality $h \geq h_* \geq \varphi$, and Itō's formula yields

$$h(t_k,x_k) \geq \mathbb{E}\Big[h\big(\rho_k, \bar{X}^k_{\rho_k}\big)\Big] \geq \mathbb{E}\Big[\varphi\big(\rho_k, \bar{X}^k_{\rho_k}\big)\Big] = \varphi(t_k,x_k) - \mathbb{E}\Big[\int_{t_k}^{\rho_k} \mathcal{L}[\varphi](u,\bar{X}^k_u)\,\mathrm{d}u\Big].$$

Upon rearranging and dividing by δ_k, it follows that

$$\frac{\gamma_k}{\delta_k} + \mathbb{E}\Big[\frac{1}{\delta_k}\int_{t_k}^{\rho_k} \mathcal{L}[\varphi](u,\bar{X}^k_u)\,\mathrm{d}u\Big] \geq 0.$$

Now $\rho_k(\omega) = t_k + \delta_k$ for eventually all $k \in \mathbb{N}$ and $\mathbb{P}$-almost every $\omega \in \Omega$. Thus, upon sending $k \to \infty$, the mean value theorem and dominated convergence imply that

$$\mathcal{L}[\varphi](\bar{t},\bar{x}) \geq 0.$$

In combination with (5.34), this means that h is a viscosity supersolution of the QVIs. It remains to establish (5.33). For this, fix $x \in \overline{\mathcal{S}}$ and choose a sequence $\{(t_k,x_k)\}_{k\in\mathbb{N}} \subset [0,T] \times \overline{\mathcal{S}}$ converging to (T,x) such that

$$\lim_{k\to\infty} h(t_k,x_k) = h_*(T,x).$$

Then $(\mathbb{H}_5)$, $(\mathbb{H}_3)$, and Fatou's lemma yield

$$h_*(T,x) = \lim_{k\to\infty} h(t_k,x_k) \geq \liminf_{k\to\infty} \mathbb{E}\Big[h\big(T,\bar{X}_T^{t_k,x_k}\big)\Big] \geq \liminf_{k\to\infty} \mathbb{E}\Big[U_{\mathrm{L}}\big(\bar{X}_T^{t_k,x_k}\big)\Big] \geq U_{\mathrm{L}}(x),$$

and the proof is complete. □

Combining the viscosity sub- and supersolution properties of $\mathbb{V}$ with the comparison principle characterizes $\mathbb{V}$ as the unique continuous viscosity solution of the QVIs.

Theorem 5.12 (Viscosity Characterization of $\mathbb{V}$)**.** *The function* $\mathbb{V}$ *defined in* (5.17) *is a continuous viscosity solution of the QVIs. It is unique in the class of nonnegative functions satisfying the growth condition* $(\mathbb{H}_2)$ *and the boundary/terminal conditions*

$$\mathbb{V}(t,x) = U_{\mathrm{L}}(x), \qquad (t,x) \in \big([0,T]\times\{0\}\big) \cup \big(\{T\}\times\overline{\mathcal{S}}\big).$$

Proof. By Proposition 5.8, Corollary 5.10, and Proposition 5.11, $\mathbb{V} = \mathbb{V}^*$ is an upper semicontinuous viscosity solution of the QVIs satisfying

$$\mathbb{V}(t,x) = U_{\mathrm{L}}(x) \leq \mathbb{V}_*(t,x), \qquad (t,x) \in \big([0,T]\times\{0\}\big) \cup \big(\{T\}\times\overline{\mathcal{S}}\big).$$

The comparison principle in Theorem 5.5 thus shows that $\mathbb{V}_* \geq \mathbb{V} = \mathbb{V}^*$, i.e. $\mathbb{V}$ is continuous. Uniqueness is also a consequence of Theorem 5.5. □

5.4 Construction of Optimal Strategies

In this final subsection we show that the value function $\mathcal{V} = \mathbb{V}$ and provide an explicit construction of optimal trading strategies for the retail investor's portfolio problem. For this, we define the continuation and intervention regions defined in terms of $\mathbb{V}$ via[15]

$$\begin{aligned}\mathcal{C} &\triangleq \big\{(t,x)\in[0,T]\times\overline{\mathcal{S}} : \mathbb{V}(t,x) > \mathcal{M}[\mathbb{V}](t,x)\big\},\\ \mathcal{I} &\triangleq \big\{(t,x)\in[0,T]\times\overline{\mathcal{S}} : \mathbb{V}(t,x) = \mathcal{M}[\mathbb{V}](t,x)\big\}.\end{aligned}$$

