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Portfolio Optimization with Optimal Expected Utility Risk Measures

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──────── ABSTRACT ────────

The purpose of this article is to evaluate optimal expected utility risk measures (OEU) in a risk-constrained portfolio optimization context where the expected portfolio return is maximized. We compare the portfolio optimization with OEU constraint to a portfolio selection model using value at risk as constraint. The former is a coherent risk measure for utility functions with constant relative risk aversion and allows individual specifications to the investor's risk attitude and time preference. In a case study with three indices we investigate how these theoretical differences influence the performance of the portfolio selection strategies. A copula approach with univariate ARMA-GARCH models is used in a rolling forecast to simulate monthly future returns and calculate the derived measures for the optimization. The results of this study illustrate that both optimization strategies perform considerably better than an equally weighted portfolio and a buy and hold portfolio. Moreover, our results illustrate that portfolio optimization with OEU constraint experiences individualized effects, e.g. less risk averse investors lose more portfolio value in the financial crises but outperform their more risk averse counterparts in bull markets.

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1 Introduction

In this article, we consider portfolio selection problems that maximize the expected portfolio return while constraining the associated risk. More precisely, a portfolio optimization problem with optimal expected utility risk measures (OEU) constraint is compared with one using value at risk (V@R) as risk restriction. The V@R-constrained optimization problem was firstly introduced by (Gaivoronski and Pflug; 1999) and (Mausser and Rosen; 1999). V@R is non-convex and thus might come along with complex computations - an issue that was studied in (Krokhmal et al.; 2002) and also confirmed by a hedge funds data application of (Chabaane et al.; 2006). The latter study proved the V@R optimization to be lengthy and difficult while the algorithms of the other considered risk measures (namely standard deviation, semi-variance and average value at risk) appeared to be fast and efficient. However, the optimal allocation results turned out to be very similar and thus the choice of risk measure does not seem to be relevant in this context. The same measures of risk are compared in (Liang and Park; 2007) for a similar time horizon and the identical asset class. In contrast to (Chabaane et al.; 2006) they found downside risk measures to be superior to the standard deviation. Another notable paper on portfolio optimization under risk constraints is (Adam et al.; 2008) where moment-based, distortion and spectral risk measures are used as risk constraints. In a bootstrapping analysis of 14 asset classes, (Xiong and Idzorek; 2011) showed that the mean-conditional value at risk optimization problem outperforms the variance-constrained optimization during the financial crisis of 2008/2009. On the contrary, an application of (Allen et al.; 2016) on European market indices did not find downside risk optimization strategies to be superior to the mean-variance problem for periods of crises.

In this article, we are introducing the portfolio optimization problem with OEU constraint and compare it to a V@R bounded problem in a case study of three indices for a twenty-year time horizon. We carry out a monthly rolling forecast from 2006 to 2016 and calculate the future expected portfolio return, V@R and OEU risk measure values using Monte Carlo simulation based on an ARMA-GARCH and copula approach for an estimation period of 10 years.

The article is structured as follows: Section 2 presents V@R and OEU in a risk-constrained portfolio optimization context. This theoretical framework is applied on a real-world data set by using a simulation and forecasting approach in section 3. After analyzing and comparing the results of the case study and some further analysis, the article closes with some concluding remarks.

2 Portfolio optimization with risk constraints

The mean-variance approach, as introduced by (Markowitz; 1952), firstly considered the risk-return tradeoff in portfolio selection by maximizing the expected portfolio return or minimizing the related variance. While the mean of the random return distribution can be seen as an acceptable measure for expected returns in the future, the variance as a measure of risk has been highly questioned and criticized in the last decades. The increasing research in this topic is attributable to the fact that the mean-variance approach only leads to an optimal decision if unrealistic assumptions such as quadratic utility functions and elliptical return distributions are met.

2.1 Setup and notations

The theoretical setup and notations are, with respect to risk measures, similar to (Geissel et al.; 2017). With the filtration $(\mathcal{F})_{0 \leq t \leq T}$, $T > 0$ that is fixed on the probability space $(\Omega, \mathcal{F}, P)$ in such a way that $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$, we can specify a *financial position* $X : \Omega \to \mathbb{R}$ as a random variable on the probability space $(\Omega, \mathcal{F}, P)$. The *financial position's payoff* is $X(\omega)$ with $\omega \in \Omega$. It is understood as the net payoff of X at the end of the regarded time horizon if ω occurs. The *set of all financial positions* is defined as $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, P)$ with left support X_{min}:=ess inf X. Further, we denote the *risk-free discount factor* as $\beta = 1/(1 + rT)$ with $\beta > 0$, the annual risk-free interest rate r and time horizon per year T. Note that this definition of β allows for positive as well as for negative risk-free interest rates.

