Working Paper Series Nº 22 - 01:

Projections of the Stochastic Discount Factor with Applications to Bubbles

Artem Dyachenko
Projections of the Stochastic Discount Factor with Applications to Bubbles

Artem Dyachenko *

January 12, 2022

Abstract

I derive projections of the stochastic discount factor (SDF) in the continuous-time stochastic volatility model by solving a system of partial differential equations. SDF projections are mixtures of exponentially affine functions. Numerical results show that if an SDF projection is U-shaped or increasing in the stock price then we have a bubble. However, if the SDF projection is monotonically decreasing then we do not have a bubble. Given an expression for the SDF projection which leads to a no-bubble situation, I propose a no-arbitrage formula to price derivatives under $\mathbb{P}$-measure.

Keywords: stochastic discount factor, projections, bubbles, asset pricing
JEL: G12, G13

*University of Trier (dyachenko@uni-trier.de), Universitätsring 15, 54296 Trier, Germany. The author thanks Martin Schweizer for helpful comments and discussions.
1 Introduction

The stochastic discount factor (SDF) is at the center of asset pricing. We use an SDF to price assets (e.g., stocks, bonds, derivatives), to understand prices of different risks (e.g., price of the market risk, price of the volatility risk), and to understand better the risk premium of various assets.

Consider a financial market which consists of a risky stock $S$ and a risk free bond $B$. We define an SDF as a stochastic process $\pi$ such that

$$\pi_t B_t \text{ and } \pi_t S_t \text{ are local martingales for } 0 \leq t \leq T$$

with respect to information at time $t$. The discounted processes $\pi B$ and $\pi S$ reflect a fair game, at least locally. In addition, assume that the dynamics of the stock $S$ is described by a continuous-time stochastic volatility model of Heston type (Heston 1993) where $V$ is the instantaneous variance of the stock $S$.

In this paper, I define projections of the stochastic discount factor using a local martingale property of discounted prices. In particular, a function $f(t, S_t, V_t)$ is an $SV$-local-projection if

$$f(t, S_t, V_t)S_t \text{ and } f(t, S_t, V_t)B_t \text{ are local martingales for } 0 \leq t \leq T$$

with respect to information at time $t$.

Similarly, a function $f(t, S_t, V_t)$ is an $SV$-projection if

$$f(t, S_t, V_t)S_t \text{ and } f(t, S_t, V_t)B_t \text{ are honest martingales for } 0 \leq t \leq T$$

with respect to information at time $t$.

Using an $SV$-local-projection and an $SV$-projection, I define an $S$-local-projection and an $S$-projection. A function $g(t, S_t)$ is an $S$-(local)-projection if

$$g(t, S_t) = \mathbb{E} [f(t, S_t, V_t) \mid S_t]$$

(4)
where \( f(t, S_t, V_t) \) is an SV-(local)-projection.

Given definitions of SV-local-projections, I show that SV-local-projections are mixtures of exponentially affine functions, with \( \lambda \) being a mixture parameter, also called the bubble parameter. Depending on the value of the bubble parameter \( \lambda \in [0, 1] \), I show that projections of an SDF may be monotonically decreasing or U-shaped. More precisely, the bubble parameter \( \lambda > 0 \) if and only if the SV-local-projection \( f(t, S_t, V_t) \) is U-shaped. Also, if the bubble parameter \( \lambda > 0 \), we have a numerical result that \( \{ f(t, S_t, V_t)S_t \} \) and \( \{ f(t, S_t, V_t)B_t \} \) are strict local martingales for \( 0 \leq t \leq T \) (i.e., we have a bubble): \( S_0 > \mathbb{E}[f(T, S_T, V_T)S_T] \) and \( B_0 > \mathbb{E}[f(T, S_T, V_T)B_T] \). In other words, assets \( S \) and \( B \) are overpriced.