Since $\mathbb{V}$ is continuous and $\mathcal{D}$ is compact-valued, a classical measurable selection argument, see [24], yields a Borel measurable function

$$\delta : [0,T]\times\big(\overline{\mathcal{S}}\setminus\mathcal{S}_\emptyset\big) \to \mathbb{R}, \qquad (t,x)\mapsto \delta(t,x),$$

such that

$$\delta(t,x)\in\mathcal{D}(x) \quad \text{and} \quad \mathcal{M}[\mathbb{V}](t,x) = \mathbb{V}\big(t,\Gamma\big(x,\delta(t,x)\big)\big), \quad (t,x)\in[0,T]\times\big(\overline{\mathcal{S}}\setminus\mathcal{S}_\emptyset\big).$$

[15]Note that we use $\mathbb{V}$ as defined in (5.17), *not* the value function $\mathcal{V}$. Our Verification Theorem 5.13 below shows that, in fact, $\mathcal{V} = \mathbb{V}$.

For any fixed $(t,x) \in [0,T] \times \overline{\mathcal{S}}$, we define a candidate optimal strategy $\Lambda^* = \{(\tau_k^*, \Delta_k^*)\}_{k\in\mathbb{N}}$ as follows: Set $(\tau_0^*, \xi_0^*) \triangleq (t,x)$ and, iteratively for all $k \in \mathbb{N}$,

$$\bar{X}^k \triangleq \bar{X}^{\tau_{k-1}^*, \xi_{k-1}^*}, \qquad \tau_k^* \triangleq \inf\{u \in [\tau_{k-1}^*, T] : (u, \bar{X}_u^k) \in \mathcal{I}\}, \tag{5.35}$$

$$\Delta_k^* \triangleq \delta\big(\tau_k^*, \bar{X}_{\tau_k^*}^k\big)\mathbb{1}_{\{\tau_k^* \le T\}}, \qquad \xi_k^* \triangleq \Gamma\big(\bar{X}_{\tau_k^*}^k, \Delta_k^*\big), \tag{5.36}$$

where we recall that for any $[0,T]$-valued stopping time τ and $\overline{\mathcal{S}}$-valued random variable ξ, we write $\bar{X}^{\tau_k^*,\xi_k^*}$ for the uncontrolled portfolio process $\{\bar{X}_t^{\tau,\xi}\}_{t\in[\tau,T]}$ with $\bar{X}_\tau^{\tau,\xi} = \xi$. From the above construction, it follows immediately that $\Lambda^* \in \mathcal{A}(t,x)$. The following verification result demonstrates rigorously that Λ^* is optimal and $\mathbb{V} = \mathcal{V}$; its proof is based on the superharmonic function technique in [7, 8, 10].

Theorem 5.13 (Verification Theorem). *For every $(t,x) \in [0,T] \times \overline{\mathcal{S}}$ we have*

$$\mathbb{V}(t,x) = \mathcal{V}(t,x) = \mathbb{E}\Big[U_{\mathrm{L}}\big(X_T^{t,x,\Lambda^*}\big)\Big]$$

where $\Lambda^* = \{(\tau_k^*, \Delta_k^*\}_{k\in\mathbb{N}}$ is the trading strategy defined via (5.35) *and* (5.36).

Proof. We fix $(t,x) \in [0,T] \times \overline{\mathcal{S}}$. Since Λ^* is admissible, we have $\mathbb{E}[U_{\mathrm{L}}(X_T^{t,x,\Lambda^*})] \le \mathcal{V}(t,x)$. As we have already shown that $\mathbb{V} \ge \mathcal{V}$, it suffices to demonstrate that

$$\mathbb{V}(t,x) = \mathbb{E}\Big[U_{\mathrm{L}}\big(X_T^{t,x,\Lambda^*}\big)\Big].$$

We set $X^* \triangleq X^{t,x,\Lambda^*}$ for ease of notation.