2.2 Risk measures for portfolio optimization

(Artzner et al.; 1999) initiated an axiomatic framework of risk measures (namely normalization, monotonicity, cash invariance and subadditivity) and introduced the notion of coherence. One of the most popular risk measures (which is, however, not subadditive) is value at risk (V@R).

Example 1. The *value at risk* at level $\lambda \in (0,1)$ of a position $X \in \mathcal{X}$ is defined by

$$V@R^\lambda(X) = \inf\{m \in \mathbb{R} : P[m + \beta X < 0] \leq \lambda\}.$$

The framework of (Artzner et al.; 1999) led to increasing research on risk measures in general as well as on their application in portfolio optimization, see e.g. (Rockafellar and Uryasev; 2000), (Föllmer and Schied; 2002) or (Acerbi and Tasche; 2002). Another classical approach towards financial risk is the expected utility theory of (Von Neumann and Morgenstern; 1947) which is based on the formalization of utility and is commonly known for decision making under uncertainty and especially accepted as a model for rational choice in economics. We use the following definition of utility functions.

Definition 1. A *utility function* is a C^3 function $u : \mathbb{R}_+ \to \mathbb{R}$ with $u'(x) > 0$ and $u''(x) < 0$ for $x \in \mathbb{R}_+$ and with concave absolute risk tolerance. Here $\mathbb{R}_+ := (0, \infty)$ and the absolute risk tolerance is given by

$$\tau_u(x) := -\frac{u'(x)}{u''(x)}, \quad x \in \mathbb{R}_+.$$

If u is a utility function, we define $u(0) := \lim_{x \downarrow 0} u(x) \in [-\infty, \infty)$, $u(\infty) := \lim_{x \uparrow \infty} u(x) \in (-\infty, \infty]$. Moreover, we denote the inverse of u by $u^{-1} : [u(0), u(\infty)) \to [0, \infty)$. u^{-1} is well-defined and continuous, and $u^{-1}(u(x)) = x$ for all $x \in [0, \infty)$. Finally, we set $u^{-1}(\mathbb{E}[u(Y)]) := -\infty$ if $P(Y < 0) > 0$. In this article we investigate portfolio optimization subjet to a risk constraint specified in terms of the following utility-based risk measure.

Example 2. Given a utility function u, the *optimal expected utility risk measure* is introduced in Geissel et al. (2017) as the map $\rho^u : \mathcal{X} \to \mathbb{R}$,

$$\rho^u(X) := -\sup_{\eta \in \mathbb{R}} \left\{ -\beta \eta + \alpha u^{-1} \left(\mathbb{E}\left[u\left(X + \eta \right) \right] \right) \right\}. \tag{OEU}$$

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α denotes the investor's subjective time preference and $0 < \alpha < \beta$. OEU is a convex risk measure for every utility function. Moreover, if u has constant relative risk aversion (CRRA), i.e. $u(x) = \frac{1}{1-\gamma}\left(x^{1-\gamma} - 1\right)$ for some $\gamma > 0$, $\gamma \neq 1$, or $u(x) = \ln(x)$, then ρ^u is a coherent risk measure. The supremum in (OEU) is attained uniquely at $\eta_X^* \in [-X_{\min}, \infty)$. We refer to (Geissel et al.; 2017) for proofs of the aforementioned properties and for a natural economic interpretation for (OEU) in terms of risk capital.

OEU considers the entire information of return distributions, while downside risk measures such as V@R consider only parts of the distribution. Moreover, additional parameters can be determined according to the investor's risk attitude and time preference within OEU. This allows for an individual parametrization. In the following, we compare OEU with CRRA utility functions to V@R in a portfolio optimization context and examine the implications of different choices of risk attitudes and time preferences.

2.3 Return-risk portfolio optimization

We treat the financial position's net payoff X as a vector containing net payoffs of the regarded assets. X is equivalent to the absolute returns of the considered indices for a particular investment and time horizon. For a general derivation we are looking at n assets that generate relative returns within a predetermined period of

$$\xi = (\xi_1, ..., \xi_n). \tag{1}$$

More precisely, in this article the relative return of asset i in T days is calculated from its daily logarithmic returns $r_{i,t}^{log}$ by

$$\xi_i = \exp\left(\sum_{t=1}^{T} r_{i,t}^{log}\right) - \frac{1}{\beta}. \tag{2}$$

Hence, X is defined as

$$X = (X_1, ..., X_n) = (\xi_1 \cdot Inv, ..., \xi_n \cdot Inv), \tag{3}$$

with Inv being the total amount invested in the n assets.

