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Christoph Belak

Daniel Hoffmann

Frank T. Seifried

Continuous-Time Mean Field Games with Finite State Space and Common Noise*

Christoph Belak† Daniel Hoffmann‡ Frank T. Seifried‡

We formulate and analyze a mathematical framework for continuous-time mean field games with finitely many states and common noise, including a rigorous probabilistic construction of the state process. The key insight is that we can circumvent the master equation and reduce the mean field equilibrium to a system of forward-backward systems of (random) ordinary differential equations by conditioning on common noise events. In the absence of common noise, our setup reduces to that of Gomes, Mohr and Souza [GMS13] and Cecchin and Fischer [CF20].

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1 Introduction

Since the seminal contributions of Lasry and Lions [LL07] and Huang, Malhamé and Caines [HMC06], mean field games have become an active field of mathematical research with a wide range of applications, including economics ([CFS15], [CDL17], [KM17], [Nut18], [EIL20], [GS20]), sociology ([GVW14]), finance ([LLLL16], [CJ19]), epidemiology ([LT15], [DGG17], [EHT20]) and computer science ([KB16]); see also the overview article [GLL11] and the monograph [CD18a].

Mean field games constitute a class of dynamic, multi-player stochastic differential games with identical agents. The key characteristic of the mean field approach is that (i) the payoff and state dynamics of each agent depend on other agents' decisions only through an aggregate statistic (typically, the aggregate distribution of states); and (ii) no individual agent's actions can change the aggregate outcome. Thus, in solving an individual agent's optimization problem, the feedback effect of his own actions on the aggregate outcome can be discarded, breaking the notorious vicious circle ("the optimal strategy depends on the aggregate outcome, which depends on the strategy, which depends ..."). This significantly facilitates the identification of rational expectations equlibria. A standard assumption that further simplifies the analysis is that randomness is idiosyncratic (equivalently, there is no common noise), i.e. that the random variables appearing in one agent's optimization are independent of those in any other's. As a result, all randomness is "averaged out" in the aggregation of individual decisions, and the equilibrium dynamics of the aggregate distribution are deterministic.

In the literature, mean field games are most often studied in settings with a continuous state space and deterministic or diffusive dynamics, i.e. stochastic differential equations (SDEs) driven by Brownian motion. The corresponding dynamic programming equations thus become parabolic partial differential equations, and the aggregate dynamics are represented by a flow of Borel probability measures; see,

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†Technische Universität Berlin, Faculty II – Mathematics and Natural Sciences, Institute of Mathematics, Straße des 17. Juni 136, 10623 Berlin, e-mail: belak@math.tu-berlin.de

‡University of Trier, Department IV – Mathematics, Universitätsring 19, 54296 Trier, e-mails: d.hoffmann@uni-trier.de, seifried@uni-trier.de

e.g., the monographs [BFY13] and [CD18a] and the references therein. Formally, the mean field game is typically formulated in terms of a controlled McKean-Vlasov SDE, where the coefficients depend on the current state and control and the distribution of the solution; intuitively, these McKean-Vlasov dynamics codify the dynamics that pertain to a representative agent. The mathematical link to N-player games is subsequently made through suitable propagation of chaos results in the mean field limit $N \to \infty$; see, e.g., [Lac15], [Fis17], [CF18], [Lac18], [DLR19]. In this context, the analysis of McKean-Vlasov SDEs has also seen significant progress recently; see, e.g., [CD13], [BP19], [CP19b], [MP19]. In the presence of common noise, i.e. sources of risk that affect all agents and do not average out in the mean field limit, the mathematical analysis becomes even more involved as the dynamics of the aggregate distribution become stochastic, leading to conditional McKean-Vlasov dynamics; see, e.g., [Ahu16], [CDL16], [CW17] and [PW17]. We refer to [CD18b] for background and further references on continuous-state mean field games with common noise.

There is also a strand of literature on mean field games with finite state spaces, including [GMS10], [GMS13], [Gué15], [BC18], [CP19a], [DGG19], [CF20], [Neu20] as well as [CD18a, §7.2]. In a recent article, [CW18] provide an extension of [GMS13] to mean field interactions that occur not only through the agents' states, but also through their controls. To the best of our knowledge, however, to date there has been no extension of these results to settings that include common noise.[1]

In this article, we set up a mathematical framework for finite-state mean field games with common noise.[2] Our setup extends that of [GMS13] and [CF20] by common noise events at fixed points in time. We provide a rigorous formulation of the underlying stochastic dynamics, and we establish a verification theorem for the optimal strategy and an aggregation theorem to determine the resulting aggregate distribution. This leads to a characterization of the mean field equilibrium in terms of a system of (random) forward-backward differential equations. The key insight is that, after conditioning on common noise configurations, we obtain classical piecewise dynamics subject to jump conditions at common noise times.

The remainder of this article is organized as follows: In Section 2 we set up the mathematical model, provide a probabilistic construction of the state dynamics, and formulate the agent's optimization problem. In Section 3 we state the dynamic programming equation and establish a verification theorem for the agent's optimization, given an *ex ante* aggregate distribution (Theorem 7). Section 4 provides the dynamics of the *ex post* distribution (Theorem 10) and, on that basis, a system of random forward-backward ODEs for the mean field equilibrium (Definition 11) as well as a corresponding existence result (Theorem 12). In Section 5 we showcase our results in two benchmark applications: agricultural production and infection control. Appendix A contains auxiliary results and the proof of Theorem 12.

2 Mean Field Model

We first provide an *informal* description of the individual agents' state dynamics, optimization problem, and the resulting mean field equilibrium. The agent's state process $X = \{X_t\}$ takes values in the finite set $\mathbb{S}$. Between common noise events, transitions from state i to state j occur with intensity $Q^{ij}(t, W_t, M_t, \nu_t)$, where W_t represents the common noise events that have occurred up to time t; M_t the time-t aggregate distribution of agents; and ν_t the agent's control. In addition, upon the realization of a common noise event W_k at time T_k, the state jumps from X_{T_k-} to $X_{T_k} = J^{X_{T_k-}}(T_k, W_{T_k}, M_{T_k-})$. With this, the agent aims to maximize

$$\mathbb{E}^{\nu}\Big[\int_0^T \psi^{X_t}(t, W_t, M_t, \nu_t)\mathrm{d}t + \Psi^{X_T}(W_T, M_T)\Big]$$

[1]In the context of finite-state mean field games, we are only aware of two contributions that include common noise (both via the master equation and with a different focus/setting): [BLL19] formulate the master equation for finite-state mean field games with common noise, and [BCCD19] add a common Gaussian noise to the aggregate distribution dynamics.

[2]We wish to point out that our focus is *not* on the mean field limit of multi-player games; rather, we directly investigate the mean field equilibrium via the corresponding McKean-Vlasov dynamics (see also Remark 8 and [CDL13] in that context).

where ψ and Ψ are suitable reward functions and the aggregate distribution process $M = \{M_t\}$ is given by

$$M_t \triangleq \mu(t, W_t) \quad \text{for } t \in [0, T].$$

Here μ represents the aggregate distribution of states as a function of the common noise factors. We obtain a rational expectations equilibrium by determining μ such that the representative agent's *ex ante* expectations equal the *ex post* aggregate distribution resulting from all agents' optimal decisions, i.e.

$$\mathbb{P}^{\widehat{\nu}}(X_t \in \,\cdot\, | W_t) = \widehat{\mu}(t, W_t) \quad \text{for all } t \in [0, T],$$

where $\widehat{\nu}$ and $\widehat{\mu}$ denote the equilibrium strategy and the equilibrium aggregate distribution. In the remainder of this section, we provide a rigorous mathematical formulation of this model.

2.1 Probabilistic Setting and Common Noise

Throughout, we fix a time horizon $T > 0$ and a finite set $\mathbb{W}$ and work on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ that carries a finite sequence $W_1, \dots, W_n$ of i.i.d. random variables that are uniformly distributed[3] on $\mathbb{W}$. We refer to $W_1, \dots, W_n$ as *common noise factors* and to $\mathbb{P}$ as the *reference probability*. The common noise factor W_k is revealed at time T_k, where

$$0 \triangleq T_0 < T_1 < T_2 < \cdots < T_n < T_{n+1} \triangleq T.$$

The piecewise constant filtration $\mathfrak{G} = \{\mathfrak{G}_t\}$ generated by common noise events is given by

$$\mathfrak{G}_t \triangleq \sigma\big(W_k : k \in [1:n],\ T_k \le t\big) \vee \mathfrak{N} \quad \text{for } t \in [0, T]$$

where $\mathfrak{N}$ denotes the set of $\mathbb{P}$-null sets. For each configuration of common noise factors $w \in \mathbb{W}^n$ we write

$$w_t \triangleq (w_1, \dots, w_k) \text{ for } t \in [T_k, T_{k+1}\rangle,\ k \in [0:n],$$

where for $0 \le s \le t \le T$ we set $[s, t\rangle \triangleq [s, t)$ if $t < T$ and $[s, T\rangle \triangleq [s, T]$. With this convention, $W = \{W_t\}$ represents a piecewise constant, $\mathfrak{G}$-adapted process.

Definition 1. A function $f : [0, T] \times \mathbb{W}^n \to \mathbb{R}^m$ is *non-anticipative* if for all $t \in [0, T]$

$$f(t, w) = f(t, \bar{w}) \quad \text{whenever } w, \bar{w} \in \mathbb{W}^n \text{ are such that } w_t = \bar{w}_t. \qquad \diamond$$

With a slight abuse of notation, if $f : [0, T] \times \mathbb{W}^n \to \mathbb{R}^m$ is non-anticipative, we write

$$f(t, w_t) \triangleq f(t, w) \quad \text{for } w \in \mathbb{W}^n,\ t \in [0, T].$$

2.2 Optimization Problem

The agent's state and action spaces are given by

$$\mathbb{S} \triangleq [1:d] \qquad \text{and} \qquad \mathbb{U} \subseteq \mathbb{R}^k, \qquad \text{where } d, k \in \mathbb{N} \text{ and } \mathbb{U} \neq \varnothing,$$

[3]While the common noise factors are i.i.d. uniformly distributed under $\mathbb{P}$, the distribution of $W_1, \dots, W_n$ in the agent's optimization problem can be modeled arbitrarily via the functions $\kappa_1, \dots, \kappa_n$ introduced below; see also Lemma 3.

and we identify the space of aggregate distributions on $\mathbb{S}$ with the space of probability vectors

$$\mathbb{M} \triangleq \Big\{ m \in [0,\infty)^{1\times d} \,:\, \sum_{i=1}^{d} m^i = 1 \Big\}.$$

We further suppose that $(\Omega, \mathfrak{A}, \mathbb{P})$ supports, for each $i, j \in \mathbb{S}$, $i \neq j$, a standard (i.e., unit intensity) Poisson process $N^{ij} = \{N_t^{ij}\}$ and an $\mathbb{S}$-valued random variable X_0 such that

$$X_0 \qquad \text{and} \qquad N^{ij},\ i,j \in \mathbb{S},\ i \neq j \qquad \text{and} \qquad W_1, \ldots, W_n \qquad \text{are independent.}$$

The corresponding full filtration $\mathfrak{F} = \{\mathfrak{F}_t\}$ is given by

$$\mathfrak{F}_t \triangleq \sigma\big(X_0,\, W_s,\, N_s^{ij} \,:\, s \in [0,t];\ i,j \in \mathbb{S},\ i \neq j\big) \vee \mathfrak{N} \quad \text{for } t \in [0,T].$$

Note that $\mathfrak{G}_t \subseteq \mathfrak{F}_t$ for all $t \in [0,T]$, that both $\mathfrak{G}$ and $\mathfrak{F}$ satisfy the usual conditions, and that N^{ij} is a standard $(\mathfrak{F}, \mathbb{P})$-Poisson process for $i, j \in \mathbb{S}$, $i \neq j$.
The agent's *optimization problem* reads[4]

$$\mathbb{E}^{\nu}\Big[\int_0^T \psi^{X_t}(t, W_t, M_t, \nu_t)\mathrm{d}t + \Psi^{X_T}(W_T, M_T)\Big] \underset{\nu \in \mathcal{A}}{\longrightarrow} \max! \tag{P$_\mu$}$$

where the class of *admissible strategies* for (P_μ) is given by the set of closed-loop controls

$$\begin{aligned}\mathcal{A} \triangleq \big\{\nu :\ [0,T] \times \mathbb{S}^{[0,T]} \times \mathbb{W}^n \to \mathbb{U} \,:\, &\nu \text{ is Borel measurable and} \\ &\nu(\,\cdot\,, x,\,\cdot\,) \text{ is non-anticipative for all } x \in \mathbb{S}^{[0,T]}\big\}.\end{aligned}$$

Note that $\mathcal{A}$ subsumes the class of Markovian feedback controls considered in, e.g., [GMS13] or [Gué15], and that each $\nu \in \mathcal{A}$ canonically induces an $\mathfrak{F}$-adapted $\mathbb{U}$-valued process via

$$\nu_t \triangleq \nu\big(t, X_{(\,\cdot\, \wedge t)-}, W_t\big) \quad \text{for } t \in [0,T].$$

The $\mathfrak{G}$-adapted, $\mathbb{M}$-valued *ex ante* aggregate distribution $M = \{M_t\}$ is given by

$$M_t \triangleq \mu(t, W_t) \quad \text{for } t \in [0,T];$$

$\mathbb{E}^{\nu}[\,\cdot\,]$ denotes the expectation operator with respect to the probability measure $\mathbb{P}^{\nu}$ given by (see Lemma 3)

$$\begin{aligned}\frac{\mathrm{d}\mathbb{P}^{\nu}}{\mathrm{d}\mathbb{P}} = \prod_{\substack{i,j \in \mathbb{S},\\ i \neq j}} \bigg(\exp\Big\{\int_0^T \big(1 - Q^{ij}(t, W_t, M_t, \nu_t)\big)\mathrm{d}t\Big\} \cdot \prod_{\substack{t \in (0,T],\\ \Delta N_t^{ij} \neq 0}} Q^{ij}(t, W_t, M_t, \nu_t)\bigg) \\ \times\ |\mathbb{W}|^n \cdot \prod_{k=1}^{n} \kappa_k\big(W_k | W_1, \ldots, W_{k-1}, M_{T_k-}\big);\end{aligned} \tag{1}$$

and the agent's state process X is given by

$$\mathrm{d}X_t = \sum_{\substack{i,j \in \mathbb{S},\\ i \neq j}} \mathbb{1}_{\{X_{t-} = i\}}(j - i)\mathrm{d}N_t^{ij} \quad \text{for } t \in [T_k, T_{k+1}\rangle,\ k \in [0:n], \tag{2}$$

[4]For notational simplicity, we write X_t instead of X_{t-}, M_t instead of M_{t-}, etc., where it does not make a difference.

subject to the jump conditions

$$X_{T_k} = J^{X_{T_k-}}\big(T_k, W_{T_k}, M_{T_k-}\big) \text{ for } k \in [1:n]. \tag{3}$$

The coefficients in the state dynamics and payoff functional are bounded and Borel measurable functions

$$\begin{aligned} &Q:\ [0,T]\times\mathbb{W}^n\times\mathbb{M}\times\mathbb{U}\to[0,\infty)^{d\times d} && J:\ [0,T]\times\mathbb{W}^n\times\mathbb{M}\to\mathbb{S}^d \\ &\psi:\ [0,T]\times\mathbb{W}^n\times\mathbb{M}\times\mathbb{U}\to\mathbb{R}^d && \Psi:\ \mathbb{W}^n\times\mathbb{M}\to\mathbb{R}^d \end{aligned}$$

such that $Q(\,\cdot\,,\,\cdot\,,m,u)$, $\psi(\,\cdot\,,\,\cdot\,,m,u)$ and $J(\,\cdot\,,\,\cdot\,,m)$ are non-anticipative for all fixed $m\in\mathbb{M}$ and $u\in\mathbb{U}$; Q satisfies the intensity matrix condition $\sum_{j\in\mathbb{S}} Q^{ij}(t,w,m,u)=0$, $(t,w,m,u)\in[0,T]\times\mathbb{W}^n\times\mathbb{M}\times\mathbb{U}$, $i\in\mathbb{S}$; and for each $k\in[1:n]$ the function

$$\kappa_k:\ \mathbb{W}^k\times\mathbb{M}\to[0,1], \qquad (w_k,w_1,\dots,w_{k-1},m)\mapsto\ \kappa_k(w_k|w_1,\dots,w_{k-1},m),$$

is Borel measurable and satisfies $\sum_{\bar{w}_k\in\mathbb{W}}\kappa_k(\bar{w}_k|w_1,\dots,w_{k-1},m)=1$ for all $w_1,\dots,w_{k-1}\in\mathbb{W}$ and $m\in\mathbb{M}$. Finally, the function μ is taken to be non-anticipative and regular in the following sense:

Definition 2. A function $f:\ [0,T]\times\mathbb{W}^n\to\mathbb{R}^m$ is *regular* if $f(\cdot,w)$ is absolutely continuous on $[T_k,T_{k+1}\rangle$ for all $k\in[0:n]$. $\diamond$

Note that for f regular, the one-sided limits $f(T_k-,w)\triangleq\lim_{t\uparrow T_k} f(t,w)$ exist for all $k\in[1:n]$, $w\in\mathbb{W}^n$. In summary, in order to formulate a mean field model within the above setting, it suffices to specify

- ▷ the agent's state space $\mathbb{S}$, action space $\mathbb{U}$ and the common noise space $\mathbb{W}$,
- ▷ the transition intensities $Q(t,w,m,u)$, transition kernels $\kappa_k(w_k|w_1,\dots,w_{k-1},m)$ and common noise jumps $J(t,w,m)$, and finally
- ▷ the reward functions $\psi(t,w,m,u)$ and $\Psi(w,m)$.