Step 1. For every $\lambda \in (0,1)$, we define

$$\mathcal{C}_\lambda \triangleq \big\{(t,x) \in [0,T] \times \overline{\mathcal{S}} : \lambda\mathbb{V}(t,x) > \mathcal{M}[\mathbb{V}](t,x)\big\},$$
$$\mathcal{I}_\lambda \triangleq \big\{(t,x) \in [0,T] \times \overline{\mathcal{S}} : \lambda\mathbb{V}(t,x) \le \mathcal{M}[\mathbb{V}](t,x)\big\}.$$

Since $\lambda\mathbb{V} - \mathcal{M}[\mathbb{V}]$ is lower semicontinuous, $\mathcal{I}_\lambda$ is closed and hence $\mathcal{C}_\lambda$ is open. Moreover, $\mathcal{I}_\lambda$ is decreasing in λ with $\mathcal{I} = \bigcap_{\lambda\in(0,1)} \mathcal{I}_\lambda$. For each $\lambda \in (0,1)$, we construct a family of stopping times via

$$\vartheta_{\bar{t},\bar{x}}^\lambda \triangleq \inf\Big\{u \in [\bar{t}, T] : \big(u, \bar{X}_u^{\bar{t},\bar{x}}\big) \in \mathcal{I}_\lambda\Big\} \wedge T, \qquad (\bar{t},\bar{x}) \in [0,T] \times \overline{\mathcal{S}}.$$

With this, we define two functions

$$h : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}_+, \qquad (\bar{t},\bar{x}) \mapsto h(\bar{t},\bar{x}) \triangleq \mathbb{E}\Big[\mathbb{V}\big(\vartheta_{\bar{t},\bar{x}}^\lambda, \bar{X}_{\vartheta_{\bar{t},\bar{x}}^\lambda}^{\bar{t},\bar{x}}\big)\Big],$$

and

$$h_\lambda : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}_+, \qquad (\bar{t},\bar{x}) \mapsto h_\lambda(\bar{t},\bar{x}) \triangleq \lambda\mathbb{V}(\bar{t},\bar{x}) + (1-\lambda)h(\bar{t},\bar{x}).$$

Step 2. We show that $h_\lambda \ge \mathbb{V}$. For this, using that h_λ is clearly Borel measurable, it suffices to show that h_λ satisfies $(\mathbb{H}_2)$ to $(\mathbb{H}_5)$; indeed, in that case Proposition 5.11 implies that h_λ is a viscosity supersolution of the QVIs, so the comparison principle in Theorem 5.5 implies that $h_\lambda \geq \mathbb{V}$. We first observe that since $\mathbb{V}$ satisfies $(\mathbb{H}_5)$, we have

$$h(\bar{t}, \bar{x}) = \mathbb{E}\Big[\mathbb{V}\big(\vartheta^\lambda_{\bar{t},\bar{x}}, \bar{X}^{\bar{t},\bar{x}}_{\vartheta^\lambda_{\bar{t},\bar{x}}}\big)\Big] \leq \mathbb{V}(\bar{t}, \bar{x}), \qquad (\bar{t}, \bar{x}) \in [0, T] \times \overline{\mathcal{S}}$$

and hence $h_\lambda \leq \mathbb{V}$, so h_λ satisfies the growth condition $(\mathbb{H}_2)$ because $\mathbb{V}$ does. The terminal condition $(\mathbb{H}_3)$ for h_λ holds because $h(T, \cdot\,) = \mathbb{V}(T, \cdot\,)$, whence $h_\lambda(T, \cdot\,) = \mathbb{V}(T, \cdot\,) = U_{\mathrm{L}}$. To establish $(\mathbb{H}_4)$ for h_λ, we fix $(\bar{t}, \bar{x}) \in [0, T] \times \overline{\mathcal{S}}$. Since $h \leq \mathbb{V}$, we have

$$\begin{aligned}\mathcal{M}[h_\lambda](\bar{t}, \bar{x}) &\leq \lambda \mathcal{M}[h](\bar{t}, \bar{x}) + (1-\lambda)\mathcal{M}[\mathbb{V}](\bar{t}, \bar{x}) \\ &\leq \lambda \mathcal{M}[\mathbb{V}](\bar{t}, \bar{x}) + (1-\lambda)\mathcal{M}[\mathbb{V}](\bar{t}, \bar{x}) = \mathcal{M}[\mathbb{V}](\bar{t}, \bar{x}).\end{aligned}$$