In the optimization problem the weights of each asset which are defined as fractions of the planned investment (Gaivoronski and Pflug; 1999, p. 4) need to be determined and have to satisfy the bound constraint $w_i \geq 0$ for all $i = 1, ..., n$ and the budget constraint $\sum_{i=1}^{n} w_i = 1$. Thus, the expected return of the portfolio is given by

$$\mathbb{E}(w^\top X) = w^\top \mathbb{E}(X). \tag{4}$$

Let's denote a risk measure by ρ and the maximum risk level an investor is willing to take by C. Then, the general risk-constrained portfolio optimization problem can be written as

$$\begin{aligned}
\max_w &\{w^\top \mathbb{E}(X)\beta\} \\
&\rho(w^\top X) \leq C \\
&w^\top \mathbb{1} = 1 \\
&w \geq 0.
\end{aligned} \tag{5}$$

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2.4 Portfolio optimization with V@R constraint

Portfolio optimization with similar downside risk measures to V@R have already been established in (Roy; 1952) and (Arzac and Bawa; 1977) where portfolio selection problems of a safety-first investor are investigated. Building on that a lot of research has been done and the introduction and following popularity of V@R as an instrument in banking regulation led to research on the suitability of V@R in portfolio optimization and was among others used by (Gaivoronski and Pflug; 1999) as a constraint in portfolio optimization. The portfolio optimization problem is given by (5) where ρ is replaced by V@R and C by a constraint value V.

2.5 Portfolio optimization with OEU constraint

While a lot of research has been done on portfolio optimization with downside risk measures only few articles, such as (Natarajan et al.; 2010), analyzed utility-based risk measures in this regard. However, to the best of our knowledge, no application on portfolio optimization with OEU constraint has been employed yet. The portfolio optimization problem with OEU constraint for CRRA utility functions is given by (5) where the constraint value C is replaced by P and ρ is defined by

$$\rho^u(w^\top X, \gamma) = -\max_{\eta > -X_{min}} \left\{ -\beta \eta + \alpha \mathbb{E}[(w^\top X + \eta)^{1-\gamma}]^{\frac{1}{1-\gamma}} \right\} \quad \text{if } \gamma \neq 1$$

and

$$\rho^u(w^\top X, \gamma) = -\max_{\eta > -X_{min}} \left\{ -\beta \eta + \alpha \exp(\mathbb{E}[\ln(w^\top X + \eta)]) \right\} \quad \text{if } \gamma = 1$$

with γ reflecting the investor's relative risk aversion.

To solve the portfolio optimization problems, meaningful constraint values for V and P have to be specified. Both, V and P, depend on the determined parameters, the regarded asset classes and the investor's risk attitude. For example (Alexander et al.; 2007) calculated the weekly V@R with $\lambda = 0.01$ for each asset by using historical data. Derived from that they consider three values for V as possible constraints. Also (Chabaane et al.; 2006) and (Gambrah and Pirvu; 2014) use different constraint levels for V. Since the OEU risk measure has not been applied in portfolio optimization before, there exists no referential work. As the two portfolio optimization strategies should be comparable, we derive the OEU constraint from the V@R constraint value in a similar way as in (Fink et al.; 2016).

3 Case study

In this case study we consider three U.S. market indices to depict different asset classes while neglecting exchange rate issues. We choose the SPX consisting of 500 stocks that represent the broad U.S. economy (Bloomberg L.P. [US]; 2017c). We further consider the BBC representing one of the most popular commodity indices. It consists of futures contracts on about 20 different raw materials (Bloomberg L.P. [US]; 2017b,d). Furthermore, we consider the AGG that particularly contains treasuries and corporate as well as government-related securities (Bloomberg L.P. [US]; 2017a). The data is taken from Reuters via Datastream. We consider the daily closing prices from 1996 to 2016 of the price index of the S&P 500 and the total return index of BBC and AGG. These price developments are illustrated in Figure 1. While the SPX and the BBC are quite volatile, the

bond index is does not exhibit large fluctuations.

Stock and commodity markets react very sensitive on incisive economic changes. For example the burst of the Dotcom-Bubble in 2000 led to a decrease in stock markets. After the markets recovered, the financial crisis of 2008/2009 caused a notable drop on the rates of the SPX. The prices on commodity markets on the other hand started to increase again after the decline caused by the Asian crisis of 1997. Since the recent financial crisis led to a worldwide economic crisis, it also had an impact on the real economy and on the commodities sector as the BBC dropped dramatically at the beginning of 2009. Soon afterwards both, the SPX as well as the BBC, started to rise again. But while the SPX has been continuously increasing due to low interest rates, bond purchases of the European Central Bank and the Federal Reserve Bank and other factors, the BBC turned around and has been falling between 2014 and 2016. Possible reasons for that are a strong USD rate, a slowdown in the Chinese economy and a higher supply than demand for almost the whole commodity sector (Bloomberg L.P. [US]; 2017d). The differences in price developments of the three indices may lead to diversification effects in portfolio optimization and thus might induce a better performance and less risk than investing in only one index.