2.3 State Dynamics

In what follows, we show that the preceding construction implies the dynamics described informally above.

Lemma 3 ($\mathbb{P}^\nu$-dynamics)**.** *For each admissible strategy $\nu\in\mathcal{A}$, $\mathbb{P}^\nu$ is a well-defined probability measure on $(\Omega,\mathfrak{A})$, absolutely continuous with respect to $\mathbb{P}$, and satisfies*

$$\mathbb{P}^\nu=\mathbb{P} \quad \textit{on } \sigma(X_0).$$

Moreover, N^{ij} is a counting process with $(\mathfrak{F},\mathbb{P}^\nu)$-intensity $\lambda^{ij}=\{\lambda^{ij}_t\}$, where

$$\lambda^{ij}_t \triangleq Q^{ij}\left(t,W_t,M_t,\nu_t\right) \quad \textit{for } t\in[0,T] \textit{ and } i,j\in\mathbb{S},\ i\neq j.$$

Finally, for all $k\in[1:n]$ we have

$$\mathbb{P}^\nu\big(W_k=w_k|\mathfrak{G}_{T_k-}\big)=\kappa_k(w_k|W_1,\dots,W_{k-1},M_{T_k-}) \quad \textit{for all } w_1,\dots,w_k\in\mathbb{W}$$

and, in particular,

$$\mathbb{P}^{\nu_1}=\mathbb{P}^{\nu_2} \quad \textit{on } \mathfrak{G}_T \quad \textit{for all admissible strategies } \nu_1,\nu_2\in\mathcal{A}.$$

Proof. We fix $\nu \in \mathcal{A}$ and split the proof into four steps.

Step 1: $\mathbb{P}^\nu$ is well-defined by (1). Since N^{ij} is a standard Poisson process under $\mathbb{P}$, the compensated process $\bar{N}_t^{ij} \triangleq N_t^{ij} - t$, $t \geq 0$, is an $(\mathfrak{F}, \mathbb{P})$-martingale for all $i, j \in \mathbb{S}$, $i \neq j$. We define $\theta^\nu = \{\theta_t^\nu\}$ via[5]

$$\theta_t^\nu \triangleq \sum_{\substack{i,j \in \mathbb{S}, \\ i \neq j}} \int_0^t \big(Q^{ij}\big(s, W_s, \mu(s, W_s), \nu_s\big) - 1\big) \mathrm{d}\bar{N}_s^{ij}, \quad t \in [0, T],$$

and observe that the Doléans-Dade exponential $\mathcal{E}[\theta^\nu]$ is a local $(\mathfrak{F}, \mathbb{P})$-martingale with

$$\mathcal{E}[\theta^\nu]_t = \prod_{\substack{i,j \in \mathbb{S}, \\ i \neq j}} \left(\exp\Big\{ \int_0^t \big(1 - Q^{ij}\big(s, W_s, \mu(s, W_s), \nu_s\big)\big) \mathrm{d}s \Big\} \cdot \prod_{\substack{s \in (0,t], \\ \Delta N_s^{ij} \neq 0}} Q^{ij}\big(s, W_s, \mu(s, W_s), \nu_s\big) \right) \tag{4}$$

for $t \in [0, T]$. Next, we define $\vartheta = \{\vartheta_t\}$ via

$$\vartheta_t \triangleq \sum_{\substack{k \in [1:n], \\ T_k \leq t}} \Big(|\mathbb{W}| \cdot \kappa_k\big(W_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big) - 1 \Big), \quad t \in [0, T],$$

and note that ϑ is an $(\mathfrak{F}, \mathbb{P})$-martingale. Indeed, for each $k \in [0:n]$ we have $\vartheta_t = \vartheta_{T_k}$ for $t \in [T_k, T_{k+1}\rangle$ and, using that W_k is independent of $\mathfrak{F}_{T_k-}$ and uniformly distributed on $\mathbb{W}$ under $\mathbb{P}$, it follows that

$$\begin{aligned} \mathbb{E}\big[\vartheta_{T_k} | \mathfrak{F}_{T_k-}\big] &= \vartheta_{T_k-} + \mathbb{E}\big[|\mathbb{W}| \cdot \kappa_k\big(W_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big) - 1 \big| \mathfrak{F}_{T_k-}\big] \\ &= \vartheta_{T_k-} - 1 + |\mathbb{W}| \cdot \sum_{w_k \in \mathbb{W}} \mathbb{P}\big(W_k = w_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big) \\ &\qquad\qquad\qquad\qquad \times \kappa_k\big(w_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big) \\ &= \vartheta_{T_k-} - 1 + |\mathbb{W}| \cdot \sum_{w_k \in \mathbb{W}} \frac{1}{|\mathbb{W}|} \cdot \kappa_k\big(w_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big) = \vartheta_{T_k-}. \end{aligned}$$

Hence the Doléans-Dade exponential $\mathcal{E}[\vartheta]$ is a local $(\mathfrak{F}, \mathbb{P})$-martingale, and we have

$$\mathcal{E}[\vartheta]_t = \prod_{s \in (0,t]} (1 + \Delta\vartheta_s) = \prod_{\substack{k \in [1:n], \\ T_k \leq t}} \Big(|\mathbb{W}| \cdot \kappa_k\big(W_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big) \Big) \tag{5}$$

for $t \in [0, T]$. Since $\Delta N_{T_k}^{ij} = 0$ for all $i, j \in \mathbb{S}$, $i \neq j$, and $k \in [1:n]$ a.s., we have $[\theta^\nu, \vartheta] = 0$, and thus the process $Z^\nu \triangleq \mathcal{E}[\theta^\nu + \vartheta] = \mathcal{E}[\theta^\nu] \cdot \mathcal{E}[\vartheta]$, i.e.

$$\begin{aligned} Z_t^\nu = \prod_{\substack{i,j \in \mathbb{S}, \\ i \neq j}} &\left(\exp\Big\{ \int_0^t \big(1 - Q^{ij}\big(s, W_s, \mu(s, W_s), \nu_s\big)\big) \mathrm{d}s \Big\} \cdot \prod_{\substack{s \in (0,t], \\ \Delta N_s^{ij} \neq 0}} Q^{ij}\big(s, W_s, \mu(s, W_s), \nu_s\big) \right) \\ &\times \prod_{\substack{k \in [1:n], \\ T_k \leq t}} \Big(|\mathbb{W}| \cdot \kappa_k\big(W_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big) \Big) \end{aligned} \tag{6}$$

is a local $(\mathfrak{F}, \mathbb{P})$-martingale. Since

$$\sup_{t \in [0,T]} |\mathcal{E}[\theta^\nu]_t| \leq \mathrm{e}^{d^2 T} \cdot \ell^Y \tag{7}$$

[5] Note that $\int_0^t Q^{ij}(s, W_s, \mu(s, W_s), \nu_s) - 1) \mathrm{d}\bar{N}_s^{ij} = \int_0^t Q^{ij}(s, W_{s-}, \mu(s, W_{s-}), \nu_s) - 1) \mathrm{d}\bar{N}_s^{ij}$ $\mathbb{P}$-a.s.

where $\ell \triangleq \max_{i,j\in\mathbb{S},\, i\neq j} \|Q^{ij}\|_\infty$ and $Y \triangleq \sum_{i,j\in\mathbb{S},\, i\neq j} N_T^{ij} \sim_{\mathbb{P}} \mathsf{Poisson}(d(d-1)T)$ and

$$\sup_{t\in[0,T]} |\mathcal{E}[\vartheta]_t| \leq |\mathbb{W}|^n \tag{8}$$

it follows that $\sup_{t\in[0,T]} |Z_t^\nu|$ is $\mathbb{P}$-integrable, so Z^ν is in fact an $(\mathfrak{F},\mathbb{P})$-martingale. Since Z^ν is non-negative with $Z_0^\nu = 1$ by construction, we conclude that $\mathbb{P}^\nu$ is a well-defined probability measure on $\mathfrak{A}$, absolutely continuous with respect to $\mathbb{P}$, with density process

$$\left.\frac{\mathrm{d}\mathbb{P}^\nu}{\mathrm{d}\mathbb{P}}\right|_{\mathfrak{F}_t} = Z_t^\nu, \quad t\in[0,T].$$

Step 2: $\mathbb{P}^\nu$-intensity of N^{ij}. Let $i,j\in\mathbb{S}$ with $i\neq j$. Since $\mathbb{P}^\nu \ll \mathbb{P}$ it is clear that N^{ij} is a $\mathbb{P}^\nu$-counting process, so it suffices to show that the process ${}^\nu\bar{N}^{ij} = \{{}^\nu\bar{N}_t^{ij}\}$ given by

$${}^\nu\bar{N}_t^{ij} \triangleq N_t^{ij} - \int_0^t Q^{ij}(s, W_s, \mu(s,W_s), \nu_s)\mathrm{d}s, \quad t\in[0,T], \tag{9}$$

is a local $(\mathfrak{F},\mathbb{P}^\nu)$-martingale. To show this, by Step 1 it suffices to demonstrate that $Z^\nu \cdot {}^\nu\bar{N}^{ij}$ is a local $(\mathfrak{F},\mathbb{P})$-martingale. Noting that

- ▹ $[N^{k\ell}, N^{ij}] = \sum_{s\in(0,\,\cdot\,]} \Delta N_s^{k\ell}\cdot\Delta N_s^{ij} = 0$ whenever $k,\ell\in\mathbb{S}$ and $(k,\ell)\neq(i,j)$,
- ▹ $\mathrm{d}Z_t^\nu = Z_{t-}^\nu \mathrm{d}\theta_t^\nu + Z_{t-}^\nu\mathrm{d}\vartheta_t = \sum_{\substack{k,\ell\in\mathbb{S},\\ k\neq\ell}} Z_{t-}^\nu\big(Q^{k\ell}(t,W_t,\mu(t,W_t),\nu_t)-1\big)\,\mathrm{d}\bar{N}_t^{k\ell} + Z_{t-}^\nu\mathrm{d}\vartheta_t$,
- ▹ $\mathrm{d}[Z^\nu, {}^\nu\bar{N}^{ij}]_t = Z_{t-}^\nu(Q^{ij}(t,W_t,\mu(t,W_t),\nu_t)-1)\mathrm{d}N_t^{ij}$,

and using integration by parts, the local martingale property follows since

$$\begin{aligned}
\mathrm{d}\big(Z_t^\nu\cdot{}^\nu\bar{N}_t^{ij}\big) &= Z_{t-}^\nu\mathrm{d}^\nu\bar{N}_t^{ij} + {}^\nu\bar{N}_{t-}^{ij}\mathrm{d}Z_t^\nu + \mathrm{d}[Z^\nu,{}^\nu\bar{N}^{ij}]_t \\
&= Z_{t-}^\nu\mathrm{d}N_t^{ij} - Z_{t-}^\nu Q^{ij}(t,W_t,\mu(t,W_t),\nu_t)\mathrm{d}t + {}^\nu\bar{N}_{t-}^{ij}\mathrm{d}Z_t^\nu \\
&\qquad\qquad + Z_{t-}^\nu Q^{ij}(t,W_t,\mu(t,W_t),\nu_t)\mathrm{d}N_t^{ij} - Z_{t-}^\nu\mathrm{d}N_t^{ij} \\
&= {}^\nu\bar{N}_{t-}^{ij}\mathrm{d}Z_t^\nu + Z_{t-}^\nu Q^{ij}(t,W_t,\mu(t,W_t),\nu_t)\mathrm{d}\bar{N}_t^{ij}.
\end{aligned}$$

Step 3: $\mathbb{P}^\nu = \mathbb{P}$ on $\sigma(X_0)$. For any function $g:\ \mathbb{S}\to\mathbb{R}$ we have

$$\mathbb{E}^\nu[g(X_0)] = \mathbb{E}[g(X_0)\cdot Z_T^\nu] = \mathbb{E}\big[g(X_0)\cdot\mathbb{E}[Z_T^\nu|\mathfrak{F}_0]\big] = \mathbb{E}[g(X_0)\cdot Z_0^\nu] = \mathbb{E}[g(X_0)]$$

by the $(\mathfrak{F},\mathbb{P})$-martingale property of Z^ν.

Step 4: Distribution of W_k under $\mathbb{P}^\nu$. Let $k\in[1:n]$ and $w_1,\ldots,w_k\in\mathbb{W}$. Since $\mathcal{E}[\theta^\nu]_{T_k} = \mathcal{E}[\theta^\nu]_{T_k-}$ a.s. and W_k is uniformly distributed on $\mathbb{W}$ and independent of $\mathfrak{F}_{T_k-}$ under $\mathbb{P}$, iterated conditioning yields

$$\begin{aligned}
&\mathbb{P}^\nu\big(W_1=w_1,\ldots,W_k=w_k\big) = \mathbb{E}\big[Z_{T_k}^\nu\cdot\mathbb{1}_{\{W_k=w_k\}}\cdot\mathbb{1}_{\{W_1=w_1,\ldots,W_{k-1}=w_{k-1}\}}\big] \\
&= \mathbb{E}\big[Z_{T_k-}^\nu\cdot|\mathbb{W}|\cdot\kappa_k(W_k|W_1,\ldots,W_{k-1},\mu(T_k-,W_{T_k-}))\cdot\mathbb{1}_{\{W_k=w_k\}}\cdot\mathbb{1}_{\{W_1=w_1,\ldots,W_{k-1}=w_{k-1}\}}\big] \\
&= |\mathbb{W}|\cdot\kappa_k(w_k|w_1,\ldots,w_{k-1},\mu(T_k-,w_{T_k-}))\cdot\mathbb{E}\big[Z_{T_k-}^\nu\cdot\mathbb{1}_{\{W_1=w_1,\ldots,W_{k-1}=w_{k-1}\}}\cdot\mathbb{P}(W_k=w_k|\mathfrak{F}_{T_k-})\big] \\
&= \kappa_k(w_k|w_1,\ldots,w_{k-1},\mu(T_k-,w_{T_k-}))\cdot\mathbb{P}^\nu\big(W_1=w_1,\ldots,W_{k-1}=w_{k-1}\big).
\end{aligned}$$

Thus we have $\mathbb{P}^\nu(W_k=w_k|\mathfrak{G}_{T_k-}) = \kappa_k(w_k|W_1,\ldots,W_{k-1},M_{T_k-})$ and the proof is complete. □

Lemma 3 implies in particular that $\mathbb{P}^\nu(\Delta N_t^{ij} \neq 0) = 0$ for every $t \in [0,T]$, so as a consequence we have

$$\Delta X_t = 0 \quad \mathbb{P}^\nu\text{-a.s. for all } t \in [0,T] \setminus \{T_1, \dots, T_n\}.$$

Moreover, since $\mathbb{P}^{\nu_1} = \mathbb{P}^{\nu_2}$ on $\mathfrak{G}_T$ for all admissible controls $\nu_1, \nu_2 \in \mathcal{A}$ and $M_t = \mu(t, W_t)$ for $t \in [0,T]$, the agent's *ex ante* beliefs concerning the common noise factors are the same, irrespective of his control.

3 Solution of the Optimization Problem

In the following, we solve the agent's maximization problem (P_μ) using the associated dynamic programming equation (DPE). This is the same methodology as in [GMS13] and [CF20]; see [CW18] for an alternative approach (to extended mean field games, but without common noise) based on backward SDEs.
The DPE for the agent's optimization problem reads

$$0 = \sup_{u \in \mathbb{U}} \left\{ \frac{\partial v^i}{\partial t}(t,w) + \psi^i\big(t,w,\mu(t,w),u\big) + Q^{i\bullet}\big(t,w,\mu(t,w),u\big) \cdot v(t,w) \right\}$$

for $i \in \mathbb{S}$, subject to suitable consistency conditions for $t = T_k$, $k \in [1:n]$, and the terminal condition

$$v(T,w) = \Psi\big(w, \mu(T,w)\big) \quad \text{for all } w \in \mathbb{W}^n.$$

Assumption 4. *There exists a Borel measurable function $h\colon\ [0,T] \times \mathbb{W}^n \times \mathbb{M} \times \mathbb{R}^d \to \mathbb{U}^d$ such that for every $i \in \mathbb{S}$ and all $(t,w,m,v) \in [0,T] \times \mathbb{W}^n \times \mathbb{M} \times \mathbb{R}^d$ we have*

$$h^i(t,w,m,v) \in \operatorname*{arg\,max}_{u \in \mathbb{U}} \big\{ \psi^i(t,w,m,u) + Q^{i\bullet}(t,w,m,u) \cdot v \big\}.$$

Assumption 4 is satisfied e.g. if $\mathbb{U}$ is compact and Q and ψ are continuous with respect to $u \in \mathbb{U}$. Note that, since $\psi^i(\,\cdot\,,\,\cdot\,,m,u)$ and $Q^{i\bullet}(\,\cdot\,,\,\cdot\,,m,u)$ are non-anticipative for $m \in \mathbb{M}$, $u \in \mathbb{U}$, we can assume without loss of generality that $h(\,\cdot\,,\,\cdot\,,m,v)$ is non-anticipative for $m \in \mathbb{M}$, $v \in \mathbb{R}^d$. With this, we define

$$\begin{aligned} &\widehat{Q}\colon\ [0,T] \times \mathbb{W}^n \times \mathbb{M} \times \mathbb{R}^d \to \mathbb{R}^{d \times d}, \qquad && \widehat{Q}^{ij}(t,w,m,v) \triangleq Q^{ij}\big(t,w,m,h^i(t,w,m,v)\big), \\ &\widehat{\psi}\colon\ [0,T] \times \mathbb{W}^n \times \mathbb{M} \times \mathbb{R}^d \to \mathbb{R}^d, && \widehat{\psi}^i(t,w,m,v) \triangleq \psi^i\big(t,w,m,h^i(t,w,m,v)\big) \end{aligned}$$

and thus obtain the following reduced-form DPE, which we use in the following:

Definition 5. Let $\mu\colon\ [0,T] \times \mathbb{W}^n \to \mathbb{M}$ be regular and non-anticipative. A function $v : [0,T] \times \mathbb{W}^n \to \mathbb{R}^d$ is called a *solution* of (DP_μ) subject to (CC_μ), (TC_μ) if v is non-anticipative and satisfies the ordinary differential equation (ODE)[6]

$$\dot{v}(t,w) = -\widehat{\psi}\big(t,w,\mu(t,w),v(t,w)\big) - \widehat{Q}\big(t,w,\mu(t,w),v(t,w)\big) \cdot v(t,w) \tag{DP_μ}$$

for $t \in [T_k, T_{k+1}\rangle$, $k \in [0:n]$, subject to the consistency and terminal conditions

$$v(T_k-,w) = \Psi_k\big(w, \mu(T_k-,w), v(T_k,\,\cdot\,)\big), \tag{CC_μ}$$

$$v(T,w) = \Psi\big(w, \mu(T,w)\big) \tag{TC_μ}$$

[6] All ODEs in this article are taken in the sense of Carathéodory; see [Hal80, §I.5].

for $k \in [1:n]$ and all $w \in \mathbb{W}^n$. Here, for $k \in [1:n]$, the jump operator Ψ_k is defined via

$$\Psi_k^i(w, m, \bar{v}) \triangleq \sum_{\bar{w}_k \in \mathbb{W}} \kappa_k\big(\bar{w}_k | w_1, \ldots, w_{k-1}, m\big) \cdot \bar{v}^{J^i(T_k, (w_{-k}, \bar{w}_k), m)}(w_{-k}, \bar{w}_k), \ i \in \mathbb{S}, \tag{Ψ_k}$$

where $\bar{v} : \mathbb{W}^n \to \mathbb{R}^d$ and $(w_{-k}, \bar{w}_k) \triangleq (w_1, \ldots, w_{k-1}, \bar{w}_k, w_{k+1}, \ldots, w_n)$ for $\bar{w}_k \in \mathbb{W}$, $w \in \mathbb{W}^n$. ⋄

Observe that (DP$_\mu$) represents a system of (random) ODEs, coupled via $w \in \mathbb{W}^n$. The ODEs run backward in time on each segment $[T_k, T_{k+1}\rangle \times \mathbb{W}^n$, $k \in [0:n]$, and their terminal conditions for $t \uparrow T_{k+1}$ are specified by (TC$_\mu$) for $k = n$ and by (CC$_\mu$) for $k < n$. Note that for $t \in [T_k, T_{k+1}\rangle$ the relevant common noise factors $W_1, \ldots, W_k$ are known.