If $(\bar{t}, \bar{x}) \in \mathcal{I}_\lambda$, then $\theta^\lambda_{\bar{t},\bar{x}} = \bar{t}$. Thus $h(\bar{t}, \bar{x}) = \mathbb{V}(\bar{t}, \bar{x})$ and therefore also $h_\lambda(\bar{t}, \bar{x}) = \mathbb{V}(\bar{t}, \bar{x})$; since $\mathbb{V}$ satisfies $(\mathbb{H}_4)$, it follows that

$$\mathcal{M}[h_\lambda](\bar{t}, \bar{x}) \leq \mathcal{M}[\mathbb{V}](\bar{t}, \bar{x}) \leq \mathbb{V}(\bar{t}, \bar{x}) = h_\lambda(\bar{t}, \bar{x}), \qquad (\bar{t}, \bar{x}) \in \mathcal{I}_\lambda.$$

If, on the other hand, $(\bar{t}, \bar{x}) \in \mathcal{C}_\lambda$, then $\mathcal{M}[\mathbb{V}](\bar{t}, \bar{x}) < \lambda \mathbb{V}(\bar{t}, \bar{x}) \leq \mathbb{V}(\bar{t}, \bar{x})$ and thus

$$\mathcal{M}[h_\lambda](\bar{t}, \bar{x}) \leq \mathcal{M}[\mathbb{V}](\bar{t}, \bar{x}) < \mathbb{V}(\bar{t}, \bar{x}), \qquad (\bar{t}, \bar{x}) \in \mathcal{C}_\lambda.$$

In summary, we have demonstrated that h_λ satisfies $(\mathbb{H}_4)$. It remains to verify $(\mathbb{H}_5)$. Since $\mathbb{V}$ satisfies $(\mathbb{H}_5)$, by linearity it is clearly sufficient to show that h satisfies $(\mathbb{H}_5)$. But this property is inherited from $\mathbb{V}$ by pathwise uniqueness and the strong Markov property of $\bar{X}$. We therefore conclude that $h_\lambda \geq \mathbb{V}$.

Step 3. For $k \in \mathbb{N}_0$, let us set

$$(\tau, \xi) \triangleq (\tau^*_k, \xi^*_k), \qquad \bar{X} \triangleq \bar{X}^{\tau,\xi}, \qquad \text{and} \qquad \vartheta^\lambda \triangleq \vartheta^\lambda_{\tau,\xi}.$$

By definition of h and the strong Markov property, we have

$$h(\tau, \xi) = \mathbb{E}\Big[\mathbb{V}\big(\vartheta^\lambda_{\bar{t},\bar{x}}, \bar{X}^{\bar{t},\bar{x}}_{\vartheta^\lambda_{\bar{t},\bar{x}}}\big)\Big]\Big|_{(\bar{t},\bar{x})=(\tau,\xi)} = \mathbb{E}\Big[\mathbb{V}\big(\vartheta^\lambda, \bar{X}_{\vartheta^\lambda}\big)\Big|\mathfrak{F}_\tau\Big] \qquad \text{on } \{\tau \leq T\}.$$

Since $h_\lambda \geq \mathbb{V}$, it follows that

$$\mathbb{V}(\tau, \xi) \leq h_\lambda(\tau, \xi) = \lambda \mathbb{V}(\tau, \xi) + (1-\lambda)\mathbb{E}\Big[\mathbb{V}\big(\vartheta^\lambda, \bar{X}_{\vartheta^\lambda}\big)\Big|\mathfrak{F}_\tau\Big] \qquad \text{on } \{\tau \leq T\}.$$

Upon rearranging, dividing by $(1-\lambda)$, and using property $(\mathbb{H}_5)$ of $\mathbb{V}$, we obtain

$$\mathbb{V}(\tau, \xi) \leq \mathbb{E}\Big[\mathbb{V}\big(\vartheta^\lambda, \bar{X}_{\vartheta^\lambda}\big)\Big|\mathfrak{F}_\tau\Big] \leq \mathbb{V}(\tau, \xi) \qquad \text{on } \{\tau \leq T\}. \tag{5.37}$$