Figure 1: Price development of SPX, BBC and AGG from 05/31/1996 to 09/23/2016.

3.1 Model

We use a copula model to describe the multivariate dependence structure between the given indices in the following simulations. For other applications of copula models in portfolio optimization see e.g. (Boubaker and Sghaier; 2013), (Ortobelli et al.; 2010) or (Autchariyapanitkul et al.; 2014). This approach allows to separate the dependence structure and the marginals of a joint distribution of multiple assets. By a famous result of Sklar, this decoupling is without loss of generality. The benefit that comes along with it is the possibility of combining any arbitrary marginal distributions with any copula and thus being able to account for individual characteristics of an asset and dependence structures between assets. This leads to more flexibility in modeling than existing multivariate distributions are capable. Thus, we use copulas combined with univariate ARMA-GARCH

models. The article follows the approach of (Fink et al.; 2017). The estimation horizon is set to 2501 price observations which is equivalent to a ten-year period. We illustrate our approach with the following example of the last estimation window that is between 01/26/2007 and 08/25/2016.

Step 1: Select and estimate suitable univariate ARMA-GARCH models

ARMA and GARCH models estimate the conditional mean and conditional variance of logarithmic returns. The GARCH part is important for financial data due to volatility clustering since the conditional variance is not assumed to be constant but varies depending on its own past and past errors over time. Thus, the logarithmic returns can be defined by the ARMA(p,q)-GARCH(r,s) model as defined in (Bollerslev; 1986) and (Würtz et al.; 2006). To decide on the distributions being considered for the model selection process, we take a look on descriptive graphs and statistics of the above defined logarithmic return series of the three indices. First, the daily logarithmic returns of the indices are illustrated over time in Figure 2. It can clearly be seen that volatility clustering is present particularly for SPX and BBC. This confirms our choice of using a GARCH model for the data on hand.

Figure 2: Daily logarithmic returns of SPX, BBC and AGG between 01/29/2007 and 08/25/2016.

More information about the distributional characteristics of the given logarithmic returns are shown in Table 1. While the annualized mean logarithmic return is positive for SPX and AGG, it exhibits a negative trend for BBC that can be explained by the declining development of the index in recent years. The volatilities of SPX and BBC are distinctly higher than the volatility of the bond index. Furthermore, the distributions of the former two show more asymmetry than AGG. All indices have a positive excess kurtosis and thus are leptokurtic. The SPX shows by far the highest value which indicates a distribution with fat tails.

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Statistics	SPX	BBC	AGG
Mean	0.00017	-0.00023	0.00020
Annualized mean	4.369%	-5.669%	5.015%
Standard deviation	0.01321	0.01112	0.00253
Minimum	-0.09470	-0.06401	-0.01359
Maximum	0.10957	0.05650	0.01425
Skewness	-0.32515	-0.31428	-0.10976
Excess kurtosis	9.99303	3.01118	2.01214

Table 1: Descriptive statistics of daily logarithmic returns of SPX, BBC and AGG between 01/29/2007 and 08/25/2016.

These findings point out that the ND is not the best fit for z_t. Hence, the SSTD that is able to approximate the STD, the SND and the ND is taken into account for the model selection process.[1] For each index the model with the smallest BIC from those that do not reject the Ljung-Box hypothesis of independence in the present time series is chosen. Therefore, we receive the model specifications listed in Table 2.

Index	Mean model	Variance model	Distribution
SPX	ARMA(1,1)	GARCH(2,1)	SSTD
BBC	ARMA(0,0)	GARCH(1,1)	SSTD
AGG	ARMA(1,1)	GARCH(1,1)	SSTD

Table 2: Specification of selected models for the estimation period 01/29/2007 to 08/25/2016.

Diagnostic analyses of the resulted standardized residuals have shown that the specified SSTD is a considerably better fit to the empirical distribution than the ND.

Step 2: Select and estimate suitable bivariate R-vine copulas
To receive the uniform distributions, the probability integral transform is applied on the standardized residuals of the three univariate ARMA-GARCH models defined in step 1. For the selection and estimation process, the same copula families are considered as in (Fink et al.; 2017) (namely the gaussian, the student's t and the gumbel copula including its rotations). The sequential estimation and selection process is done via maximum likelihood and AIC. For the given data a D-vine with two trees is estimated. From the first tree we obtain two bivariate t-copulas. The former one measures the dependency between SPX and BBC and the latter one between AGG and SPX. The second tree reflects the dependency between AGG and BBC conditioned on SPX. Figure 3 and 4 illustrate the relationship between the standardized residuals of the fitted marginals on the left-hand side and the associated contour plots of the estimated bivariate copulas on the right-hand side.