Remark 6. While the significance of the DPE (DP$_\mu$) and the terminal condition (TC$_\mu$) are clear, the consistency conditions (CC$_\mu$) warrant a brief comment: For $i \in \mathbb{S}$, $k \in [1:n]$ and $w \in \mathbb{W}^n$ the state process jumps from state i to state $j \triangleq J^i(T_k, (w_{-k}, W_k), \mu(T_k-, w_{T_k-}))$ on $\{X_{T_k-} = i\} \cap \{W_{T_k-} = w_{T_k-}\}$ when the common noise factor W_k is revealed at time T_k. ⋄

We next link the solution of the DPE to the underlying stochastic control problem.

Theorem 7 (Verification)**.** *Suppose $\mu : [0,T] \times \mathbb{W}^n \to \mathbb{M}$ is regular and non-anticipative and v is a solution of* (DP$_\mu$) *subject to* (CC$_\mu$) *and* (TC$_\mu$)*. Then v is the agent's value function for problem* (P$_\mu$)*, i.e.*

$$\sum_{i \in \mathbb{S}} \mathbb{P}(X_0 = i) v^i(0) = \sup_{\nu \in \mathcal{A}} \mathbb{E}^\nu \Big[\int_0^T \psi^{X_t}(t, W_t, M_t, \nu_t) \mathrm{d}t + \Psi^{X_T}(W_T, M_T)\Big],$$

and an optimal control is given by $\widehat{\nu} \in \mathcal{A}$ with

$$\widehat{\nu}\left(t, X_{(\,\cdot\,\wedge t)-}, W_t\right) = h^{X_{t-}}\big(t, W_t, \mu(t, W_t), v(t, W_t)\big) \quad \textit{for } t \in [0,T].$$

Proof. Let $\nu \in \mathcal{A}$ be an admissible strategy. Until further notice we fix $k \in [0:n]$.
Step 1: Dynamics on $[T_k, T_{k+1}\rangle$. From Itô's lemma, applicable due to regularity of v, we obtain

$$\begin{aligned} v^{X_{T_k}}(T_k, W_{T_k}) &= v^{X_{T_{k+1}-}}\big(T_{k+1}-, W_{T_{k+1}-}\big) \\ &\quad - \sum_{\substack{i,j \in \mathbb{S}, \\ i \neq j}} \int_{T_k}^{T_{k+1}} \mathbb{1}_{\{X_{s-} = i\}} \big(v^j(s, W_s) - v^i(s, W_s)\big)\, \mathrm{d}^\nu \bar{N}_s^{ij} \\ &\quad - \int_{T_k}^{T_{k+1}} \sum_{i=1}^d \mathbb{1}_{\{X_s = i\}} \big(\dot{v}^i(s, W_s) + Q^{i\bullet}(s, W_s, \mu(s, W_s), \nu_s) \cdot v(s, W_s)\big)\, \mathrm{d}s. \end{aligned} \tag{10}$$

Step 2: Jump dynamics at T_k. We recall from Lemma 3 that

$$\begin{aligned} \mathbb{P}^\nu(W_k = \bar{w}_k | X_{T_k-}, W_1, \ldots, W_{k-1}) &= \mathbb{P}^\nu(W_k = \bar{w}_k | W_1, \ldots, W_{k-1}) \\ &= \kappa_k\big(\bar{w}_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big). \end{aligned}$$

In view of the jump dynamics (3) and the consistency condition (CC$_\mu$), we thus obtain

$$\begin{aligned} &\mathbb{E}^\nu\big[v^{X_{T_k}}(T_k, W_{T_k}) \big| \sigma(X_{T_k-}, W_{T_k-})\big] \\ &= \mathbb{E}^\nu\big[v^{J^{X_{T_k-}}(T_k, (W_{T_k-}, W_k), \mu(T_k-, W_{T_k-}))}\big(T_k, (W_{T_k-}, W_k)\big) \big| X_{T_k-}, W_{T_k-}\big] \\ &= \sum_{\bar{w}_k \in \mathbb{W}} \kappa_k\big(\bar{w}_k | W_{T_k-}, \mu(T_k-, W_{T_k-})\big) v^{J^{X_{T_k-}}(T_k, (W_{T_k-}, \bar{w}_k), \mu(T_k-, W_{T_k-}))}\big(T_k, (W_{T_k-}, \bar{w}_k)\big) \\ &= \Psi_k^{X_{T_k-}}\big(W_{T_k-}, \mu(T_k-, W_{T_k-}), v(T_k, \,\cdot\,)\big) = v^{X_{T_k-}}(T_k-, W_{T_k-}). \end{aligned} \tag{11}$$

Step 3: Optimality. Combining (10) and (11) for $k = [1:n]$ and using (TC_μ) yields

$$
\begin{aligned}
v^{X_0}(0) &= v^{X_T}(T, W_T) + \sum_{k=1}^{n} \big(v^{X_{T_k -}}(T_k -, W_{T_k -}) - v^{X_{T_k}}(T_k, W_{T_k}) \big) \\
&\qquad - \sum_{k=0}^{n} \Bigg[\sum_{\substack{i,j\in\mathbb{S},\\ i\neq j}} \int_{T_k}^{T_{k+1}} \mathbb{1}_{\{X_{s-}=i\}} \big(v^j(s,W_s) - v^i(s,W_s) \big)\, \mathrm{d}^\nu \bar{N}_s^{ij} \\
&\qquad\qquad + \int_{T_k}^{T_{k+1}} \sum_{i=1}^{d} \mathbb{1}_{\{X_s=i\}} \big(\dot{v}^i(s,W_s) + Q^{i\bullet}(s, W_s, \mu(s,W_s), \nu_s) \cdot v(s,W_s) \big)\, \mathrm{d}s \Bigg] \\
&= \Psi^{X_T}(W_T, \mu(T, W_T)) + \sum_{k=1}^{n} \Big(\mathbb{E}^\nu \big[v^{X_{T_k}}(T_k, W_{T_k}) \big| \sigma(X_{T_k -}, W_{T_k -}) \big] - v^{X_{T_k}}(T_k, W_{T_k}) \Big) \\
&\qquad - \sum_{\substack{i,j\in\mathbb{S},\\ i\neq j}} M_T^{ij} - \int_0^T \sum_{i=1}^{d} \mathbb{1}_{\{X_s=i\}} \big(\dot{v}^i(s,W_s) + Q^{i\bullet}(s, W_s, \mu(s,W_s), \nu_s) \cdot v(s,W_s) \big)\, \mathrm{d}s,
\end{aligned} \tag{12}
$$

where for $i, j \in \mathbb{S}$, $i \neq j$ the local $(\mathfrak{F}, \mathbb{P}^\nu)$-martingale M^{ij} is given by

$$
M_t^{ij} \triangleq \int_0^t \mathbb{1}_{\{X_{s-}=i\}} \big(v^j(s,W_s) - v^i(s,W_s) \big)\, \mathrm{d}^\nu \bar{N}_s^{ij} \quad \text{for } t \in [0,T].
$$

Since ${}^\nu\bar{N}^{ij}$ is a compensated counting process and v and Q are bounded, M^{ij} is in fact an $(\mathfrak{F}, \mathbb{P}^\nu)$-martingale. Hence taking $\mathbb{P}^\nu$-expectations in (12), using the tower property of conditional expectation and the fact that $\mathbb{P}^\nu$ and $\mathbb{P}$ coincide on $\sigma(X_0)$ by Lemma 3, and finally that v solves the DPE, we obtain

$$
\begin{aligned}
\sum_{i\in\mathbb{S}} \mathbb{P}(X_0 = i) v^i(0) &= \mathbb{E}\big[v^{X_0}(0)\big] = \mathbb{E}^\nu\big[v^{X_0}(0)\big] \\
&= \mathbb{E}^\nu \Bigg[\Psi^{X_T}(W_T, \mu(T, W_T)) \\
&\qquad - \int_0^T \sum_{i=1}^{d} \mathbb{1}_{\{X_s=i\}} \big(\dot{v}^i(s,W_s) + Q^{i\bullet}(s, W_s, \mu(s,W_s), \nu_s) \cdot v(s,W_s) \big)\, \mathrm{d}s \Bigg] \\
&\geq \mathbb{E}^\nu \Bigg[\Psi^{X_T}(W_T, M_T) + \int_0^T \psi^{X_s}(s, W_s, M_s, \nu_s)\, \mathrm{d}s \Bigg].
\end{aligned} \tag{13}
$$

If we replace ν with $\widehat{\nu}$, the same argument applies with equality in (13); we thus conclude that v is the value function of (P_μ), and that the strategy $\widehat{\nu}$ is optimal. $\square$

The optimal strategy is Markovian in the agent's state; this is unsurprising given the literature, see e.g. [GMS13, Theorem 1] or [CW18, Proposition 3.9] and [CF20, Theorem 4]. Note, however, that the time-t optimal strategy may depend on *all* common noise events that have occurred up to time t, as $W_t = (W_1, \ldots, W_k)$ for $t \in [T_k, T_{k+1}\rangle$. In the following, we denote by $\widehat{\mathbb{P}}$ the probability measure

$$
\widehat{\mathbb{P}} \triangleq \mathbb{P}^{\widehat{\nu}}
$$

where $\widehat{\nu}$ is the optimal control specified in Theorem 7. It follows from Lemma 3 that N^{ij} has $\widehat{\mathbb{P}}$-intensity $\widehat{\lambda}^{ij} = \{\widehat{\lambda}_t^{ij}\}$ for $i, j \in \mathbb{S}$, $i \neq j$, where

$$
\widehat{\lambda}_t^{ij} \triangleq Q^{ij}\big(t, W_t, \mu(t, W_t), h^{X_{t-}}(t, W_t, \mu(t,W_t), v(t,W_t))\big) \quad \text{for } t \in [0,T]. \tag{14}
$$

4 Equilibrium

Having solved the agent's optimization problem for a given *ex ante* function μ, we now turn to the resulting mean field equilibrium. We first identify the aggregate distribution resulting from the optimal control.

Remark 8. This paper generally adopts a "representative agent" point of view; an alternative justification of mean field equilibrium is via convergence of Nash equilibria of symmetric N-player games in the limit $N \to \infty$; see, among others, [Fis17], [BC18], [CF18], [CW18], [CP19a], [CPFP19], [DGG19] and [CF20]. In the setting of this article (albeit under additional regularity conditions) a mean field limit justification can be provided along the lines of the proof of Theorem 7 in [GMS13] by conditioning on common noise configurations, similarly as in the proof of Theorem 10 below. $\diamond$

4.1 Aggregation

Given an *ex ante* aggregate distribution specified in terms of a regular, non-anticipative function μ and a corresponding solution v of (DP$_\mu$) subject to (CC$_\mu$), (TC$_\mu$), Theorem 7 yields an optimal strategy $\widehat{\nu}$ for the agent's optimization problem (P$_\mu$). With $\widehat{\mathbb{P}}$ denoting the probability measure associated with $\widehat{\nu}$, the resulting *ex post* aggregate distribution is given by the $\mathbb{M}$-valued, $\mathfrak{G}$-adapted process $\widehat{M} = \{\widehat{M}_t\}$ given by

$$\widehat{M}_t \triangleq \widehat{\mathbb{P}}(X_t \in \cdot \mid \mathfrak{G}_t) \quad \text{for } t \in [0, T].$$

We note that $\widehat{M}$ is càdlàg since $\mathfrak{G}$ is piecewise constant and X is càdlàg. Equilibrium obtains if $\widehat{M}_t = \mu(t, W_t)$ for all $t \in [0, T]$. To proceed, we aim for a more explicit description of $\widehat{M}$. Thus we define for $k \in [1:n]$

$$\Phi_k : \mathbb{W}^n \times \mathbb{M} \times \mathbb{M} \to \mathbb{M}, \quad \Phi_k(w, m, \bar{m}) \triangleq m \cdot P_k(w, \bar{m}), \tag{Φ_k}$$

where $P_k : \mathbb{W}^n \times \mathbb{M} \to \{0,1\}^{d \times d}$ is given by

$$P_k^{ij}(w, \bar{m}) \triangleq \mathbb{1}_{\{J^i(T_k, w_1, \ldots, w_k, \bar{m}) = j\}} \quad \text{for } i, j \in \mathbb{S}$$

and we set

$$m_0 \triangleq \mathbb{P}(X_0 \in \cdot) = \widehat{\mathbb{P}}(X_0 \in \cdot) \in \mathbb{M}.$$

Lemma 9. *Let $\mu : [0,T] \times \mathbb{W}^n \to \mathbb{M}$ and $v : [0,T] \times \mathbb{W}^n \to \mathbb{R}^d$ be regular and non-anticipative, and suppose that $Y = \{Y_t\}$ is an $\mathbb{M}$-valued stochastic process with dynamics*

$$Y_0 = m_0, \quad Y_t = Y_{T_k} + \int_{T_k}^t Y_s \cdot \widehat{Q}\big(s, W_s, \mu(s, W_s), v(s, W_s)\big) \mathrm{d}s \quad \textit{for } t \in [T_k, T_{k+1}\rangle,\ k \in [0:n] \tag{15}$$

that satisfies the consistency conditions

$$Y_{T_k} = \Phi_k\big(W_{T_k}, Y_{T_k -}, \mu(T_k -, W_{T_k -})\big) \quad \textit{for } k \in [1:n].$$

Then Y is $\mathfrak{G}$-adapted.