However, the bubble parameter \( \lambda = 0 \) if and only if the SV-local-projection is monotonically decreasing in the stock price. Also, if the bubble-parameter \( \lambda = 0 \), we have a numerical result that \( \{ f(t, S_t, V_t)S_t \} \) and \( \{ f(t, S_t, V_t)B_t \} \) are honest martingales for \( 0 \leq t \leq T \). In other words, we have a fair pricing and \( S_0 = \mathbb{E}[f(T, S_T, V_T)S_T] \) and \( B_0 = \mathbb{E}[f(T, S_T, V_T)B_T] \), in particular.

This paper extends the literature on a U-shaped SDF, i.e., the kernel puzzle, and on the SDF recovery: Christoffersen et al. (2013), Song & Xiu (2016), Borovička et al. (2016), Park (2016), Walden (2017), Schneider & Trojani (2019), to name few recent papers\(^1\). In particular, this paper is related to Christoffersen et al. (2013) where the authors consider a discrete-time stochastic volatility model of Heston type, assume a functional form of an SDF, and show that the SDF projection on the stock is U-shaped (given a negative variance risk premium). In this paper, I consider the stochastic volatility model of Heston type, use simplicity of continuous time, and impose no assumption on an SDF. The asset pricing equations lead to the system of PDEs which imply projections. As mentioned above, an SDF projection may be U-shaped (which indicates a bubble) or monotonically decreasing (which indicates an absence of a bubble).

The paper has the following structure. In Section 2, I specify assumptions and definitions. In Section 3, I derive an SV-local-projection by solving the system of pricing PDEs. Section 4 describes S-local-projections. In Section 5, I relate SV-local-projections to bubbles.

\(^1\)The actual list of papers dealing with the SDF recovery and the kernel puzzle is much larger. I do not aim to cite all of the papers on this issue.
derive a pricing formula under $\mathbb{P}$-measure, and discuss the volatility risk premium. Section 6 concludes.

2 Assumptions and Definitions

Assumption 1. Consider a probability space $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t; 0 \leq t \leq T\}, \mathbb{P})$ where $\{\mathcal{F}_t; 0 \leq t \leq T\}$ is a filtration. A financial market is given by a stochastic process

$$\{B_t, S_t; 0 \leq t \leq T\},$$

where $B$ is a risk free bond and $S$ is a risky stock.

Definition 1. A stochastic discount factor (SDF) is a strictly positive semimartingale $\pi$ such that

$$\{\pi_t S_t, \mathcal{F}_t; 0 \leq t \leq T\} \text{ and } \{\pi_t B_t, \mathcal{F}_t; 0 \leq t \leq T\} \text{ are local martingales with } \pi_0 := 1.$$

In the following discussion, we will use (explicitly or implicitly) the following two results.

Fact 1. Let $M = \{M_t, \mathcal{F}_t; 0 \leq t \leq T\}$ be a one-dimensional adapted stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ such that the quadratic variation $\langle M, M \rangle_t(\omega)$ is a continuous function of time $t$ $\mathbb{P}$-almost every $\omega$. Then $M$ is a continuous local martingale if and only if

$$M_t = \int_0^t X_s dW_s, \ 0 \leq t \leq T,$$

where $W = \{W_t, \mathcal{F}_t; 0 \leq t \leq T\}$ is a Brownian motion defined on an extension $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{P})$ and $X = \{X_t, \mathcal{F}_t; 0 \leq t \leq T\}$ is a measurable adapted process with

$$\bar{\mathbb{P}} \left[ \int_0^T X_s^2 ds < \infty \right] = 1.$$  

Fact 1 tells us that continuous local martingales are stochastic integrals with respect to Brownian motion (for details see Karatzas & Shreve (1998), p. 170). Consequently, the sum
of stochastic integrals is a local martingale.

The next result will be useful to distinguish local martingales from honest ones.