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[1]ND: normal distribution, SSTD: skew-student's t distribution, STD: student's t distribution, SND: skew-normal distribution

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Figure 3: Dependence structures of standardized residuals from the univariate models versus associated bivariate t-copulas - top: SPX vs. BBC; bottom: AGG vs. SPX.

Figure 4: Conditioned dependence structure of standardized residuals from the univariate models versus associated conditioned bivariate copula: AGG vs. BBC conditioned on SPX.

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3.2 Simulation and forecasting approach

The information of the estimated univariate models and copulas is used for a simulation and forecasting approach. We employ a fixed rolling window size of $n = 2500$ daily logarithmic return data and set the number of simulations to $n_{sim} = 100,000$ as done in (Giot and Laurent; 2003) or (Fantazzini; 2008). Typical forecasting horizons for V@R are one day or one month. Since longer term forecasts are usually more interesting in portfolio optimization, the regarded forecasting window is fixed at $T = 20$ trading days. We set the V@R confidence level to $\lambda = 0.05$. The constraint value V is determined by the fraction v of the total amount invested in the three indices. We follow the approach of (Alexander et al.; 2007) and calculate the 20-day V@R$^{0.05}$ of the historical data of the first estimation period for all three indices. These and the V@R$^{0.05}$ of the associated continuously adjusted equally weighted portfolio (EWP) and the buy and hold portfolio (B&H[2]) are presented in Table 3. Since the individual V@R values range from 1.64% to 6.31% and the one of the EWP and B&H are given by 3.19% and 3.09%, a fraction of $v = 2.5\%$ seems to be an acceptable assumption for our case study. This corresponds to a loss of approximately 30% per year.

Index	SPX	BBC	AGG	EWP	B&H
V@R$^{0.05}$	6.31%	6.07%	1.64%	3.19%	3.09%

Table 3: Historical monthly V@R$^{0.05}$ of SPX, BBC, AGG, EWP and B&H for the period between 06/03/1996 and 12/30/2005.

To calculate the discounting factor β_i for every forecasting period i, we use the annualized daily effective rate of the Federal Reserve (FFR) to compute an average annualized rate ffr_i for the previous forecasting period. The subjective rate of time preference of the investor is naturally higher than the risk-free rate. In the context of portfolio optimization we assume the investor to place emphasis on her well-being in the further future and thus fix the parameters

$$\alpha_i = \frac{1}{1 + (3\% + ffr_i) \cdot T/365} \quad \text{and} \quad \beta_i = \frac{1}{1 + ffr_i \cdot T/365},$$

where T denotes the above defined estimation period in days. The corresponding subjective rate of time preference of $3\% + ffr_i$ is reasonably bigger than ffr_i. As a sanity check, in section 3.4 we present results for a subjective rate of time preference of $6\% + ffr_i$ as well.

The relative risk aversion parameter γ refelects the investor's personal attitude towards risk. In the present case study we choose $\gamma = 35$ as a relatively risk-averse specification; again, in section 3.4, carry out a sensitivity analysis with regard to γ. We refer to (Meyer and Meyer; 2005) for an overview on well-known articles on possible specifications of γ. The rolling forecast starts with a budget of 100,000 USD that is invested in the first period according to the weights of the three indices which are calculated in the portfolio optimizations with V@R and OEU constraint, respectively. The basis for the optimization is simulated expected returns for the given forecasting horizon. At the end of each forecasting horizon, the new portfolio value for each approach is calculated and taken as investment for a re-allocation in the next period.

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[2]B&H means that the budget is divided equally between the three assets in the beginning of the investment period. These specified number of shares are hold for the entire investment period without any adjustments.

3.3 Results

In order to compare the performance of portfolio optimization with OEU constraint to portfolio optimization with V@R constraint, we evaluate the data from the before described forecasting approach for the following strategies:

Strategy 1: Investment is allocated to the three indices through portfolio optimization with V@R constraint in every forecasting period.

Strategy 2: Investment is allocated to the three indices through portfolio optimization with OEU constraint in every forecasting period.

EWP: Investment is allocated to the three indices with equal weights in every forecasting period.

B&H: The initial investment is allocated to the three indices with equal weights. These shares are hold for the entire forecasting horizon without any adjustments.