Proof. Step 1: Existence and uniqueness of Carathéodory solutions. For each $k \in [0:n]$ and $w \in \mathbb{W}^n$, since μ and v are regular and Q is bounded, the function

$$f : [T_k, T_{k+1}] \times \mathbb{R}^{1 \times d} \to \mathbb{R}^{1 \times d}, \quad f(t, y) \triangleq y \cdot \widehat{Q}\big(t, w, \mu(t, w), v(t, w)\big)$$

is measurable in the first and Lipschitz continuous in the second argument. Thus, using that μ, v and $\widehat{Q}$ are non-anticipative, a classical result, see [Hal80, Theorem I.5.3], implies that for each initial condition

$y \in \mathbb{R}^{1\times d}$ there exists a unique Carathéodory solution $\varphi_k^{y,w_{T_k}} : [T_k, T_{k+1}\rangle \to \mathbb{R}^{1\times d}$ of

$$\dot{y}(t) = y(t) \cdot \widehat{Q}\big(t, w_{T_k}, \mu(t, w_{T_k}), v(t, w_{T_k})\big) \text{ for } t \in [T_k, T_{k+1}\rangle, \qquad y(T_k) = y.$$

Step 2: Y is $\mathfrak{G}$-adapted. First note that $Y_0 = m_0$ is clearly $\mathfrak{G}_0$-measurable. Next, suppose that Y_{T_k} is $\mathfrak{G}_{T_k}$-measurable, and note that for $t \in [T_k, T_{k+1}\rangle$ we have $W_t = W_{T_k}$, so

$$Y_t = Y_{T_k} + \int_{T_k}^t Y_s \cdot \widehat{Q}\big(s, W_{T_k}, \mu(s, W_{T_k}), v(s, W_{T_k})\big)\mathrm{d}s.$$

Thus from uniqueness in part (a) it follows that we have the representation

$$Y_t = \varphi_k^{Y_{T_k}, W_{T_k}}(t) \quad \text{for } t \in [T_k, T_{k+1}\rangle.$$

Hence Y_t is $\mathfrak{G}_{T_k}$-measurable for all $t \in [T_k, T_{k+1}\rangle$. Finally, for all $k \in [0 : (n-1)]$ the consistency condition implies that $Y_{T_{k+1}} = \Phi_{k+1}(W_{T_{k+1}}, Y_{T_{k+1}-}, \mu(T_{k+1}-, W_{T_{k+1}-}))$ is $\mathfrak{G}_{T_{k+1}}$-measurable, so the claim follows by induction on $k \in [0 : n]$. □

Theorem 10 (Aggregation)**.** *Let $\mu : [0,T] \times \mathbb{W}^n \to \mathbb{M}$ be regular and non-anticipative with $\mu(0) = m_0$. Suppose v is a solution of* (DP$_\mu$) *subject to* (CC$_\mu$), (TC$_\mu$)*, and the agent implements his optimal strategy $\widehat{\nu}$ as defined in Theorem 7. Then the aggregate distribution $\widehat{M}$ has the $\widehat{\mathbb{P}}$-dynamics*

$$\mathrm{d}\widehat{M}_t = \widehat{M}_t \cdot \widehat{Q}\big(t, W_t, \mu(t, W_t), v(t, W_t)\big)\mathrm{d}t \quad \textit{for } t \in [T_k, T_{k+1}\rangle,\ k \in [0 : n], \tag{M}$$

and satisfies the initial condition

$$\widehat{M}_0 = m_0 \tag{M$_0$}$$

and the jump conditions

$$\widehat{M}_{T_k} = \Phi_k\big(W_{T_k}, \widehat{M}_{T_k-}, \mu(T_k-, W_{T_k-})\big) \quad \textit{for } k \in [1 : n]. \tag{M$_k$}$$

Proof. Let $w \in \mathbb{W}^n$ be a common noise configuration. Since X is defined path by path, see (2) and (3), we first note that $X = X^w$ on $\{W_T = w\}$, where X^w satisfies (2) and

$$X_{T_k}^w = J^{X_{T_k-}^w}\big(T_k, w_{T_k}, \mu(T_k-, w_{T_k-})\big) \text{ for } k \in [1 : n]. \tag{16}$$

We define $\zeta(w) = \{\zeta(w)_t\}$ via

$$\zeta(w)_t \triangleq \prod_{\substack{i,j\in\mathbb{S},\\ i\neq j}} \Bigg(\exp\Big\{\int_0^t \Big(1 - Q^{ij}\big(s, w_s, \mu(s, w_s), h^{X_{s-}^w}(s, w_s, \mu(s, w_s), v(s, w_s))\big)\Big)\mathrm{d}s\Big\}$$
$$\times \prod_{\substack{s\in(0,t],\\ \Delta N_s^{ij}\neq 0}} Q^{ij}\big(s, w_s, \mu(s, w_s), h^{X_{s-}^w}(s, w_s, \mu(s, w_s), v(s, w_s))\big)\Bigg).$$

Using analogous arguments as in Step 1 of the proof of Lemma 3 (see in particular (4) and (7)), it follows that there exists a probability measure $\widehat{\mathbb{P}}^w$ with density process

$$\frac{\mathrm{d}\widehat{\mathbb{P}}^w}{\mathrm{d}\mathbb{P}}\bigg|_{\mathfrak{H}_t} \triangleq \zeta(w)_t \quad \text{for } t \in [0, T],$$

where the filtration $\mathfrak{H} = \{\mathfrak{H}_t\}$ is given by

$$\mathfrak{H}_t \triangleq \sigma\big(X_0,\, N_s^{ij} \,:\, s \in [0,t];\; i,j \in \mathbb{S},\; i \neq j\big) \vee \mathfrak{N} \quad \text{for } t \in [0,T].$$

Furthermore, in view of (4) and (14) we have

$$\zeta(w) = \mathcal{E}[\theta^{\widehat{\nu}}] \quad \text{on } \{W_T = w\}. \tag{17}$$

Step 1: Conditional Kolmogorov dynamics. Throughout Step 1, we fix a common noise configuration $w \in \mathbb{W}^n$. It follows exactly as in the proof of Lemma 3 (with $\widehat{\mathbb{P}}^w$ in place of $\widehat{\mathbb{P}}$) that

$$\widehat{\mathbb{P}}^w \ll \mathbb{P}, \qquad \widehat{\mathbb{P}}^w = \mathbb{P} \quad \text{on } \sigma(X_0),$$

and that for $i,j \in \mathbb{S}$, $i \neq j$, the process N^{ij} is a counting process with $(\mathfrak{H}, \widehat{\mathbb{P}}^w)$-intensity

$$Q^{ij}\big(t, w_t, \mu(t,w_t), h^{X_{t-}^w}(t, w_t, \mu(t,w_t), v(t,w_t))\big) \quad \text{for } t \in [0,T].$$

Boundedness of Q implies that for each $z \in \mathbb{R}^d$ the process $L^w[z] = \{L_t^w[z]\}$,

$$L_t^w[z] \triangleq \sum_{\substack{i,j\in\mathbb{S},\\ i\neq j}} \int_0^t \mathbb{1}_{\{X_{s-}^w = i\}} \cdot (z^j - z^i)\mathrm{d}^w\bar{N}_s^{ij} \quad \text{for } t \in [0,T],$$

is an $(\mathfrak{H}, \widehat{\mathbb{P}}^w)$-martingale, where ${}^w\bar{N}^{ij} = \{{}^w\bar{N}_t^{ij}\}$ is given by

$${}^w\bar{N}_t^{ij} \triangleq N_t^{ij} - \int_0^t Q^{ij}\big(s, w_s, \mu(s,w_s), h^{X_{s-}^w}(s, w_s, \mu(s,w_s), v(s,w_s))\big)\mathrm{d}s, \quad t \in [0,T].$$

Using Itô's lemma and the fact that $\widehat{\lambda}_t^{ij} = \widehat{Q}^{ij}(t, W_t, \mu(t,W_t), v(t,W_t))$ on $\{X_{t-} = i\}$, $t \in [0,T]$, by (14), we have for each $z \in \mathbb{R}^d$, $k \in [0:n]$ and $t \in [T_k, T_{k+1}\rangle$

$$z^{X_t^w} = z^{X_{T_k}^w} + L_t^w[z] - L_{T_k}^w[z] + \sum_{i=1}^d \int_{T_k}^t \mathbb{1}_{\{X_s^w = i\}} \cdot \widehat{Q}^{i\bullet}(s, w_s, \mu(s,w_s), v(s,w_s)) \cdot z \;\mathrm{d}s.$$

Taking expectations with respect to $\widehat{\mathbb{P}}^w$ and using Fubini's theorem yields

$$\widehat{\mathbb{E}}^w\big[z^{X_t^w}\big] = \widehat{\mathbb{E}}^w\big[z^{X_{T_k}^w}\big] + \sum_{i=1}^d \int_{T_k}^t \widehat{\mathbb{P}}^w(X_s^w = i) \cdot \widehat{Q}^{i\bullet}(s, w_s, \mu(s,w_s), v(s,w_s)) \cdot z \;\mathrm{d}s,$$

so with $z = e_i$, $i \in \mathbb{S}$, we get

$$\widehat{\mathbb{P}}^w(X_t^w = i) = \widehat{\mathbb{P}}^w(X_{T_k}^w = i) + \sum_{j=1}^d \int_{T_k}^t \widehat{\mathbb{P}}^w(X_s^w = j) \cdot \widehat{Q}^{ji}(s, w_s, \mu(s,w_s), v(s,w_s))\mathrm{d}s. \tag{18}$$

It follows from (18) that $\eta(w) = \{\eta(w)_t\}$,

$$\eta(w)_t \triangleq \widehat{\mathbb{P}}^w(X_t^w \in \,\cdot\,), \quad t \in [0,T] \tag{19}$$

satisfies, for all $i \in \mathbb{S}$ and $k \in [0:n]$,

$$\eta(w)_t^i = \eta(w)_{T_k}^i + \int_{T_k}^t \eta(w)_s \cdot \widehat{Q}^{\bullet i}\,(s, w_s, \mu(s,w_s), v(s,w_s))\,\mathrm{d}s \quad \text{for } t \in [T_k, T_{k+1}\rangle. \tag{20}$$

Moreover, since $\widehat{\mathbb{P}}^w = \mathbb{P}$ on $\sigma(X_0)$ and $X_0^w = X_0$, $\eta(w)$ satisfies the initial condition

$$\eta(w)_0 = \widehat{\mathbb{P}}^w(X_0^w \in \cdot\,) = \mathbb{P}(X_0^w \in \cdot\,) = \mathbb{P}(X_0 \in \cdot\,) = m_0. \tag{21}$$

Finally, consider a common noise time $t = T_k$ and note that for all $i \in \mathbb{S}$ the jump condition (16) implies

$$\begin{aligned}
\eta(w)^i_{T_k} &= \widehat{\mathbb{P}}^w\big(X^w_{T_k} = i\big) = \widehat{\mathbb{P}}^w\big(J^{X^w_{T_k-}}(T_k, w_{T_k}, \mu(T_k-, w_{T_k-})) = i\big) \\
&= \sum_{j=1}^d \widehat{\mathbb{P}}^w\big(J^j(T_k, w_{T_k}, \mu(T_k-, w_{T_k-})) = i \big| X^w_{T_k-} = j\big) \cdot \widehat{\mathbb{P}}^w(X^w_{T_k-} = j) \\
&= \sum_{j=1}^d \mathbb{1}_{\left\{J^j(T_k, w_{T_k}, \mu(T_k-, w_{T_k-})) = i\right\}} \cdot \widehat{\mathbb{P}}^w(X^w_{T_k-} = j) \\
&= \sum_{j=1}^d P_k^{ji}(w_{T_k}, \mu(T_k-, w_{T_k-})) \cdot \eta(w)^j_{T_k-} = \Phi^i_k\big(w_{T_k}, \eta(w)_{T_k-}, \mu(T_k-, w_{T_k-})\big).
\end{aligned} \tag{22}$$

Since $\eta(W_T) = \sum_{w \in \mathbb{W}^n} \mathbb{1}_{\{W_T = w\}} \cdot \eta(w)$, in view of (20), (21) and (22) it follows from Lemma 9 that the process $\eta(W_T)$ is $\mathfrak{G}$-adapted.

Step 2: Identification of $\eta(W_T)$. Recall that $\mathfrak{G}_T = \sigma(W_T) \vee \mathfrak{N}$ and let $w \in \mathbb{W}^n$. For $t \in [0, T]$ and $i \in \mathbb{S}$ we have by (6) and (17)

$$\begin{aligned}
&\widehat{\mathbb{E}}\big[\mathbb{1}_{\{W_T = w\}} \cdot \mathbb{1}_{\{X_t = i\}}\big] = \mathbb{E}\big[\mathbb{1}_{\{W_T = w\}} \cdot \mathbb{1}_{\{X^w_t = i\}} \cdot Z^{\widehat{\nu}}_T\big] = \mathbb{E}\big[\mathbb{1}_{\{W_T = w\}} \cdot \mathbb{1}_{\{X^w_t = i\}} \cdot \zeta(w)_T \cdot \mathcal{E}[\vartheta]_T\big] \\
&= \prod_{k=1}^n \big(|\mathbb{W}| \cdot \kappa_k(w_k | w_1, \ldots, w_{k-1}, \mu(T_k-, w_{T_k-}))\big) \cdot \mathbb{E}\big[\mathbb{1}_{\{W_T = w\}} \cdot \mathbb{1}_{\{X^w_t = i\}} \cdot \zeta(w)_T\big] \\
&= |\mathbb{W}|^n \cdot \widehat{\mathbb{P}}(W_T = w) \cdot \mathbb{P}(W_T = w) \cdot \widehat{\mathbb{P}}^w(X^w_t = i) = \widehat{\mathbb{E}}\big[\mathbb{1}_{\{W_T = w\}} \cdot \eta(W_T)^i_t\big],
\end{aligned}$$

where in the final line the first identity is due to Lemma 3 and $\mathbb{P}$-independence of $(\zeta(w), X^w)$ and $\mathfrak{G}_T$; and the second is due to (19) and the fact that $\mathbb{P}(W_T = w) = 1/|\mathbb{W}|^n$. Thus

$$\widehat{\mathbb{P}}(X_t \in \cdot\,|\mathfrak{G}_T) = \eta(W_T)_t \quad \widehat{\mathbb{P}}\text{-a.s. for } t \in [0, T].$$

Step 3: Dynamics of $\widehat{M}$. By Step 2 and the tower property of conditional expectation, we find that for each $i \in \mathbb{S}$ and $t \in [0, T]$

$$\widehat{M}^i_t = \widehat{\mathbb{P}}(X_t = i | \mathfrak{G}_t) = \widehat{\mathbb{E}}\big[\widehat{\mathbb{E}}[\mathbb{1}_{\{X_t = i\}} | \mathfrak{G}_T] | \mathfrak{G}_t\big] = \widehat{\mathbb{E}}\big[\eta(W_T)^i_t | \mathfrak{G}_t\big] = \eta(W_T)^i_t \quad \widehat{\mathbb{P}}\text{-a.s.},$$

where the final identity is due to the fact that $\eta(W_T)$ is $\mathfrak{G}$-adapted by Step 1 and $\widehat{\mathbb{E}}$ denotes $\widehat{\mathbb{P}}$-expectation. Since both $\widehat{M}$ and $\eta(W_T)$ are càdlàg, it follows that $\widehat{M} = \eta(W_T)$ $\widehat{\mathbb{P}}$-a.s., and (M), (M$_0$) and (M$_k$) follow from (20), (21) and (22). □

As a by-product, the preceding proof yields the alternative representation

$$\widehat{M}_t = \widehat{\mathbb{P}}(X_t \in \cdot \mid \mathfrak{G}_T) \quad \text{for } t \in [0, T] \ \widehat{\mathbb{P}}\text{-a.s.}$$

4.2 Mean Field Equilibrium System

As discussed above, equilibrium obtains if the agents' ex ante beliefs coincide with the ex post outcome. This holds if and only if the ex post aggregate distribution process $\widehat{M}$ from (M) satisfies

$$\widehat{\mathbb{P}}(X_t \in \cdot\,|\mathfrak{G}_t) = \widehat{M}_t \overset{!}{=} M_t = \mu(t, W_t) \quad \text{for all } t \in [0, T].$$

Definition 11 (Equilibrium System). A pair (μ, v) of regular and non-anticipative functions

$$\mu:\ [0,T]\times\mathbb{W}^n \to \mathbb{M} \qquad \text{and} \qquad v:\ [0,T]\times\mathbb{W}^n \to \mathbb{R}^d$$

is called a *rational expectations equilibrium*, or briefly an *equilibrium*, if for all $w \in \mathbb{W}^n$

$$\dot{\mu}(t,w) = \mu(t,w)\cdot \widehat{Q}\big(t,w,\mu(t,w),v(t,w)\big) \tag{E1}$$

$$\dot{v}(t,w) = -\widehat{\psi}\big(t,w,\mu(t,w),v(t,w)\big) - \widehat{Q}\big(t,w,\mu(t,w),v(t,w)\big)\cdot v(t,w) \tag{E2}$$

for $t \in [T_k, T_{k+1}\rangle$, $k \in [0:n]$, subject to the consistency conditions[7]

$$\mu(T_k,w) = \Phi_k\big(w,\mu(T_k-,w)\big) \tag{E3}$$

$$v(T_k-,w) = \Psi_k\big(w,\mu(T_k-,w),v(T_k,\ \cdot\ )\big) \tag{E4}$$

for $k \in [1:n]$, and the initial/terminal conditions

$$\mu(0,w) = m_0 \tag{E5}$$

$$v(T,w) = \Psi\big(w,\mu(T,w)\big). \tag{E6}$$

We also refer to (E1)-(E6) as the *equilibrium system.* ◇

In combination, Theorem 7 and Theorem 10 demonstrate that, given a solution (μ, v) of the equilibrium system, v is the value function of the agent's optimization problem (P_μ) with *ex ante* aggregate distribution μ; and the *ex post* distribution resulting from the corresponding optimal strategy is given by μ itself. Under standard Lipschitz conditions, see Assumption A.1 in Appendix A, we have the following existence result for the equilibrium system; the proof is reported in Appendix A and is a ramification of that in [CF20, Theorem 6] based on Banach's fixed point theorem.

Theorem 12 (Existence of Equilibria). *Let Assumption A.1 be satisfied and fix $n \in \mathbb{N}_0$. Then there exists $T^\star > 0$ such that for every time horizon $T \le T^\star$ and every choice of common noise times $0 = T_0 < T_1 < \cdots < T_n < T_{n+1} = T$ there is a unique solution of the equilibrium system* (E1)-(E6).[8]

Using Theorems 7 and 10 we can thus identify a mean field equilibrium with common noise by producing a solution of the equilibrium system (E1)-(E6); under the conditions of Theorem 12, a solution exists and can be computed numerically by a fixed point iteration. We provide some illustrations in Section 5.

5 Applications

Before we illustrate our results in two showcase examples, we briefly discuss our numerical approach to the equilibrium system (E1)-(E6). (E1)-(E2) is a forward-backward system of $2d$ ODEs with boundary conditions (E3)-(E6), coupled through the parameter $w \in \mathbb{W}^n$ representing common noise configurations. The special case $n = 0$ (no common noise) corresponds to the setting of [GMS13] and [CF20], with the equilibrium system reducing to a single $2d$-dimensional forward-backward ODE. For $n \ge 1$, the consistency conditions (E3)-(E4) specify initial conditions for μ on $[T_k, T_{k+1}\rangle$ and terminal conditions for v on $[T_{k-1}, T_k\rangle$, $k \in [1:n]$; since these conditions are interconnected, there is in general *no* segment $[T_k, T_{k+1}\rangle \times \mathbb{W}^n$ where the equilibrium system yields both an explicit initial condition for μ and an explicit terminal condition for v, so we cannot simply split the problem into subintervals. Rather, the

[7] With a slight abuse of notation, here and subsequently we set $\Phi_k(w,m) \triangleq \Phi_k(w,m,m)$ for $k \in [1:n]$, $w \in \mathbb{W}^n$, $m \in \mathbb{M}$.