**Definition 2.** A stochastic process $M = \{M_t, \mathcal{F}_t; 0 \leq t \leq T\}$ is a supermartingale if

$$\mathbb{E}[M_t | \mathcal{F}_s] \leq M_s \text{ for } 0 \leq s \leq t \leq T.$$  \hspace{1cm} (8)

**Fact 2.** Let $M = \{M_t, \mathcal{F}_t; 0 \leq t \leq T\}$ be a non-negative local martingale. Suppose $\mathbb{E}[M_0] < \infty$. Then $M$ is a supermartingale. If

$$\mathbb{E}[M_T] = \mathbb{E}[M_0],$$ \hspace{1cm} (9)

then $M$ is a martingale.

Fact 2 tells us if $M$ is a non-negative martingale and if condition (9) holds, then $M$ is a martingale (for details see Steele (2012), p. 106).

**Assumption 2.** The dynamics for the stock $S$ and its variance $V$ under $\mathbb{P}$-measure is given by

$$\frac{dS_t}{S_t} = (r + \eta_S V_t) dt + \sqrt{V_t} dW_{S,t},$$

$$dV_t = \kappa (\theta - V_t) dt + \sigma_V \sqrt{V} \left( \rho dW_{S,t} + \sqrt{1 - \rho^2} dW_{V,t} \right),$$ \hspace{1cm} (10)

where $W_S$ and $W_V$ are two independent Brownian motions, $\rho$ is the correlation between $W_S$ and $W_V$, $\eta_S$ is the market risk premium parameter, $\theta$ is the long run level of the variance $V_t$, $\kappa$ is the speed of reversion to the long run variance level $\theta$, $\sigma_V$ is the volatility of the variance, and $r$ is the risk free interest rate.

Given Assumption 2, we now specify the filtration $\{\mathcal{F}_t; 0 \leq t \leq T\}$ in Assumption 1.

**Assumption 3.** In what follows, we set

$$\mathcal{F}_t = \sigma \left( \{(W_{S,s}, W_{V,s}); 0 \leq s \leq t\} \right) \text{ for } 0 \leq t \leq T,$$ \hspace{1cm} (11)
where $W_S$ and $W_V$ are Brownian motions in Assumption 2. Here, the big $S$ denotes the stock whereas the small $s$ stands for time.

We define projection of an SDF on the stock price $S_T$ and its instantaneous variance $V_T$ as follows.

**Definition 3.** An SV-local-projection is a strictly positive stochastic process $f$ such that $f_t$ is $\sigma(S_t, V_t)$-measurable and

$$\{f_t B_t, \mathcal{F}_t; 0 \leq t \leq T\} \text{ and } \{f_t S_t, \mathcal{F}_t; 0 \leq t \leq T\} \text{ are local martingales with } f_0 := 1.$$  

**Definition 4.** The SV-projection is a strictly positive stochastic process $f$ such that $f_t$ is $\sigma(S_t, V_t)$-measurable and

$$\{f_t B_t, \mathcal{F}_t; 0 \leq t \leq T\} \text{ and } \{f_t S_t, \mathcal{F}_t; 0 \leq t \leq T\} \text{ are honest martingales with } f_0 := 1.$$  

Next, we define an $S$-local-projection and $S$-projection as conditional expectations of an SV-local-projection and of the $S$-projection, respectively.

**Definition 5.** Let $f = f(T, S_T, V_T)$ be an SV-local-projection of an SDF $\pi$. An $S$-local projection is a conditional expectation function $g: \mathbb{R}_{\geq 0} \to \mathbb{R}_{> 0}$:

$$g(T, S_T) = \mathbb{E}[f(T, S_T, V_T) | \sigma(S_T)] =: \mathbb{E}[f(T, S_T, V_T) | S_T]. \quad (12)$$

**Definition 6.** Let $f = f(T, S_T, V_T)$ be the SV-projection of an SDF $\pi$. The $S$-projection is a conditional expectation function $g: \mathbb{R}_{\geq 0} \to \mathbb{R}_{> 0}$:

$$g(T, S_T) = \mathbb{E}[f(T, S_T, V_T) | \sigma(S_T)] =: \mathbb{E}[f(T, S_T, V_T) | S_T]. \quad (13)$$
3 SV-Local-Projection