The performances of the four strategies are illustrated in Figure 5. From 2006 to 2008 all strategies show a similar trend and increase quite steadily. During the financial crisis of 2008/2009 the EWP and B&H values are dropping dramatically while the other two strategies (especially strategy 1) lose much less in portfolio value. The reason for that is the different allocation of the investment to the three indices as shown in Figure 6. Due to the fact that the SPX rate dropped drastically while the price of the AGG stayed comparatively stable, strategy 1 and 2 reduced their investment in SPX to very small fractions during this period. On the other hand, EWP and B&H remain the same weights of each index and thus lose in value. Since strategy 1 invests a higher part in the bond index than strategy 2 does, the drop of the former is less intense. In the following years from 2009 to 2016 strategy 2 outperforms every other strategy and ends up with the largest portfolio value at the end of the considered investment period. This again is due to the different asset allocations. Strategy 2 is more volatile compared to strategy 1 since smaller fractions are invested in AGG on average. Also, since the OEU-based strategy 2 also considers the upside potential of the available assets, the portfolio weight of the (best performing) SPX clearly dominates every other asset in the last years of the investment period. This explains why strategy 2 experiences stronger drops in bearish markets and outperforms in bullish markets.

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Figure 5: Performance development of strategy 1, strategy 2, EWP and B&H over the forecasting horizon.

Figure 6: Development of portfolio weights of SPX, BBC and AGG for strategy 1, strategy 2, the EWP and the B&H over the forecasting horizon.

Regarding the whole forecasting horizon, EWP and B&H proceed almost equivalently. Their relatively weak performance from 2010 to 2016 is due to the negative performance of BBC (strategy 1 and 2 only invest in SPX and AGG in these years). Just during the last two years the B&H performs slightly better due to the smaller fraction in BBC. A comparative statistics of average asset weights as well as mean and standard deviation of portfolio returns for all four strategies is illustrated in Table 4. We note that strategy 2 performs best for the given forecasting horizon but is also the most volatile option. Strategy 1 is the least volatile with a clearly better average portfolio return than EWP and B&H. In a next step, we take a closer look on the relative performance of the two main strategies. Therefore the first graph of Figure 7 illustrates the relative performance of strategy 1 and 2 compared to the EWP over the entire forecasting horizon. In times where the line is above 1, the respective strategy performs better than the EWP. The second graph compares the relative performance of strategy 2 versus strategy 1. If the blue line is above the grey horizontal this indicates that the portfolio optimization with an OEU constraint performs better than the one with a V@R constraint.

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Figure 7: Development of the relative performance over the forecasting horizon: Performance of strategy 1 and 2 relatively to the EWP (top) and of strategy 2 relatively to strategy 1 (bottom).

We note that both, strategy 1 and 2, perform considerably better than the EWP during the financial crisis of 2008/2009. Furthermore, both strategies show a superior relative performance in recent years which has already been noted in Figure 5. The better performance of strategy 1 compared to strategy 2 in 2008 is highlighted in the second graph of Figure 7. On the contrary, strategy 2 outperforms strategy 1 on average between 2009 and 2011 as well as during the last 3 to 4 years of the regarded forecasting horizon, while it has been alternating in times of the European crisis in 2011 and 2012.

To get a better understanding of the strategies' investment allocations, we take a closer look at the risk measures of the portfolio optimizations. Figure 8 illustrates the V@R$^{0.05}$ and ρ^u values that are based on the simulated data if the available budget is fully invested in the regarded index. The constraints V_i and P_i are sketched in a grey dashed line.

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Figure 8: V@R$^{0.05}$ (top) and ρ^u (bottom) of simulated values in case of a full investment in each index compared to the given constraints of strategy 1 and 2, respectively.

The developments of the V@R$^{0.05}$ and ρ^u values are quite similar for the first half of the forecasting horizon. Afterwards, the ρ^u of SPX is decreasing and even takes on negative values below P_i and the corresponding ρ^u of AGG. Negative risks mean that the investor expects a gain from her investment in SPX. According to ρ^u, this makes SPX (in these years) a more attractive investment opportunity to an investor with α and γ as specified in section 3.2 compared to the other two risky assets. The values of BBC on the other hand are rising rapidly in the beginning of 2015 and hence the investor has to carry high costs to hedge this position in OEU. This may explain why in recent years often the whole budget is invested in the SPX when using strategy 2. Also the V@R$^{0.05}$ of the SPX is getting closer to V_i while the one of the BBC increases during the last few years. However, the effect is by far not the same as for ρ^u. This may cause the on average higher investment in AGG when using strategy 1 since it constantly has a lower V@R$^{0.05}$ value than the other two indices. The middle part of the forecasting horizon shows that the ρ^u of BBC is - if regarded relatively - much closer to P_i than the according V@R$^{0.05}$ is to V_i. This might explain

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the result of the on average higher investment in this index by strategy 2 compared to strategy 1. To sum up, the portfolio optimization with OEU finds BBC and SPX in bullish markets a more reasonable investment than the portfolio optimization with V@R does by only considering downside risk through the 95% loss quantile.