[8] Although Banach's fixed-point theorem also yields uniqueness of the fixed point, the resulting mean field equilibrium need not be unique unless the maximizer h in Assumption 4 is uniquely determined.

equilibrium system can be regarded as a *multi-point boundary value problem* where for each of the $|\mathbb{W}|^k$ conceivable combinations of common noise factors on $[T_k, T_{k+1}\rangle$, $k \in [0:n]$, we have to solve a coupled forward-backward system of ODEs in $2d$ dimensions, resulting in a tree of such systems of size

$$\sum_{k=0}^{n} |\mathbb{W}|^k = \frac{|\mathbb{W}|^{n+1} - 1}{|\mathbb{W}| - 1} \in \mathcal{O}(|\mathbb{W}^n|).$$

Our approach to solving (E1)-(E6) is to rely on the probabilistic interpretation as a fixed-point system, based on Theorem 12. Thus, starting from an initial flow of probability weights $\mu_0(t,w)$, $(t,w) \in [0,T] \times \mathbb{W}^n$ with $\mu_0(0,w) = m_0$ for all $w \in \mathbb{W}^n$, we solve (DP_μ) subject to (TC_μ) and (CC_μ) backward in time for all non-negligible common noise configurations $w \in \mathbb{W}^n$ to obtain the value $v_0(t,w)$, $(t,w) \in [0,T] \times \mathbb{W}^n$, of the agents' optimal response to the given belief μ_0. This, in turn, is used to solve (M) subject to (M_0) and (M_k) forward in time. As a result, we obtain an ex post aggregate distribution $\mu_1(t,w)$, $(t,w) \in [0,T] \times \mathbb{W}^n$; we then iterate this with μ_1 in place of μ_0, etc. Note that Theorem 12 guarantees the convergence of this methodology for appropriate parameter settings.

5.1 A Decentralized Agricultural Production Model

As a first (stylized) example we consider a mean field game of agents, each of which owns (an infinitesimal amount of) land of identical size and quality within a given area. If it is farmed, each field has a productivity $f(w_k) > 0$ depending on the common weather condition w_k. We assume that weather is either good, bad or catastrophic, so $w_k \in \mathbb{W} \triangleq \{\uparrow, \downarrow, \lightning\}$, and changes at given common noise times $T_1, \ldots, T_n$.

Each agent is in exactly one state $i \in \mathbb{S} \triangleq \{0, 1\}$ depending on whether he grows crops on his field ($i = 1$, the agent is a farmer) or not ($i = 0$). The selling price p for his harvest depends on *aggregate* production, and thus in particular on the proportion $m^1 \in [0,1]$ of farmers; the mean field interaction is transmitted through the market price of the crop. We assume that p is a strictly decreasing function of overall production $f(w_k) \cdot m^1$; see Figure 1 for illustration.

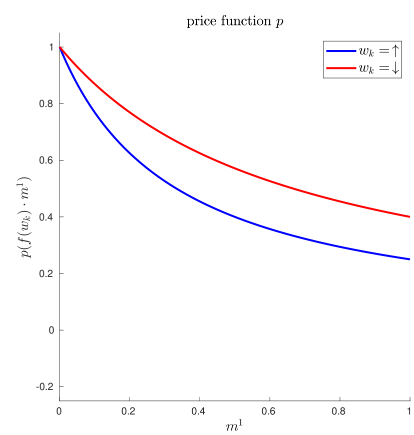

Figure 1: Price function p (parameters as in Table 1).

We assume that $f(\uparrow) \geq f(\downarrow) = f(\lightning) \geq 0$. Moreover, on the catastrophic event $\{W_k = \lightning\}$ all agents are reduced to being non-farmers, and thus

$$J^i(t,w,m) \triangleq \begin{cases} 0 & \text{if } t \in \{T_1, \ldots, T_n\},\, t = T_k,\, w_k = \lightning, \\ i & \text{else} \end{cases}$$

Parameter	T	n	T_k	$q_{\text{entry}},\ q_{\text{exit}}$	$f(\uparrow)$	$f(\downarrow)$	$p(q)$	c_{prod}	c_{entry}
Value	1	4	$k/5$	0.7	1	0.5	$1/(1+3q)$	0.3	0.1

Table 1: Coefficients in the agricultural production model.

for $(i,t,w,m) \in \mathbb{S} \times [0,T] \times \mathbb{W}^n \times \mathbb{M}$. Each agent can make an effort $u \in \mathbb{U} \triangleq [0,1]$ to become or stop being a farmer; the intensity matrix for state transitions is given by

$$Q(t,w,m,u) = u \cdot \begin{bmatrix} -q_{\text{entry}} & q_{\text{entry}} \\ q_{\text{exit}} & -q_{\text{exit}} \end{bmatrix} \quad \text{for } (t,w,m,u) \in [0,T] \times \mathbb{W}^n \times \mathbb{M} \times \mathbb{U},$$

where $q_{\text{entry}}, q_{\text{exit}} \geq 0$ are given maximum rates. The running rewards capture the fact that both efforts to building up farming capacities and production itself are costly, while revenues from the sales of the crop generate profits; thus

$$\psi^0(t,w,m,u) = -\frac{1}{2} c_{\text{entry}} \cdot u^2 \qquad \text{and} \qquad \psi^1(t,w,m,u) = p\big(f(w_k) \cdot m^1\big) \cdot f(w_k) - c_{\text{prod}}$$

for $t \in [T_k, T_{k+1}\rangle$, $k \in [0:n]$, where $w_0 \triangleq \uparrow$ and $c_{\text{entry}}, c_{\text{prod}} \geq 0$. The terminal reward is zero. It follows that the unique maximizer in Assumption 4 is given by

$$h^0(t,w,m,v) = \Big[\frac{q_{\text{entry}}}{c_{\text{entry}}}(v^1 - v^0)^+\Big] \wedge 1 \qquad \text{and} \qquad h^1(t,w,m,v) = \begin{cases} 0 & \text{if } v^1 \geq v^0, \\ 1 & \text{else.} \end{cases}$$

We choose $m_0^1 \triangleq 10\%$ for the initial proportion of farmers, and report the relevant coefficients in Table 1. Our results for the evolution of the mean field equilibrium are shown in Figures 2 through 5 for various common noise configurations $w \in \mathbb{W}^n$ and the following two baseline models:

(nC) Catastrophic weather conditions do not occur; we use

$$\kappa_k(\uparrow | w_1, \ldots, w_{k-1}, m) = \kappa_k(\downarrow | w_1, \ldots, w_{k-1}, m) = 0.5$$

for all $w \in \mathbb{W}^n$ and $m \in \mathbb{M}$.

(C) Catastrophic events are likely; we use

$$\kappa_k(\uparrow | w_1, \ldots, w_{k-1}, m) = 0.25, \quad \kappa_k(\downarrow | w_1, \ldots, w_{k-1}, m) = 0.25, \quad \kappa_k(\lightning | w_1, \ldots, w_{k-1}, m) = 0.5$$

for all $w \in \mathbb{W}^n$ and $m \in \mathbb{M}$.

Figures 2, 3 and 4 illustrate the resulting equilibrium proportions of farmers, optimal actions, and market prices for some fixed common noise configurations. In particular, equilibrium prices are stochastically modulated by the prevailing weather conditions. To illustrate the effect of uncertainty about future weather conditions we also show, for each common noise configuration, the theoretical perfect-foresight equilibria that would pertain if future weather conditions were known; these are plotted using dashed lines in Figures 2 through 4, and the subscript $\circ$ indicates the relevant *deterministic* common noise path. Finally, Figure 5 illustrates the tree of all possible equilibrium evolutions in model (C).

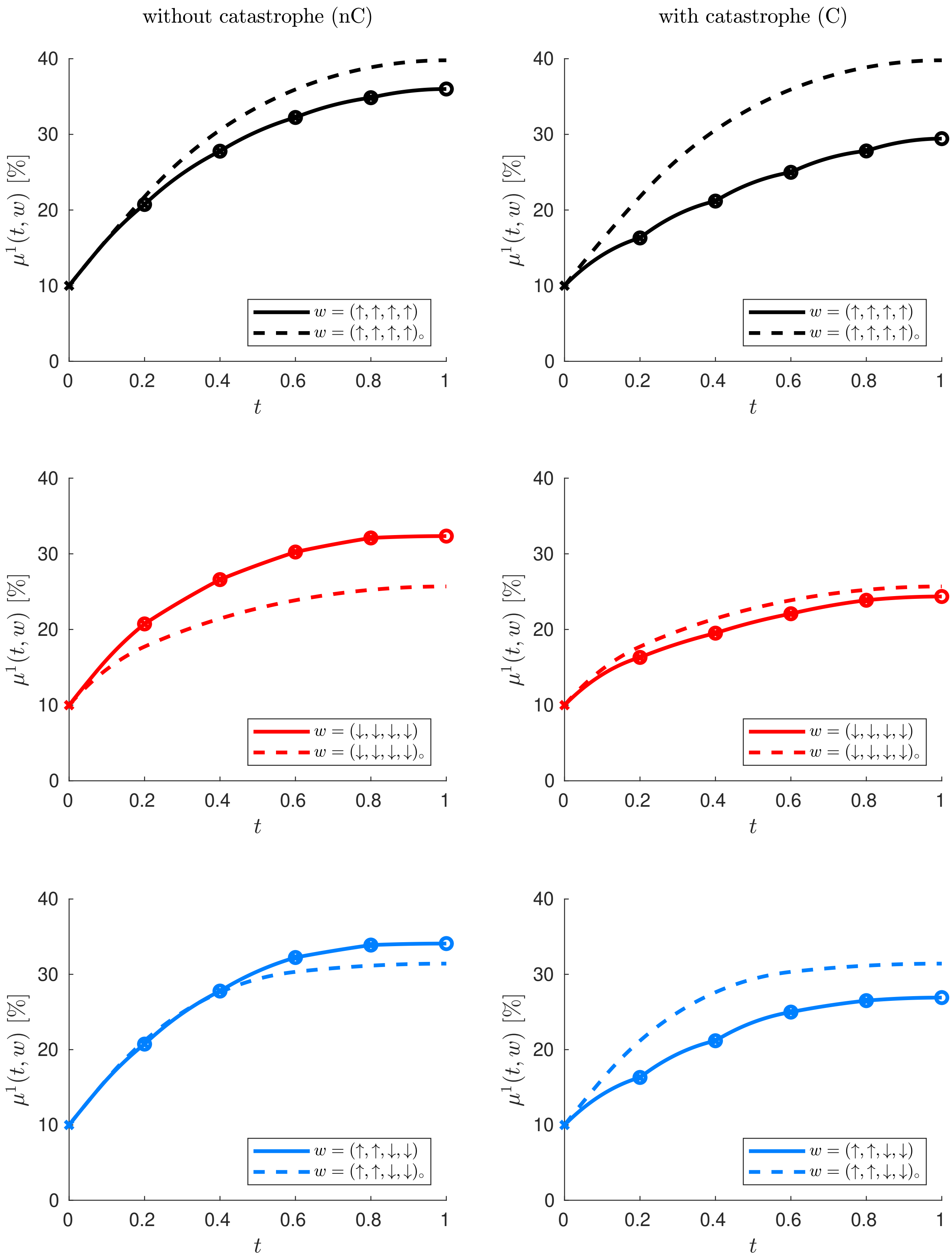

Figure 2: Proportion of farmers in models (nC) and (C).

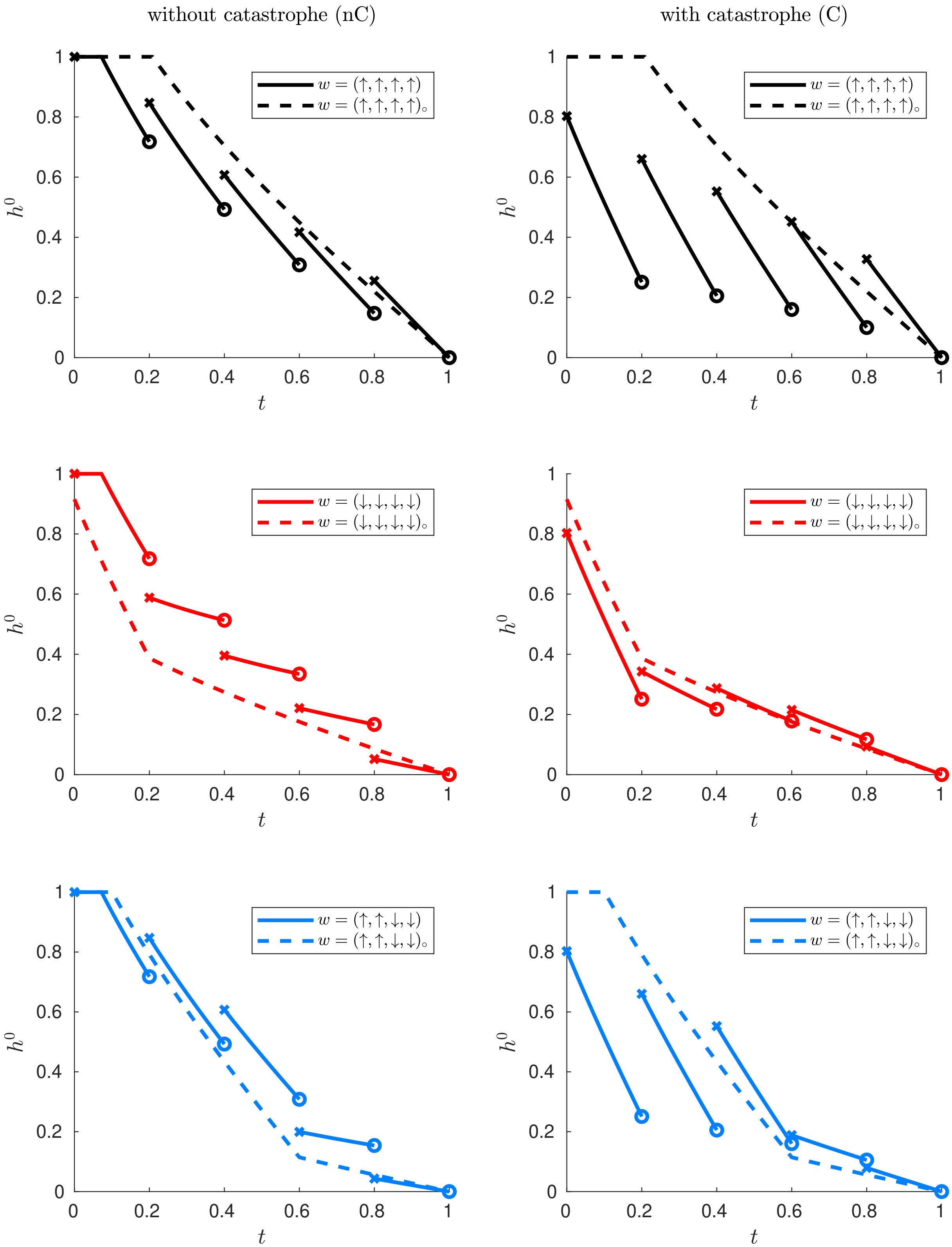

Figure 3: Optimal action h^0 of non-farmers in models (nC) and (C).

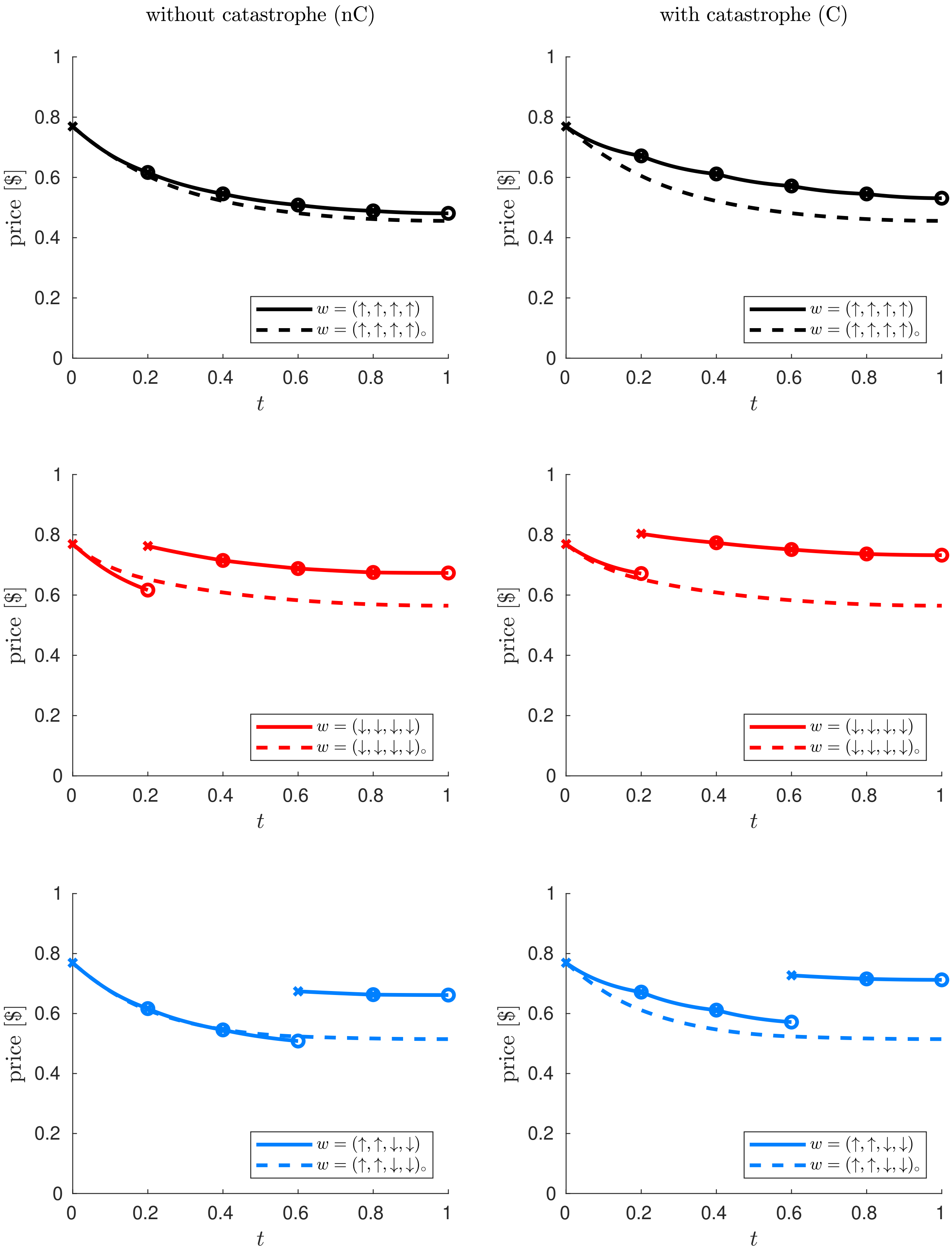

Figure 4: Equilibrium market prices in models (nC) and (C).