Step 4. Since $\vartheta^\lambda \leq \tau^*_{k+1} \wedge T$ and the mapping $\lambda \mapsto \vartheta^\lambda$ is increasing, it follows that $\vartheta \triangleq \lim_{\lambda\uparrow 1} \vartheta^\lambda$ exists and satisfies $\vartheta \leq \tau^*_{k+1} \wedge T$. On the other hand, since $\mathbb{V}$ is continuous and satisfies $(\mathbb{H}_4)$, $(\vartheta^\lambda, \bar{X}_{\vartheta^\lambda}) \in \mathcal{I}_\lambda$, and $\mathcal{M}[\mathbb{V}]$ is upper semicontinuous, we have

$$\begin{aligned}\mathcal{M}[\mathbb{V}](\vartheta, \bar{X}_\vartheta) \leq \mathbb{V}(\vartheta, \bar{X}_\vartheta) = \lim_{\lambda\uparrow 1} \mathbb{V}(\vartheta^\lambda, \bar{X}_{\vartheta^\lambda}) \\ \leq \limsup_{\lambda\uparrow 1} \frac{1}{\lambda}\mathcal{M}[\mathbb{V}](\vartheta^\lambda, \bar{X}_{\vartheta^\lambda}) \leq \mathcal{M}[\mathbb{V}](\vartheta, \bar{X}_\vartheta) \quad \text{on } \{\tau \leq T\},\end{aligned}$$

which is only possible if $\mathcal{M}[\mathbb{V}](\vartheta, \bar{X}_\vartheta) = \mathbb{V}(\vartheta, \bar{X}_\vartheta)$, i.e. $\vartheta = \tau^*_{k+1}$ on $\{\tau^*_{k+1} \leq T\}$. As a consequence, using (5.37) and the fact that $(\vartheta^\lambda, \bar{X}_{\vartheta^\lambda}) \in \mathcal{I}_\lambda$, dominated convergence and upper semicontinuity of $\mathcal{M}[\mathbb{V}]$, and finally $(\mathbb{H}_4)$ and $(\mathbb{H}_5)$, it follows that

$$\begin{aligned}\mathbb{V}(\tau, \xi) &= \lim_{\lambda\uparrow 1} \mathbb{E}\Big[\mathbb{V}\big(\vartheta^\lambda, \bar{X}_{\vartheta^\lambda}\big)\Big|\mathfrak{F}_\tau\Big] \\ &\leq \limsup_{\lambda\uparrow 1} \frac{1}{\lambda}\mathbb{E}\Big[\mathcal{M}[\mathbb{V}]\big(\vartheta^\lambda, \bar{X}_{\vartheta^\lambda}\big)\Big|\mathfrak{F}_\tau\Big] \\ &\leq \mathbb{E}\Big[\mathcal{M}[\mathbb{V}]\big(\tau^*_{k+1}, \bar{X}_{\tau^*_{k+1}}\big)\Big|\mathfrak{F}_\tau\Big] \leq \mathbb{E}\Big[\mathbb{V}\big(\tau^*_{k+1}, \bar{X}_{\tau^*_{k+1}}\big)\Big|\mathfrak{F}_\tau\Big] \leq \mathbb{V}(\tau, \xi)\end{aligned}$$

on $\{\tau^*_{k+1} \leq T\}$, where in fact we have equality everywhere. Now by definition of Λ^* and using $(\tau^*_k, \xi^*_k) = (\tau, \xi)$, we obtain

$$\mathbb{V}(\tau^*_k, \xi^*_k) = \mathbb{E}\Big[\mathcal{M}[\mathbb{V}]\big(\tau^*_{k+1}, \bar{X}_{\tau^*_{k+1}}\big)\Big|\mathfrak{F}_{\tau^*_k}\Big] = \mathbb{E}\Big[\mathbb{V}\big(\tau^*_{k+1}, \xi^*_{k+1}\big)\Big|\mathfrak{F}_{\tau^*_k}\Big] \quad \text{on } \{\tau^*_{k+1} \leq T\}.$$

Iteratively applying this equality, using the definition of X^*, the fact $\mathbb{P}[\lim_{k\to\infty} \tau^*_k > T] = 1$ and dominated convergence, and finally the terminal condition $\mathbb{V}(T, \cdot\,) = U_{\mathrm{L}}$, it follows that

$$\begin{aligned}\mathbb{V}(t, x) = \lim_{k\to\infty} \mathbb{E}\Big[\mathbb{V}\big(\tau^*_k, \xi^*_k\big)\mathbb{1}_{\{\tau^*_k \leq T\}} + \mathbb{V}\big(T, X^*_T\big)\mathbb{1}_{\{\tau^*_k > T\}}\Big] \\ = \mathbb{E}\big[\mathbb{V}\big(T, X^*_T\big)\big] = \mathbb{E}\Big[U_{\mathrm{L}}\big(X^*_T\big)\Big],\end{aligned}$$

and the proof is complete. □

Acknowledgments. Lukas Mich and Frank Seifried gratefully acknowledge financial support from Deutsche Forschungsgemeinschaft (DFG) within the Research Training Group 2126.