Furthermore, we want to investigate if the portfolio risk measured by the two risk measures always satisfies the defined constraints of the portfolio optimization problems. For both, strategy 1 and strategy 2, deviations from the constraints P_i and V_i are depicted in Figure 9. While P_i is satisfied for most portfolios found in the first half of the forecasting horizon, it is not met in the second half due to the attractiveness of SPX and the corresponding negative values of OEU for SPX. Strategy 1 shows only few portfolios not satisfying the constraint V_i but in contrast to strategy 2, it once exceeds the risk constraint. A reason for this might be that no portfolio exists for the simulated returns of this period that satisfies the given constraint.

Figure 9: Deviations from constraints V_i in strategy 1 (left) and from constraints P_i in strategy 2 (right) over the forecasting horizon.

To further explore the outlier in strategy 1, the efficient frontier of the portfolio with the highest positive deviation from V_i (in November 2008) is drawn in Figure 10 on the top left. The return-risk combinations of the simulated values in 5% weight steps are calculated and delineated by '+' signs in the graph where the ones having the highest expected return for a given $V@R^{0.05}$ build the efficient frontier. The constraint is sketched by the red vertical and the determined optimal portfolio by the green rhombus. The zoomed version below clearly shows that there is no portfolio meeting the constraint and thus we get the minimum-$V@R^{0.05}$ portfolio as the optimal solution in this case. On the right-hand side of Figure 10 the associated portfolio of strategy 2 is depicted. It shows the usual approach where the expected portfolio return is maximized while the portfolio risk is constrained on a prespecified level.

3.4 Sensitivity analysis

This section provides a sensitivity analysis concerning the investor's relative risk aversion γ, the investor's subjective time preference α and the percentage $V@R^{0.05}$ constraint value v. While the

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Figure 10: Return-risk graphs of various weight combinations with risk constraints in red and the optimal portfolios in green for strategy 1 and 2 (top) and their zoomed versions (bottom).

former two only influence the results of the portfolio optimization with OEU constraint, the latter implies a more conservative boundary in strategy 1 and also influences the constraint value of strategy 2.

For the relative risk aversion parameter γ, besides the earlier chosen value of $\gamma = 35$, we also consider investors who are less risk averse ($\gamma = 15$ and $\gamma = 25$). With respect to α, a subjective rate of time preference of $6\% + rff_i$ is considered besides the previously assumed rate of $3\% + rff_i$. The change from 3% to 6% implies that the investor focuses more on her well-being in the present and is more averse to invest her money in the portfolio with future payoff. Hence, she has to be more convinced of the given investment opportunity for being willing to spend her money on it. The third parameter v that was previously set to 2.5% implies a yearly potential loss of about 30% of the respective investment in 5% of cases. We choose a second specification for v of 1.25% to take a more conservative portfolio strategy into account. In total, we have 3 specifications of γ and 2 assumptions for α as well as for v and thus we receive two different portfolio selection approaches for strategy 1 and 12 distinctive ones for strategy 2. These 14 approaches are illustrated in Figure 11. We notice that the performance of strategy 2 with $\gamma = 15$ is affected to a much greater extent by the financial crisis of 2008/2009 than any other strategy; most notably for the subjective rate of time preference of $3\% + rff_i$. This is explained by the corresponding asset allocations: Strategy 2 with

γ = 15 and $\alpha = \frac{1}{1+(3\%+rff_i)\cdot T/365}$ is most invested in BBC during the financial crisis, whereas strategy 2 with $\alpha = \frac{1}{1+(6\%+rff_i)\cdot T/365}$ invests more in AGG. Then again, strategy 2 with γ = 15 outperforms every other strategy in 2013 and 2014 due to the fact that it invests most in SPX in these times. This is a perfect exemplary demonstration of OEU's ability to include the investor's individual risk attitude into the risk assessment. The risk taking investor (γ = 15) experiences more drastic losses during the financial crisis than her more risk averse counterparts (γ = 25 and γ = 35). In the following years the risky investments in SPX pay back and the overall portfolio value of strategy 2 with γ = 15 finally exceeds the values of most other strategies.

The performance of strategy 1 for a more conservative V@R constraint v = 1.25% is smoother than for v = 2.5%. While the drop of portfolio value during the financial crisis is barely recognizable, the following value growth is less than before. This is explainable by the distinctly higher investment in AGG over the entire forecasting horizon caused by the lower constraint value V_i.