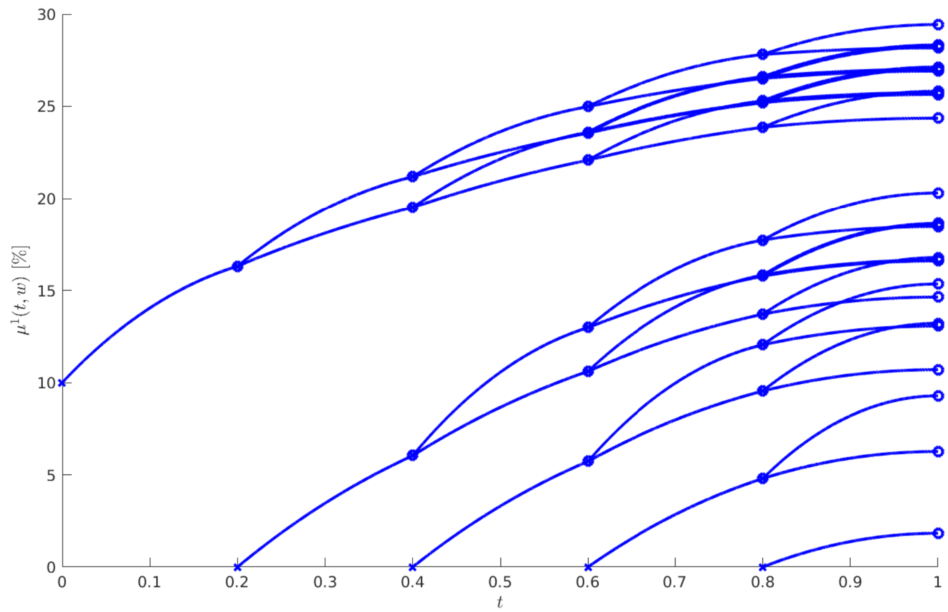

Figure 5: Proportion of farmers in model (C) for all possible common noise configurations $w \in \mathbb{W}^n$.

5.2 An SIR Model with Random One-Shot Vaccination

Our second application is a mean field game of agents that are confronted with the spread of an infectious disease. Our main focus is to illustrate the qualitative effects of common noise on the equilibrium behavior of the system. We consider a classical SIR model setup with $\mathbb{S} = \{\mathrm{S}, \mathrm{I}, \mathrm{R}\}$: Each agent can be either *susceptible* to infection (S), *infected* and simultaneously infectious for other agents (I), or *recovered* and thus immune to (re-)infection (R); see Figure 6.

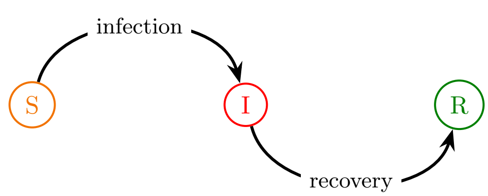

Figure 6: State space and transitions in the SIR model.

The infection rate is proportional to the prevalence of the disease, i.e. the percentage of currently infected agents. Susceptible agents can make individual efforts of size $u \in \mathbb{U} \triangleq [0, 1]$ to protect themselves against infection and thus reduce intensity of infection. The transition intensities are given by

$$Q(t, w, m, u) \triangleq \begin{bmatrix} -q_{\mathrm{inf}}(t, w, m, u) & q_{\mathrm{inf}}(t, w, m, u) & 0 \\ 0 & -q_{\mathrm{IR}} & q_{\mathrm{IR}} \\ 0 & 0 & 0 \end{bmatrix}$$

for $(t, w, m, u) \in [0, T] \times \mathbb{W}^n \times \mathbb{M} \times \mathbb{U}$, where $q_{\mathrm{IR}} \geq 0$ denotes the recovery rate of infected agents and the infection rate is given by

$$q_{\inf}(t,w,m,u) \triangleq q_{\mathrm{SI}} \cdot m^{\mathrm{I}} \cdot (1-u) \cdot \mathbb{1}_{\{t<\tau^\star\}}(w)$$

with a given maximum rate $q_{\mathrm{SI}} \geq 0$. The running reward penalizes both protection efforts and time spent in the infected state; with $c_{\mathrm{P}}, \psi_{\mathrm{I}} \geq 0$ we set

$$\psi^{\mathrm{S}}(t,w,m,u) \triangleq -c_{\mathrm{P}}\frac{u}{1-u}, \qquad \psi^{\mathrm{I}}(t,w,m,u) \triangleq -\psi_{\mathrm{I}}, \qquad \psi^{\mathrm{R}}(t,w,m,u) \triangleq 0.$$

In addition, we include the possibility of a one-shot vaccination that becomes available, simultaneously to all agents, at a random point of time $\tau^\star \in \{T_1,\dots,T_n\} \subset (0,T)$. We set $\mathbb{W} \triangleq \{0,1\}$ and identify the k^{th} unit vector $e_k = (\delta_{kj})_{j\in[1:n]} \in \mathbb{W}^n$, $k \in [1:n]$ with the indicator of the event $\{\tau^\star = T_k\}$. The event that no vaccine is available until T is represented by $0 \in \mathbb{W}^n$; we set $\tau^\star \triangleq +\infty$ in this case.[9] If and when it is available, all susceptible agents are vaccinated instantaneously, rendering them immune to infection; thus

$$J^{\mathrm{S}}(t,w,m) \triangleq \begin{cases} \mathrm{R} & \text{if } t \in \{T_1,\dots,T_n\},\, t = T_k = \tau^\star, \\ \mathrm{S} & \text{otherwise} \end{cases} \qquad \text{and} \qquad J^i(t,w,m) \triangleq i \quad \text{for } i \in \{\mathrm{I},\mathrm{R}\}.$$

The probability of vaccination becoming available is proportional to the percentage of agents that have already recovered from the disease. Thus for $k \in [1:n]$, $w_1,\dots,w_k \in \mathbb{W}$ and $m \in \mathbb{M}$ we set

$$\kappa_k(1\,|w_1,\dots,w_{k-1},m) \triangleq \begin{cases} \alpha \cdot m^{\mathrm{R}} & \text{if } w_1,\dots,w_{k-1} = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $\kappa_k(0\,|w_1,\dots,w_{k-1},m) \triangleq 1 - \kappa_k(1\,|w_1,\dots,w_{k-1},m)$ where $\alpha \in (0,1]$. As a consequence, for all $(i,t,w,m,v) \in \mathbb{S} \times [0,T] \times \mathbb{W} \times \mathbb{M} \times \mathbb{R}^3$, a maximizer as required in Assumption 4 is given by[10]

$$h^{\mathrm{S}}(t,w,m,v) \triangleq \begin{cases} \left[1 - \sqrt{\frac{c_{\mathrm{P}}}{q_{\mathrm{SI}}\cdot m^{\mathrm{I}}\cdot(v^{\mathrm{S}}-v^{\mathrm{I}})}}\right]^+ & \text{if } v^{\mathrm{S}} > v^{\mathrm{I}} \text{ and } m^{\mathrm{I}} > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $h^i(t,w,m,v) \triangleq 0$ for $i \in \{\mathrm{I},\mathrm{R}\}$.

Remark 13 (SIR Models in the Literature)**.** Note that, given the above specification of the transition matrix Q, the forward dynamics (E1) within the equilibrium system (E1)-(E6) read as follows:

$$\begin{cases} \dot\mu^{\mathrm{S}}(t,w) = -q_{\mathrm{SI}} \cdot \mu^{\mathrm{I}}(t,w) \cdot \big(1 - h^{\mathrm{S}}(t,w,\mu(t,w),v(t,w))\big) \cdot \mathbb{1}_{\{t<\tau^\star\}}(w) \cdot \mu^{\mathrm{S}}(t,w) \\ \dot\mu^{\mathrm{I}}(t,w) = \quad q_{\mathrm{SI}} \cdot \mu^{\mathrm{I}}(t,w) \cdot \big(1 - h^{\mathrm{S}}(t,w,\mu(t,w),v(t,w))\big) \cdot \mathbb{1}_{\{t<\tau^\star\}}(w) \cdot \mu^{\mathrm{S}}(t,w) - q_{\mathrm{IR}} \cdot \mu^{\mathrm{I}}(t,w) \\ \dot\mu^{\mathrm{R}}(t,w) = \qquad q_{\mathrm{IR}} \cdot \mu^{\mathrm{I}}(t,w). \end{cases}$$

Disregarding common noise, these constitute a ramification of the classical SIR dynamics, which are a basic building block of numerous compartmental epidemic models in the literature; see, among others, [Het00], [HLM14], [Mil17] or [GMS20] and the references therein. The SIR mean field game with controlled infection rates, albeit without common noise, has recently been studied in the independent article [EHT20]; we also refer to [LT15] and [DGG17] for mean field models with controlled vaccination rates. Mathematically similar contagion mechanisms also appear in, e.g., [KB16], [KM17], §7.2.3 in [CD18a], §7.1.10 in [CD18b],

[9] The specification of κ_k, $k \in [1:n]$, below implies that $\tau^\star = +\infty$ is equivalent to $w = 0 \in \mathbb{W}^n$ $\mathbb{P}$-a.s., i.e., the configurations $\mathbb{W}^n \setminus \big(\{0\} \cup \{e_k : k \in [1:n]\}\big)$ are $\mathbb{P}$-negligible.

[10] Note that for given $w \in \mathbb{W}^n$ the stated maximizer h^{S} is unique for times $t < \tau^\star$; otherwise its specification is immaterial. The latter applies likewise to h^{I} and h^{R}.

or §4.4 in [Wan19]. ◇

For our numerical results, the initial distribution of agents is given by $m_0 \triangleq (0.995, 0.005, 0.00)$, and the model coefficients are reported in Table 2. Note that there are $n = 1999$ common noise times $T_k = k \cdot 0.01$, $k = 1, \ldots, 1999$, at which a vaccine can be administered. The specifications of q_{SI} and q_{IR} imply a basic reproduction number $R_0 \triangleq q_{\mathrm{SI}}/q_{\mathrm{IR}} = 15$ in the absence of vaccination and protection efforts.

Parameter	T	n	T_k	α	q_{SI}	q_{IR}	c_{P}	ψ_{I}
Value	20	1999	$k \cdot 0.01$	0.1	7.5	0.5	25	100

Table 2: Coefficients in the SIR model.

Our results for the mean field equilibrium distributions of agents μ and the corresponding optimal protection efforts of susceptible agents h^{S} are displayed in Figures 7 through 9 for different common noises configurations, i.e. vaccination times $\tau^\star$. As in Section 5.1, we also display the corresponding (theoretical) perfect-foresight equilibria, marked by the subscript $\circ$.

Note that an agent's running reward is the same in state S with zero protection effort and in state R; agents are penalized relative to these in state I and hence aim to avoid that state. Susceptible agents can reach the state R of immunity by two ways: First, they can become infected and overcome the disease; second, they can be vaccinated and jump instantly from state S to state R. While the first alternative is painful, the second comes at no cost and is hence clearly preferable. However, as the availability of a vaccine cannot be directly controlled by the agents, they can only protect themselves against infection at a certain running cost until the vaccine becomes available.

Figures 7 to 9 demonstrate that the possibility of vaccination as a common noise event can dampen the spread of the disease and lower the peak infection rate. This is due to an increase in agents' protection efforts *during the time period when the proportion of infected agents is high.* By contrast, in the perfect-foresight equilibria where the vaccination date is known, agents do not make substantial protection efforts until the vaccination date is imminent, see Figures 8 and 9; in the scenario without vaccination, see Figure 7, protection efforts are only ever made by a very small fraction of the population. With perfect foresight, the agents' main rationale is to avoid being in state I when the vaccine becomes available. This highlights the importance of being able to model the vaccination date as a (random) common noise event. Finally, observe that our numerical results indicate convergence to the stationary distribution $\bar{\mu} = (0, 0, 1) \in \mathbb{M}$, showing that the model is able to capture the entire evolution of an epidemic.

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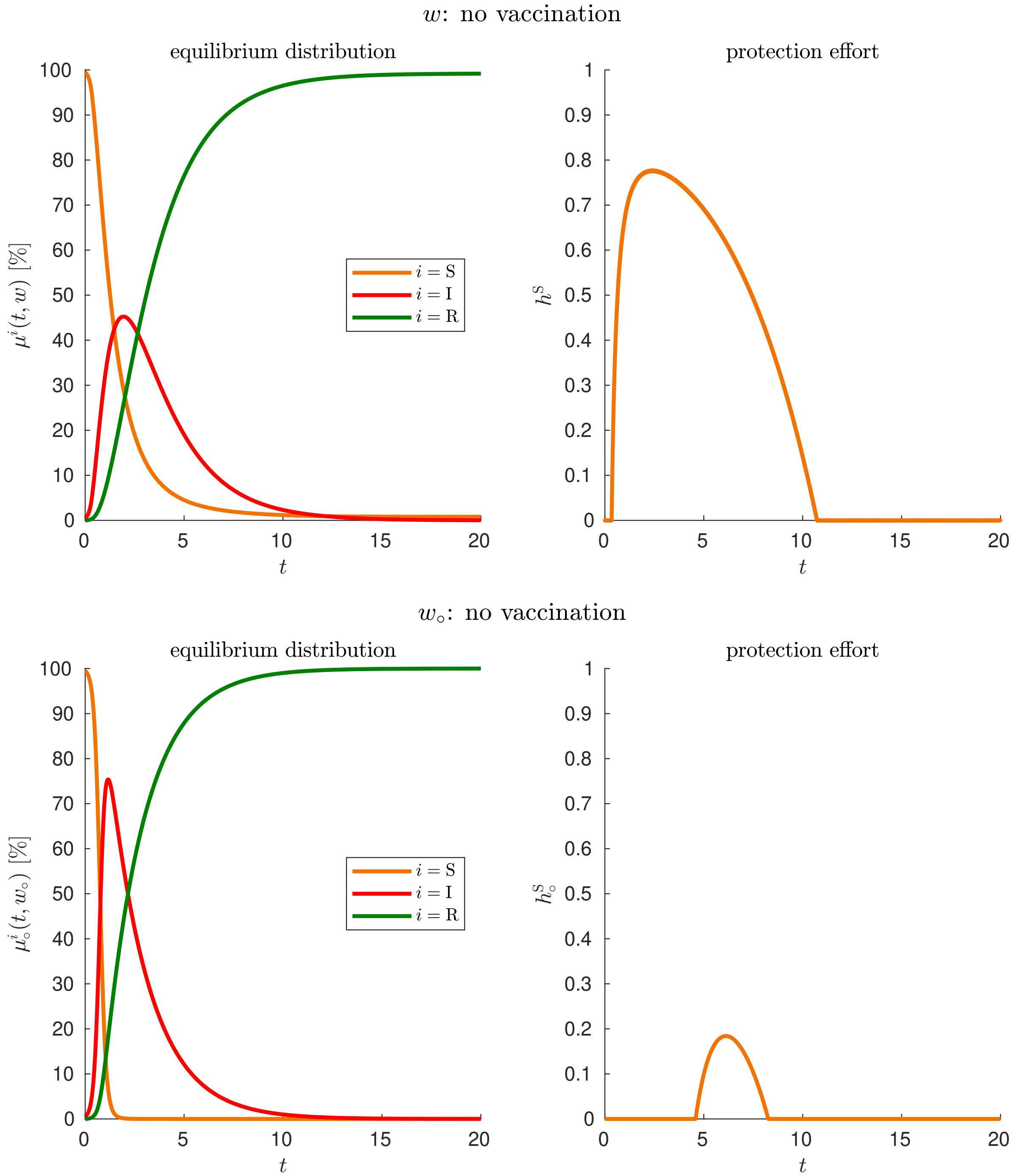

Figure 7: Equilibrium distribution and protection effort for $\tau^\star = +\infty$: Mean field game with common noise (top) and corresponding perfect-foresight equilibrium (bottom).