3.1 Main Result

Proposition 1. Under Assumption 2, an SV-local-projection $f$ of an SDF $\pi$ is

$$f(t, \log S_t, V_t; \lambda) = \lambda \exp \left( \alpha_1 (\log S_t - \log S_0) + \beta_1 (V_t - V_0) - (r + \alpha_1 r + \kappa \theta \beta_1) t \right)$$
$$+ (1 - \lambda) \exp \left( \alpha_2 (\log S_t - \log S_0) + \beta_2 (V_t - V_0) - (r + \alpha_2 r + \kappa \theta \beta_2) t \right),$$

(14)

where the mixture parameter $\lambda \in [0, 1]$ and

$$\alpha_i = -\eta - \sigma \rho \beta_i \text{ with } i \in \{1, 2\},$$
$$\beta_1 = \frac{-b + \sqrt{D}}{\sigma^2 (1 - \rho^2)},$$
$$\beta_2 = \frac{-b - \sqrt{D}}{\sigma^2 (1 - \rho^2)},$$
$$b = \frac{1}{2} \sigma \rho - \kappa - \sigma \rho \eta S,$$
$$D = \left( \frac{1}{2} \sigma \rho - \kappa \right)^2 + (\sigma \eta S + \kappa \rho)^2 - \kappa^2 \rho^2 - \sigma^2 \eta S.$$

Proof. Let $x_t := \log S_t$ and $f_t := f(t, x_t, V_t)$ denote the SV-projection of an SDF $\pi$.

First, consider the following three integrals.

1. First integral:

$$\int_0^T f_t dS_t = \int_0^T f_t \left[ S_t (r + \eta S V_t) dt + S_t \sqrt{V_t} dW_{S,t} \right].$$

(15)
2. Second integral:

\[
\int_0^T S_t df_t = \int_0^T S_t \left[ \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} \left[ (r + \eta S_t - \frac{1}{2} V_t) dt + \sqrt{V_t} dW_{S,t} \right] + \frac{\partial f}{\partial V} \left[ \kappa (\theta - V_t) dt + \sigma \sqrt{V_t} \left( \rho dW_{S,t} + \sqrt{1 - \rho^2} dW_{V,t} \right) \right] \right] + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} V_t dt + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} \sigma^2 V_t dt + \frac{\partial^2 f}{\partial x \partial V} \sigma \rho V_t dt. \]

(16)

3. Third integral:

\[
\int_0^T dS_t df_t = \int_0^T \frac{\partial f}{\partial x} S_t V_t dt + \frac{\partial f}{\partial V} \sigma S_t V_t dt.
\]

(17)

Since \( f \) is the \( SV \)-local-projection of an SDF \( \pi \), \( f_S \) is a local martingale. Since \( f_S \) is a local martingale, \( f_S \) has a zero drift.

\[
f_T S_T - f_0 S_0 = \int_0^T f_t S_t dt + \int_0^T S_t df_t + \int_0^T dS_t df_t =
\]

\[
= \int_0^T \left[ f_t r + \frac{\partial f}{\partial t} + r \frac{\partial f}{\partial x} + \frac{\partial f}{\partial V} \right] S_t dt +
\]

\[
= 0 + \int_0^T \left[ f_t \eta_S + \frac{\partial f}{\partial x} (\eta_S - \frac{1}{2}) - \kappa \frac{\partial f}{\partial V} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial V^2} + \sigma \rho \frac{\partial^2 f}{\partial x \partial V} + \frac{\partial f}{\partial x} + \sigma \rho \frac{\partial f}{\partial V} \right] S_t V_t dt +
\]

\[
+ \int_0^T f_t S_t \sqrt{V_t} dW_{S,t} + \int_0^T S_t \frac{\partial f}{\partial x} \sqrt{V_t} dW_{S,t} + \int_0^T S_t \frac{\partial f}{\partial V} \sqrt{V_t} \rho dW_{S,t} + \int_0^T S_t \frac{\partial f}{\partial V} \sqrt{V_t} \sqrt{1 - \rho^2} dW_{V,t}.
\]

(18)