References

[1] A. Altarovici, J. Muhle-Karbe, and H. M. Soner. Asymptotics for fixed transaction costs. *Finance Stoch.*, 19(2):363–414, 2015.

[2] A. Altarovici, M. Reppen, and H. M. Soner. Optimal consumption and investment with fixed and proportional transaction costs. *SIAM J. Control Optim.*, 55(3):1673–1710, 2017.

[3] P. Azimzadeh and P. A. Forsyth. Weakly chained matrices, policy iteration, and impulse control. *SIAM J. Numer. Anal.*, 54(3):1341–1364, 2016.

[4] E. Bayraktar and M. Sîrbu. Stochastic Perron's method and verification without smoothness using viscosity comparison: the linear case. *Proc. Amer. Math. Soc.*, 140(10):3645–3654, 2012.

[5] E. Bayraktar and M. Sîrbu. Stochastic Perron's method for Hamilton-Jacobi-Bellman equations. *SIAM J. Control Optim.*, 51(6):4274–4294, 2013.

[6] E. Bayraktar and M. Sîrbu. Stochastic Perron's method and verification without smoothness using viscosity comparison: obstacle problems and Dynkin games. *Proc. Amer. Math. Soc.*, 142(4):1399–1412, 2014.

[7] C. Belak and S. Christensen. Utility maximisation in a factor model with constant and proportional transaction costs. *Finance Stoch.*, 23(1):29–96, 2019.

[8] C. Belak, S. Christensen, and F. T. Seifried. A general verification result for stochastic impulse control problems. *SIAM J. Control Optim.*, 55(2):627–649, 2017.

[9] T. R. Bielecki and S. R. Pliska. Risk sensitive asset management with transaction costs. *Finance Stoch.*, 4(1):1–33, 2000.

[10] S. Christensen. On the solution of general impulse control problems using superharmonic functions. *Stoch. Proc. Appl.*, 124(1):709–729, 2014.

[11] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[12] C. Czichowsky and W. Schachermayer. Portfolio optimisation beyond semimartingales: shadow prices and fractional Brownian motion. *Ann. Appl. Probab.*, 27(3):1414–1451, 2017.

[13] M. H. A. Davis and A. R. Norman. Portfolio selection with transaction costs. *Math. Oper. Res.*, 15(4):676–713, 1990.

[14] J. F. Eastham and K. J. Hastings. Optimal impulse control of portfolios. *Math. Oper. Res.*, 13(4):588–605, 1988.

[15] M.-R. Feodoria. *Optimal investment and utility indifference pricing in the presence of small fixed transaction costs.* PhD thesis, Christian-Albrechts-Universität Kiel, 2016.

[16] N. Garleanu and L. H. Pedersen. Dynamic portfolio choice with frictions. *J. Econ. Theory*, 165:487–516, 2016.

[17] K. Ishii. Viscosity solutions of nonlinear second order elliptic PDEs associated with impulse control problems. *Funkcial. Ekvac.*, 36(1):123–141, 1993.

[18] J. Kallsen and J. Muhle-Karbe. On using shadow prices in portfolio optimization with transaction costs. *Ann. Appl. Probab.*, 20(4):1341–1358, 2010.

[19] R. Korn. Portfolio optimisation with strictly positive transaction costs and impulse control. *Finance Stoch.*, 2(2):85–114, 1998.

[20] H. Liu. Optimal consumption and investment with transaction costs and multiple risky assets. *J. Finance*, 59(1):289–338, 2004.

[21] A. J. Morton and S. R. Pliska. Optimal portfolio management with fixed transaction costs. *Math. Finance*, 5(4):337–356, 1995.

[22] B. Øksendal and A. Sulem. Optimal consumption and portfolio with both fixed and proportional transaction costs. *SIAM J. Control Optim.*, 40(6):1765–1790, 2002.

[23] J. Palczewski and Ł. Stettner. Impulsive control of portfolios. *Appl. Math. Optim.*, 56(1):67–103, 2007.

[24] M. Schäl. A selection theorem for optimization problems. *Arch. Math. (Basel)*, 25:219–224, 1974.

[25] S. E. Shreve and H. M. Soner. Optimal investment and consumption with transaction costs. *Ann. Appl. Probab.*, 4(3):609–692, 1994.