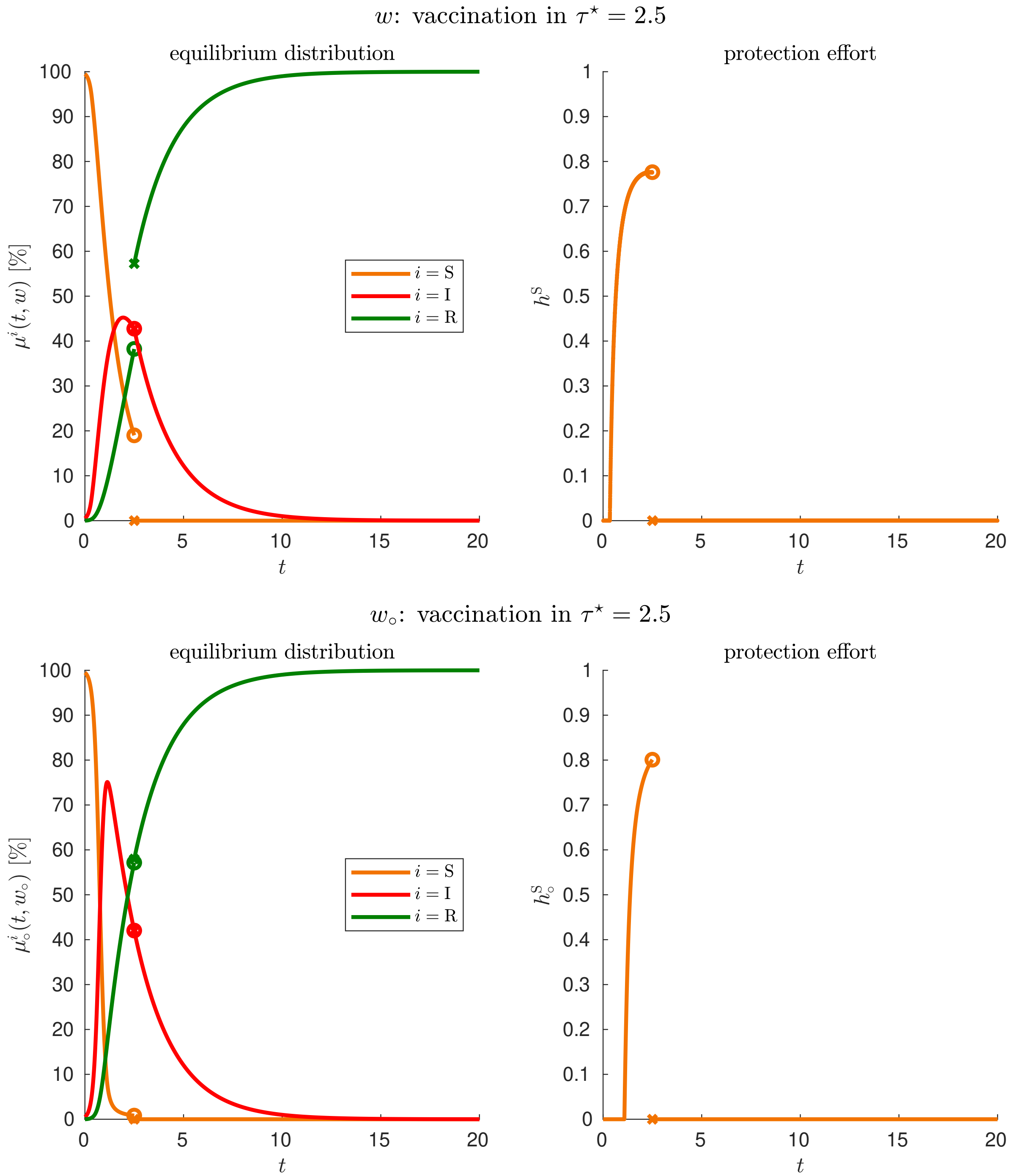

Figure 8: Equilibrium distribution and protection effort for $\tau^\star = 2.5$: Mean field game with common noise (top) and corresponding perfect-foresight equilibrium (bottom).

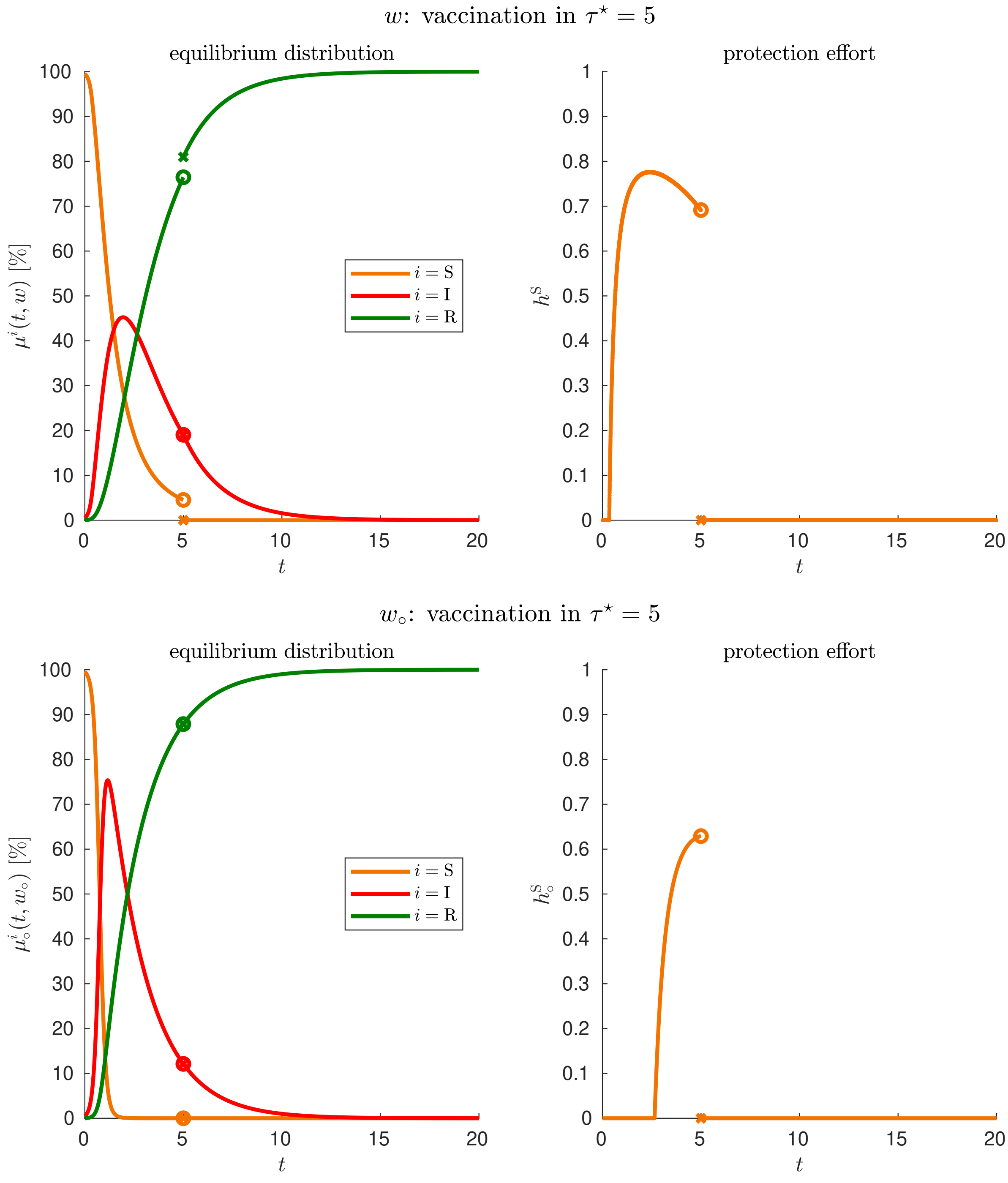

Figure 9: Equilibrium distribution and protection effort for $\tau^\star = 5$: Mean field game with common noise (top) and corresponding perfect-foresight equilibrium (bottom).

A Appendix: Proof of Theorem 12

Throughout this appendix, we fix a time horizon $T > 0$ and a number of common noise events $n \in \mathbb{N}_0$. We set $\Pi \triangleq \{\pi = [T_0, T_1, \ldots, T_n, T_{n+1}] : 0 = T_0 < T_1 < \cdots < T_n < T_{n+1} = T\}$.

Assumption A.1.

(i) The terminal reward function Ψ is Lipschitz with respect to m, i.e. there exists $L_\Psi \geq 0$ with

$$\|\Psi(w, m_1) - \Psi(w, m_2)\| \leq L_\Psi \cdot \|m_1 - m_2\| \tag{L_Ψ}$$

for all $w \in \mathbb{W}^n$ and $m_1, m_2 \in \mathbb{M}$.

(ii) The reduced-form running reward function $\widehat{\psi}$ is jointly Lipschitz with respect to (m, v), i.e.

$$\|\widehat{\psi}(t, w, m_1, v_1) - \widehat{\psi}(t, w, m_2, v_2)\| \leq L_{\widehat{\psi}} \cdot \big(\|m_1 - m_2\| + \|v_1 - v_2\|\big) \tag{$\mathrm{L}_{\widehat{\psi}}$}$$

for all $t \in [0, T]$, $w \in \mathbb{W}^n$, $m_1, m_2 \in \mathbb{M}$ and $v_1, v_2 \in \mathbb{R}^d$, where $L_{\widehat{\psi}} \geq 0$.

(iii) The reduced-form intensity matrix function $\widehat{Q}$ is jointly Lipschitz with respect to (m, v), i.e.

$$\big\|\widehat{Q}(t, w, m_1, v_1) - \widehat{Q}(t, w, m_2, v_2)\big\| \leq L_{\widehat{Q}} \cdot \big(\|m_1 - m_2\| + \|v_1 - v_2\|\big) \tag{$\mathrm{L}_{\widehat{Q}}$}$$

for all $t \in [0, T]$, $w \in \mathbb{W}^n$, $m_1, m_2 \in \mathbb{M}$ and $v_1, v_2 \in \mathbb{R}^d$, where $L_{\widehat{Q}} \geq 0$.

(iv) For each $k \in [1 : n]$ the transition kernels κ_k satisfy the Lipschitz condition

$$\begin{aligned}\Big| \sum_{\bar{w}_k \in \mathbb{W}} & \kappa_k\big(\bar{w}_k | w_1, \ldots, w_{k-1}, m_1\big) v^{J^i(T_k, (w_{-k}, \bar{w}_k), m_1)} \\ & - \kappa_k\big(\bar{w}_k | w_1, \ldots, w_{k-1}, m_2\big) v^{J^i(T_k, (w_{-k}, \bar{w}_k), m_2)} \Big| \leq L_\kappa \cdot \|m_1 - m_2\|\end{aligned} \tag{L_κ}$$

for all $i \in \mathbb{S}$, $w \in \mathbb{W}^n$, $m_1, m_2 \in \mathbb{M}$ and $v \in \mathbb{R}^d$ with $\|v\| \leq v_{\max}$, where $L_\kappa \geq 0$ and

$$v_{\max} \triangleq \big(\Psi_{\max} + T \cdot \psi_{\max}\big) \cdot \mathrm{e}^{Q_{\max} \cdot T}. \tag{$\mathrm{v}_{\max}$}$$

(v) For each $k \in [1 : n]$ and $w \in \mathbb{W}^n$ the map $\Phi_k(w, \cdot\,)$ is Lipschitz, i.e. there exists $L_\Phi \geq 0$ with

$$\|\Phi_k(w, m_1) - \Phi_k(w, m_2)\| \leq L_\Phi \cdot \|m_1 - m_2\| \tag{L_Φ}$$

for all $w \in \mathbb{W}^n$ and $m_1, m_2 \in \mathbb{M}$. ◇

Since all norms on $\mathbb{R}^d$ are equivalent, the concrete specification is immaterial for Assumption A.1. However, the following results partially depend on the sizes of the relevant constants; to be concrete, in the following we use the maximum norm on $\mathbb{R}^d$ and a compatible matrix norm on $\mathbb{R}^{d \times d}$; moreover, we suppose that ($\mathrm{L}_{\widehat{Q}}$) holds for both $\widehat{Q}$ and $\widehat{Q}^\top$. The constants in ($\mathrm{v}_{\max}$) are given by

$$\begin{aligned} Q_{\max} &\triangleq \sup_{\substack{t \in [0,T],\, w \in \mathbb{W}^n, \\ m \in \mathbb{M},\, v \in \mathbb{R}^d}} \big\|\widehat{Q}(t, w, m, v)\big\|, \\ \psi_{\max} &\triangleq \sup_{\substack{t \in [0,T],\, w \in \mathbb{W}^n, \\ m \in \mathbb{M},\, v \in \mathbb{R}^d}} \big\|\widehat{\psi}(t, w, m, v)\big\| \quad \text{and} \quad \Psi_{\max} \triangleq \sup_{\substack{m \in \mathbb{M}, \\ w \in \mathbb{W}^n}} \|\Psi(w, m)\|. \end{aligned} \tag{23}$$

Remark A.2. Sufficient conditions for Assumption A.1(i)-(iii) in terms of the model's primitives can be found in, e.g., [GMS13] or [CF20]. Furthermore, in the special case where the jump map J is independent of $m \in \mathbb{M}$, Assumption A.1(v) is trivially satisfied, and Assumption A.1(iv) is equivalent to

$$\mathbb{M} \ni m \mapsto \sum_{\bar{w}_k \in \,\cdot} \kappa_k(\bar{w}_k | w_1, \ldots, w_{k-1}, m) \in \mathsf{Prob}(\mathbb{W})$$

being Lipschitz continuous in total variation norm. $\diamond$

Let $E \subseteq \mathbb{R}^d$ and define the space

$$\mathsf{D}(E) \triangleq \big\{ f: \ [0,T] \times \mathbb{W}^n \to E : \ f \text{ is càdlàg and non-anticipative} \big\}$$

together with the norm $\|f\|_{\sup} \triangleq \sup_{t \in [0,T],\, w \in \mathbb{W}^n} \|f(t,w)\|$ for $f \in \mathsf{D}(E)$. It is clear that $\mathsf{D}(E)$ is a Banach space provided $E \subseteq \mathbb{R}^d$ is closed; the linear subspace of regular non-anticipative functions is denoted by $\mathsf{Reg}(E)$.

Lemma A.3 (Forward/backward Gronwall estimates)**.** *Let $f \in \mathsf{D}([0,\infty))$ and $\delta \geq 0$.*

(a) *Let $\vec{\alpha}, \vec{\beta} \geq 0$ and $\vec{\gamma} \geq 1$, and suppose that $f(0) = 0$,*

$$f(t,w) \leq f(T_k, w) + \vec{\alpha}(t - T_k) \cdot \delta + \int_{T_k}^t \vec{\beta} \cdot f(s,w) \mathrm{d}s \quad \text{for } t \in [T_k, T_{k+1}\rangle,\ w \in \mathbb{W}^n, \tag{24}$$

for $k \in [0:n]$, and

$$f(T_k, w) \leq \vec{\gamma} \cdot f(T_k-, w) \quad \text{for all } w \in \mathbb{W}^n, \tag{25}$$

for $k \in [1:n]$. Then we have

$$f(t,w) \leq \vec{C} \cdot \delta \quad \text{for all } (t,w) \in [0,T] \times \mathbb{W}^n, \qquad \text{where} \qquad \vec{C} \triangleq \vec{\alpha} \cdot \big(\vec{\gamma}\big)^n \cdot \mathrm{e}^{\vec{\beta} T} \cdot T.$$

(b) *Let $\overleftarrow{\rho}, \overleftarrow{\alpha}, \overleftarrow{\beta}, \overleftarrow{\varepsilon} \geq 0$ and suppose that $f(T,w) \leq \overleftarrow{\rho} \cdot \delta$ for all $w \in \mathbb{W}^n$,*

$$f(t,w) \leq f(T_{k+1}-, w) + \overleftarrow{\alpha}(T_{k+1} - t) \cdot \delta + \int_t^{T_{k+1}} \overleftarrow{\beta} \cdot f(s,w) \mathrm{d}s, \quad t \in [T_k, T_{k+1}\rangle,\ w \in \mathbb{W}^n, \tag{26}$$

for $k \in [0:n]$, and

$$f(T_k-, w) \leq \sum_{\bar{w}_k \in \mathbb{W}} \overleftarrow{\gamma}_k(w_{-k}, \bar{w}_k) \cdot f(T_k, (w_{-k}, \bar{w}_k)) + \overleftarrow{\varepsilon} \cdot \delta, \quad w \in \mathbb{W}^n, \tag{27}$$

for $k \in [1:n]$, where for all $w_1, \ldots, w_{k-1} \in \mathbb{W}$ the family $\{\overleftarrow{\gamma}_k(w_{-k}, \bar{w}_k)\}_{\bar{w}_k \in \mathbb{W}}$ consists of probability weights on $\mathbb{W}$. Then we have

$$f(t,w) \leq \overleftarrow{C} \cdot \delta \quad \text{for all } (t,w) \in [0,T] \times \mathbb{W}^n, \qquad \text{where} \qquad \overleftarrow{C} \triangleq \mathrm{e}^{\overleftarrow{\beta} T} \cdot \overleftarrow{\rho} + \big(\overleftarrow{\alpha} T + n \overleftarrow{\varepsilon}\big) \cdot \mathrm{e}^{\overleftarrow{\beta} T}.$$

Importantly, the constants $\vec{C}$ and $\overleftarrow{C}$ do not depend on $\pi \in \Pi$. In particular, we have

$$C \triangleq \vec{C} \cdot \overleftarrow{C} < 1 \quad \text{whenever } T > 0 \text{ is sufficiently small.} \tag{28}$$

Proof. We only prove part (a); part (b) follows by similar arguments upon time reversal. We recursively define $\vec{C}_0 \triangleq 0$ and

$$\vec{C}_{k+1} \triangleq \vec{\gamma} \cdot (\vec{C}_k + \vec{\alpha}(T_{k+1} - T_k)) \cdot \mathrm{e}^{\vec{\beta}(T_{k+1} - T_k)} \quad \text{for } k \in [0:n].$$

Note that the sequence $\{\vec{C}_k\}_{k\in[0:(n+1)]}$ is non-decreasing and $\vec{C}_{n+1} \leq \vec{C}$. Hence it suffices to show that

$$f(t,w) \leq \vec{C}_{k+1} \cdot \delta \quad \text{for all } (t,w) \in [T_k, T_{k+1}] \times \mathbb{W}^n,\ k \in [0:n]. \tag{29}$$

It is clear that $f(T_0) = f(0) = 0$. Next let $k \in [0:(n-1)]$ and

$$f(T_k, w) \leq \vec{C}_k \cdot \delta \quad \text{for all } w \in \mathbb{W}^n.$$

It follows from (24) and Gronwall's inequality that for $(t,w) \in [T_k, T_{k+1}\rangle \times \mathbb{W}^n$

$$f(t,w) \leq \big(f(T_k,w) + \vec{\alpha}(t-T_k)\cdot\delta\big)\cdot \mathrm{e}^{\vec{\beta}(t-T_k)} \leq \big(\vec{C}_k + \vec{\alpha}(T_{k+1}-T_k)\big)\cdot \mathrm{e}^{\vec{\beta}(T_{k+1}-T_k)}\cdot\delta. \tag{30}$$

In particular, (25) yields

$$f(T_{k+1},w) \leq \vec{\gamma}\cdot f(T_{k+1}-,w) \leq \vec{\gamma}\cdot\big(\vec{C}_k + \vec{\alpha}(T_{k+1}-T_k)\big)\cdot \mathrm{e}^{\vec{\beta}(T_{k+1}-T_k)}\cdot\delta = \vec{C}_{k+1}\cdot\delta,$$

and consequently

$$f(t,w) \leq \vec{C}_{k+1} \cdot \delta \quad \text{for all } (t,w) \in [T_k, T_{k+1}] \times \mathbb{W}^n. \qquad \square$$

In the following, we first consider the backward system (E2), (E4), (E6) and subsequently the forward system (E1), (E3), (E5).