Since \( f_S \) is a local martingale, \( f \) has to satisfy the following two partial differential equations:

- \( S_t dt \):

\[
r f + \frac{\partial f}{\partial t} + r \frac{\partial f}{\partial x} + \kappa \frac{\partial f}{\partial V} = 0,
\]

(19)
• $S_tV_t dt$:

$$f \eta + \frac{\partial f}{\partial x} \left( \eta - \frac{1}{2} \right) - \kappa \frac{\partial f}{\partial V} + \left( \frac{1}{2} \right) \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial V^2} + \sigma \rho \frac{\partial^2 f}{\partial x \partial V} + \frac{\partial f}{\partial x} + \sigma \rho \frac{\partial f}{\partial V} = 0, \quad (20)$$

with the initial condition $f_0 = 1$.

Next, consider the following two integrals.

1. First integral:

$$\int_0^T f_t dB_t = \int_0^T f_t B_t dt. \quad (21)$$

2. Second integral:

$$\int_0^T B_t df_t = \int_0^T B_t \left[ \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} \left( r + \eta S_t - \frac{1}{2} V_t \right) dt + \sqrt{V_t} dW_{S,t} \right] +$$

$$\int_0^T \left[ \frac{\partial f}{\partial V} \left( \kappa \theta - V_t \right) dt + \sigma \sqrt{V_t} \left( \rho dW_{S,t} + \sqrt{1 - \rho^2} dW_{V,t} \right) \right] +$$

$$\int_0^T \left[ \frac{1}{2} \frac{\partial^2 f}{\partial x^2} V_t dt + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} \sigma^2 V_t dt + \frac{\partial^2 f}{\partial x \partial V} \sigma \rho V_t dt \right] \quad (22)$$

Since $f$ is the $SV$-local-projection of an SDF $\pi$, $fB$ is a local martingale. Since $fB$ is a local martingale, $fB$ has a zero drift.

$$f_T B_T - f_0 B_0 = \int_0^T f_t dB_t + \int_0^T B_t df_t =$$

$$\int_0^T \left[ r f_t + \frac{\partial f}{\partial t} + r \frac{\partial f}{\partial x} + \kappa \theta - \frac{\partial f}{\partial V} \right] B_t dt + \int_0^T \left[ \left( \eta - \frac{1}{2} \right) \frac{\partial f}{\partial x} - \kappa \frac{\partial f}{\partial V} + \left( \frac{1}{2} \right) \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial V^2} + \sigma \rho \frac{\partial^2 f}{\partial x \partial V} \right] B_t V_t dt +$$

$$\int_0^T B_t \frac{\partial f}{\partial x} \sqrt{V_t} dW_{S,t} + \int_0^T B_t \frac{\partial f}{\partial V} \sqrt{V_t} dW_{S,t} + \int_0^T B_t \frac{\partial f}{\partial V} \sigma \sqrt{V_t} \sqrt{1 - \rho^2} dW_{V,t} \quad (23)$$
Since \( fB \) is a local martingale, \( fB \) has to satisfy the following two partial differential equations:

- \( B_t dt \):
  \[
  fr + \frac{\partial f}{\partial t} + r \frac{\partial f}{\partial x} + \kappa \theta \frac{\partial f}{\partial V} = 0, \tag{24}
  \]

- \( B_t V_t dt \):
  \[
  (\eta S - \frac{1}{2}) \frac{\partial f}{\partial x} - \kappa \frac{\partial f}{\partial V} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} + \frac{1}{2} \frac{\sigma^2}{\partial x \partial V} + \frac{1}{2} \frac{\sigma^2}{\partial V} = 0, \tag{25}
  \]

with the initial condition \( f_0 = 1 \).