To get a better overview on the key characteristics of the different allocation strategies, Table 4 presents the related average portfolio weights and performances for the entire forecasting horizon. Furthermore, the average Herfindahl Index (HI) is calculated to evaluate the diversification effect. The HI values range from 1 divided by the number of assets (meaning perfectly diversified) to 1. The HI is also used in other studies with a comparable context, see e.g. (Chen and Wang; 2008).

The highest mean portfolio return is reached for the first specification of strategy 2. The conservative V@R-constrained portfolio has the smallest variation. Generally, it can be said that portfolio strategies with v = 2.5% have a higher mean return and variation than the associated portfolios with v = 1.25%. Portfolios of strategy 2 with a subjective rate of time preference of 6% + rff_i have smaller variations than those with 3% + rff_i, ceteris paribus. Moreover, the performance gets more volatile with decreasing risk aversion as less shares of the AGG are bought. Furthermore, portfolios of strategy 2 show on average higher investments in the SPX and the BBC than the associated allocation of strategy 1. The EWP has by far the worst mean portfolio return with a comparatively high volatility. On the contrary, it is obviously the most diversified portfolio and thus the benchmark for the others. Besides EWP, the two specifications of strategy 1 are the ones with smallest HI followed by strategy 2 with γ = 35, $\alpha = \frac{1}{1+(6\%+ffr_i)\cdot T/365}$ and v = 1.25%. All of these index values are considerably larger than the HI of the EWP which is among others explainable by the constantly negative performance of the BBC during the last few years.

	Parameters			Portfolio weights			Portfolio returns		
Strategy	γ	r_{add}	v	SPX	BBC	AGG	Mean	SD	HI
1			2.5%	0.49	0.12	0.39	0.00588	0.02160	0.57
2	35	3%	2.5%	0.59	0.21	0.20	0.00645	0.03219	0.72
2	25	3%	2.5%	0.60	0.22	0.18	0.00627	0.03462	0.76
2	15	3%	2.5%	0.60	0.27	0.13	0.00580	0.03912	0.82
2	35	6%	2.5%	0.52	0.14	0.34	0.00580	0.02496	0.64
2	25	6%	2.5%	0.52	0.15	0.33	0.00593	0.02620	0.66
2	15	6%	2.5%	0.54	0.17	0.29	0.00624	0.02949	0.70
1			1.25%	0.26	0.06	0.68	0.00468	0.01252	0.59
2	35	3%	1.25%	0.49	0.13	0.38	0.00595	0.02486	0.67
2	25	3%	1.25%	0.53	0.15	0.32	0.00616	0.02873	0.70
2	15	3%	1.25%	0.56	0.22	0.22	0.00577	0.03487	0.75
2	35	6%	1.25%	0.36	0.08	0.56	0.00492	0.01695	0.61
2	25	6%	1.25%	0.43	0.09	0.48	0.00512	0.02041	0.65
2	15	6%	1.25%	0.49	0.13	0.38	0.00595	0.02503	0.67
EWP				0.33	0.33	0.33	0.00200	0.02698	0.33
B&H				0.34	0.27	0.39	0.00236	0.02434	0.35

Table 4: Summary statistics of portfolio weights, portfolio returns and HI of all strategy 1 and strategy 2 specifications, the B&H and the EWP for the forecasting horizon.

Strategy 1 and 2 lead, for most of the regarded parameter specifications, to much better performances than the EWP. While the more conservative constraint value v has a large effect on strategy 1, it only leads to recognizable effects of strategy 2 for more risk averse investors during the financial crisis. We observe that the less risk averse the investor, the less she invests in AGG and the more volatile is her portfolio performance. The increase of the subjective rate of time preference causes that the investor is more averse to invest in a risky portfolio with future payoff.

4   Conclusion

The results for the different parameter specifications show that by including information of the entire return distribution, the portfolio optimization with OEU constraint finds the BBC and SPX in bullish markets a more reasonable investment than the portfolio optimization with V@R does. Moreover, both strategies outperform the B&H and EWP strategies. Our sensitivity analysis of the investor's relative risk aversion, her time preference and the effect of a more conservative V@R constraint value show that almost all specifications of V@R- and OEU-based strategies lead to distinctly better performances than the EWP strategy. In addition to that, the findings of the different specifications highlight the main advantage of the OEU risk measure over V@R in our case study. Due to the possibility of diverse specifications of the investor's risk attitude and time preference it allows for an individually tailored optimal portfolio strategy.

Figure 11: Performance development of all specifications of strategy 1 and strategy 2 and of the EWP over the forecasting horizon.

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