Lemma A.4. *Suppose that Assumption A.1 holds and let $\mu \in \mathsf{D}(\mathbb{M})$. Then there exists a unique solution $\bar{v}$ of* (E2) *subject to* (E4) *and* (E6). *Moreover, we have $\bar{v} \in \mathsf{Reg}(\mathbb{R}^d)$ and $\|\bar{v}(t,w)\| \leq v_{\max}$ for all $(t,w) \in [0,T]\times\mathbb{W}^n$ where $v_{\max}$ is given by* ($\mathrm{v}_{\max}$).

Proof. Step 1: Construction of $\bar{v}$. We construct $\bar{v}$ by backward induction on $k \in [0:n]$ on each segment $[T_k, T_{k+1}\rangle \times \mathbb{W}^n$. First, we set $\bar{v}(T,w) \triangleq \Psi(w, \mu(T,w))$ for $w \in \mathbb{W}^n$. Suppose that $k \in [0:n]$, fix $w \in \mathbb{W}^n$, and let $\tilde{v}(T_{k+1}, w_{T_k}) \in \mathbb{R}^d$ be given and independent of $w_{k+1},\dots,w_n$. Using ($\mathrm{L}_{\widehat{\psi}}$) and ($\mathrm{L}_{\widehat{Q}}$) it follows that the Carathéodory conditions are satisfied, so [Hal80, Theorem I.5.3] yields the unique Carathéodory solution $\tilde{v}(\,\cdot\,, w_{T_k})\colon [T_k, T_{k+1}] \to \mathbb{R}^d$ of

$$\begin{aligned}
\tilde{v}(t, w_{T_k}) &= \tilde{v}(T_{k+1}, w_{T_k}) + \int_t^{T_{k+1}} \Big(\widehat{\psi}\big(s, w_{T_k}, \mu(s, w_{T_k}), \tilde{v}(s, w_{T_k})\big) \\
&\qquad\qquad + \widehat{Q}\big(s, w_{T_k}, \mu(s, w_{T_k}), \tilde{v}(s, w_{T_k})\big)\cdot \tilde{v}(s, w_{T_k})\Big)\mathrm{d}s \\
&= \tilde{v}(T_{k+1}, w_{T_k}) + \int_t^{T_{k+1}} \Big(\widehat{\psi}\big(s, w, \mu(s,w), \tilde{v}(s, w_{T_k})\big) \\
&\qquad\qquad + \widehat{Q}\big(s, w, \mu(s,w), \tilde{v}(s, w_{T_k})\big)\cdot \tilde{v}(s, w_{T_k})\Big)\mathrm{d}s, \quad t \in [T_k, T_{k+1}],
\end{aligned}$$

where the final identity is due to the fact that $\widehat{\psi}(\,\cdot\,,\,\cdot\,,\bar{m},\bar{v})$ and $\widehat{Q}(\,\cdot\,,\,\cdot\,,\bar{m},\bar{v})$ are non-anticipative. Define

$$\bar{v}(t,w) \triangleq \tilde{v}(t, w_{T_k}) \quad \text{for } t \in [T_k, T_{k+1}\rangle \quad \text{and each } w \in \mathbb{W}^n.$$

By construction, $\bar{v}(\,\cdot\,, w)$ solves (E2) on $[T_k, T_{k+1}\rangle$ and does not depend on $w_{k+1},\dots,w_n$. Having constructed $\bar{v}$ on $[T_k, T_{k+1}\rangle \times \mathbb{W}^n$, we use (E4) and define

$$\bar{v}(T_k, w_{T_{k-1}}) \triangleq \Psi_k(w, \mu(T_k-, w), \bar{v}(T_k, \,\cdot\,))$$

for $w \in \mathbb{W}^n$. By (Ψ_k) and the fact that μ and J are non-anticipative, it follows that this definition does not depend on $w_k,\dots,w_n$. Consequently, the above construction can be iterated, and hence we obtain $\bar{v}$

as the unique solution of (E2) subject to (E4) and (E6). By definition, $\bar{v}$ is non-anticipative and regular, i.e. $\bar{v} \in \mathsf{Reg}(\mathbb{R}^d)$.

Step 2: A priori bound. For $k \in [0:n]$ and $(t,w) \in [T_k, T_{k+1}\rangle \times \mathbb{W}^n$ we have

$$\begin{aligned}\|\bar{v}(t,w)\| &\leq \|\bar{v}(T_{k+1}-,w)\| + \int_t^{T_{k+1}} \big\|\widehat{\psi}(s,w,\mu(s,w),\bar{v}(s,w))\big\| + \big\|\widehat{Q}(s,w,\mu(s,w),\bar{v}(s,w))\big\| \cdot \|\bar{v}(s,w)\|\mathrm{d}s \\ &\leq \|\bar{v}(T_{k+1}-,w)\| + \psi_{\max} \cdot (T_{k+1}-t) + Q_{\max} \cdot \int_t^{T_{k+1}} \|\bar{v}(s,w)\|\mathrm{d}s. \end{aligned} \tag{31}$$

On the other hand, for $k \in [1:n]$, $w \in \mathbb{W}^n$ and $i \in \mathbb{S}$ we observe from (Ψ_k) that

$$\|\bar{v}(T_k-,w)\| \leq \sum_{\bar{w}_k \in \mathbb{W}} \kappa_k(\bar{w}_k | w_1, \ldots, w_{k-1}, \mu(T_k-, w_{T_k-})) \cdot \|\bar{v}(T_k, (w_{-k}, \bar{w}_k))\|. \tag{32}$$

Since $\|\bar{v}(T,w)\| = \|\Psi(w,\mu(T,w))\| \leq \Psi_{\max}$ it follows from (31), (32) and Lemma A.3(b) with $\delta \triangleq \psi_{\max}$ and $\overleftarrow{\rho} \triangleq \Psi_{\max}/\psi_{\max}$ that

$$\|\bar{v}(t,w)\| \leq \overleftarrow{C} \cdot \delta \leq \big(\Psi_{\max} + T \cdot \psi_{\max}\big) \cdot \mathrm{e}^{Q_{\max} \cdot T} = v_{\max} \quad \text{for all } (t,w) \in [0,T] \times \mathbb{W}^n. \qquad \square$$

Lemma A.5. *Suppose that Assumption A.1 is satisfied and let $v \in \mathsf{D}(\mathbb{R}^d)$. Then there is a unique solution $\bar{\mu}$ of* (E1) *subject to* (E3) *and* (E5)*, and we have $\bar{\mu} \in \mathsf{Reg}(\mathbb{M})$.*

Proof. The proof is analogous to (but somewhat simpler than) that of Lemma A.4. $\square$

Proof of Theorem 12. Let $T > 0$ be as in (28) below Lemma A.3 with the relevant coefficients given by

$$\begin{aligned} &\quad & \vec{\alpha} &= L_{\widehat{Q}}, & \vec{\beta} &= Q_{\max} + L_{\widehat{Q}}, & \vec{\gamma} &= L_\Phi + 1, \\ \overleftarrow{\rho} &= L_\Psi, & \overleftarrow{\alpha} &= L_{\widehat{\psi}} + L_{\widehat{Q}} \cdot v_{\max}, & \overleftarrow{\beta} &= L_{\widehat{\psi}} + Q_{\max} + L_{\widehat{Q}} \cdot v_{\max}, & \overleftarrow{\varepsilon} &= L_\kappa, \end{aligned}$$

and fix $\pi \in \Pi$.

Step 1: Solution operators. We define

$$\overleftarrow{\chi} : \ \mathsf{D}(\mathbb{M}) \to \mathsf{Reg}(\mathbb{R}^d), \quad \overleftarrow{\chi}[\mu] \triangleq \bar{v},$$

where $\bar{v} \in \mathsf{Reg}(\mathbb{R}^d)$ is the unique solution of (E2) subject to (E4) and (E6) given $\mu \in \mathsf{D}(\mathbb{M})$; $\overleftarrow{\chi}$ is well-defined by Lemma A.4. Moreover, let

$$\vec{\chi} : \ \mathsf{D}(\mathbb{R}^d) \to \mathsf{Reg}(\mathbb{M}), \quad \vec{\chi}[v] \triangleq \bar{\mu},$$

where $\bar{\mu} \in \mathsf{Reg}(\mathbb{M})$ is the unique solution of (E1) subject to (E3) and (E5) given $v \in \mathsf{D}(\mathbb{R}^d)$; $\vec{\chi}$ is well-defined by Lemma A.5.

Step 2: Lipschitz continuity of $\overleftarrow{\chi}$. Let $\mu_1, \mu_2 \in \mathsf{D}(\mathbb{M})$ and set $\bar{v}_1 \triangleq \overleftarrow{\chi}[\mu_1]$ and $\bar{v}_2 \triangleq \overleftarrow{\chi}[\mu_2]$. Using $(\mathrm{L}_{\widehat{\psi}})$ and $(\mathrm{L}_{\widehat{Q}})$ it follows that for all $k \in [0:n]$ and $(t,w) \in [T_k, T_{k+1}\rangle \times \mathbb{W}^n$ we have

$$\begin{aligned} &\big\|\bar{v}_1(t,w) - \bar{v}_2(t,w)\big\| \leq \big\|\bar{v}_1(T_{k+1}-,w) - \bar{v}_2(T_{k+1}-,w)\big\| \\ &\quad + \int_t^{T_{k+1}} \big\|\widehat{\psi}\big(s,w,\mu_1(s,w),\bar{v}_1(s,w)\big) - \widehat{\psi}\big(s,w,\mu_2(s,w),\bar{v}_2(s,w)\big)\big\|\mathrm{d}s \\ &\quad + \int_t^{T_{k+1}} \big\|\widehat{Q}\big(s,w,\mu_1(s,w),\bar{v}_1(s,w)\big) \cdot \bar{v}_1(s,w) - \widehat{Q}\big(s,w,\mu_2(s,w),\bar{v}_2(s,w)\big) \cdot \bar{v}_2(s,w)\big\|\mathrm{d}s \\ &\leq \big\|\bar{v}_1(T_{k+1}-,w) - \bar{v}_2(T_{k+1}-,w)\big\| + \big(L_{\widehat{\psi}} + L_{\widehat{Q}} \cdot v_{\max}\big) \cdot (T_{k+1}-t) \cdot \|\mu_1 - \mu_2\|_{\sup} \end{aligned}$$

$$+\big(L_{\widehat{\psi}}+Q_{\max}+L_{\widehat{Q}}\cdot v_{\max}\big)\cdot\int_t^{T_{k+1}}\big\|\bar v_1(s,w)-\bar v_2(s,w)\big\|\mathrm{d}s. \tag{33}$$

On the other hand for $k\in[1:n]$ we obtain from (Ψ_k) and (L_κ) that for every $w\in\mathbb{W}^n$

$$\begin{aligned}&\big\|\bar v_1(T_k-,w)-\bar v_2(T_k-,w)\big\|\\&\le\sum_{\bar w_k\in\mathbb{W}}\kappa_k\big(\bar w_k|w_1,\dots,w_{k-1},\mu_1(T_k-,w_{T_k-})\big)\cdot\big\|\bar v_1(T_k,(w_{-k},\bar w_k))-\bar v_2(T_k,(w_{-k},\bar w_k))\big\|\\&\qquad+L_\kappa\cdot\|\mu_1-\mu_2\|_{\sup}.\end{aligned}\tag{34}$$

Finally, for $w\in\mathbb{W}^n$ by (L_Ψ) we have

$$\|\bar v_1(T,w)-\bar v_2(T,w)\|=\|\Psi(w,\mu_1(T,w))-\Psi(w,\mu_2(T,w))\|\le L_\Psi\cdot\|\mu_1-\mu_2\|_{\sup}. \tag{35}$$

In view of (33), (34) and (35) it follows from Lemma A.3(b) with $\delta\triangleq\|\mu_1-\mu_2\|_{\sup}$ that

$$\|\bar v_1(t,w)-\bar v_2(t,w)\|\le\overleftarrow{C}\cdot\|\mu_1-\mu_2\|_{\sup}\quad\text{for all }(t,w)\in[0,T]\times\mathbb{W}^n,$$

and thus

$$\big\|\overleftarrow{\chi}[\mu_1]-\overleftarrow{\chi}[\mu_2]\big\|_{\sup}=\|\bar v_1-\bar v_2\|_{\sup}\le\overleftarrow{C}\cdot\|\mu_1-\mu_2\|_{\sup}. \tag{36}$$

Step 3: Lipschitz continuity of $\vec{\chi}$. Let $v_1,v_2\in\mathsf{D}(\mathbb{R}^d)$ and set $\bar\mu_1\triangleq\vec{\chi}[v_1]$ and $\bar\mu_2\triangleq\vec{\chi}[v_2]$. By $(\mathrm{L}_{\widehat{Q}})$ we have for $k\in[0:n]$ and $(t,w)\in[T_k,T_{k+1}\rangle\times\mathbb{W}^n$

$$\begin{aligned}&\big\|\bar\mu_1(t,w)-\bar\mu_2(t,w)\big\|\le\big\|\bar\mu_1(T_k,w)-\bar\mu_2(T_k,w)\big\|\\&\qquad+\int_{T_k}^t\big\|\bar\mu_1(s,w)\cdot\widehat{Q}(s,w,\bar\mu_1(s,w),v_1(s,w))-\bar\mu_2(s,w)\cdot\widehat{Q}(s,w,\bar\mu_2(s,w),v_2(s,w))\big\|\mathrm{d}s\\&\le\|\bar\mu_1(T_k,w)-\bar\mu_2(T_k,w)\|+L_{\widehat{Q}}\cdot(t-T_k)\cdot\|v_1-v_2\|_{\sup}\\&\qquad+(Q_{\max}+L_{\widehat{Q}})\cdot\int_{T_k}^t\big\|\bar\mu_1(s,w)-\bar\mu_2(s,w)\big\|\mathrm{d}s.\end{aligned}\tag{37}$$

On the other hand, by (L_Φ) we have for $k\in[1:n]$ and $w\in\mathbb{W}^n$

$$\begin{aligned}\|\bar\mu_1(T_k,w)-\bar\mu_2(T_k,w)\|&=\|\Phi_k(w,\bar\mu_1(T_k-,w))-\Phi_k(w,\bar\mu_2(T_k-,w))\|\\&\le L_\Phi\cdot\|\bar\mu_1(T_k-,w)-\bar\mu_2(T_k-,w)\|.\end{aligned}\tag{38}$$

Since $\bar\mu_1(0)=\bar\mu_2(0)=m_0$, it follows from (37), (38) and Lemma A.3(a) with $\delta\triangleq\|v_1-v_2\|_{\sup}$ that

$$\|\bar\mu_1(t,w)-\bar\mu_2(t,w)\|\le\vec{C}\cdot\|v_1-v_2\|_{\sup}\quad\text{for all }(t,w)\in[0,T]\times\mathbb{W}^n,$$

and consequently

$$\big\|\vec{\chi}[v_1]-\vec{\chi}[v_2]\big\|_{\sup}=\|\bar\mu_1-\bar\mu_2\|_{\sup}\le\vec{C}\cdot\|v_1-v_2\|_{\sup}. \tag{39}$$

Step 4: Construction of the fixed point. Let $\chi:\ \mathsf{D}(\mathbb{M})\to\mathsf{Reg}(\mathbb{M})$, $\chi\triangleq\vec{\chi}\circ\overleftarrow{\chi}$ and note that by (36) and (39) we have

$$\|\chi[\mu_1]-\chi[\mu_2]\|_{\sup}\le C\cdot\|\mu_1-\mu_2\|_{\mathsf{op}}\quad\text{for all }\mu_1,\mu_2\in\mathsf{D}(\mathbb{M}),$$

where $C=\vec{C}\cdot\overleftarrow{C}<1$ by (28). Thus Banach's fixed point theorem yields a unique fixed point $\mu\in\mathsf{D}(\mathbb{M})$. Finally, setting $v\triangleq\overleftarrow{\chi}[\mu]\in\mathsf{Reg}(\mathbb{R}^d)$ it follows that $\mu=\vec{\chi}[v]\in\mathsf{Reg}(\mathbb{M})$ and that (μ,v) is a solution of (E1)-(E6). □

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