To summarize, the SV-local-projection \( f \) has to satisfy the following three partial differential equations and the initial condition:

\[
\begin{cases}
  f\eta S + \frac{\partial f}{\partial x} (\eta S - \frac{1}{2}) - \kappa \frac{\partial f}{\partial V} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} + \frac{1}{2} \frac{\sigma^2}{\partial x \partial V} + \frac{1}{2} \frac{\sigma^2}{\partial V} + \frac{1}{2} \frac{\sigma^2}{\partial x \partial V} = 0, \tag{26a}
  \\
  (\eta S - \frac{1}{2}) \frac{\partial f}{\partial x} - \kappa \frac{\partial f}{\partial V} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} + \frac{1}{2} \frac{\partial^2 f}{\partial V^2} + \frac{1}{2} \frac{\sigma^2}{\partial x \partial V} = 0, \tag{26b}
  \\
  rf + \frac{\partial f}{\partial t} + r \frac{\partial f}{\partial x} + \kappa \theta \frac{\partial f}{\partial V} = 0, \tag{26c}
  \\
  f(0, x_0, V_0) := 1. \tag{26d}
\end{cases}
\]

Suppose a solution to PDEs (26c) - (26b) is of the form

\[
f(t, x_t, V_t) = \exp \left( \alpha x_t + \beta V_t + \gamma(t) \right), \tag{27}
\]

where \( \alpha, \beta, d \in \mathbb{R} \) and \( \gamma : \mathbb{R}_{>0} \to \mathbb{R} \).

Next, subtracting equation (26b) from equation (26a) yields the PDE

\[
f\eta S + \frac{\partial f}{\partial x} + \sigma \rho \frac{\partial f}{\partial V} = 0. \tag{28}
\]
Equations (27) - (28) imply 
\[ f(\eta S + \alpha + \sigma \rho \beta) = 0 \] (29) 
so that 
\[ \alpha = -\eta S - \sigma \rho \beta. \] (30)

If we plug in our guess (27) for \( f \) and the result (30) into equation (26a), we get the following quadratic equation that pins down the parameter \( \beta \) in our guess (27):
\[ \frac{1}{2} \sigma^2 (1 - \rho^2) \beta^2 + \left( \frac{1}{2} \sigma \rho - \kappa - \sigma \rho \eta S \right) \beta + \frac{1}{2} \eta S (1 - \eta S) = 0. \] (31)

The solutions to equation (31) are
\[ \beta_1 = \frac{-b + \sqrt{D}}{\sigma^2 (1 - \rho^2)}, \] (32a) 
and
\[ \beta_2 = \frac{-b - \sqrt{D}}{\sigma^2 (1 - \rho^2)}, \] (32b) 
where
\[ b := \frac{1}{2} \sigma \rho - \kappa - \sigma \rho \eta S \] (33a) 
and
\[ D := \left( \frac{1}{2} \sigma \rho - \kappa \right)^2 + (\sigma \eta S + \kappa \rho)^2 - \kappa^2 \rho^2 - \sigma^2 \eta S. \] (33b)

The two particular solutions to PDEs (26a) - (26b) are 
\[ f^{(1)}(t, x_t, V_t) = \exp \left( \alpha_1 x_t + \beta_1 V_t + \gamma_1(t) \right) \] (34a) 
and
\[ f^{(2)}(t, x_t, V_t) = \exp \left( \alpha_2 x_t + \beta_2 V_t + \gamma_2(t) \right) \] (34b)
where \( \beta_1 \) and \( \beta_2 \) are given by equations (32a) - (32b), \( \alpha_1 \) and \( \alpha_2 \) correspond to \( \beta_1 \) and \( \beta_2 \), and \( \gamma_1(t) \) and \( \gamma_2(t) \) are some functions of time \( t \).
Using the PDE (26c) and the initial condition (26d), the first particular solution to the PDE system (26a) - (26d) is thus

\[ f^{(1)} = \exp \left( \alpha_1 (x_t - x_0) + \beta_1 (V_t - V_0) - (r + r\alpha_1 + \kappa\theta_1) t \right). \] (35)

Similarly, the second particular solution to the PDE system (26a) - (26d) is thus

\[ f^{(2)} = \exp \left( \alpha_2 (x_t - x_0) + \beta_2 (V_t - V_0) - (r + r\alpha_2 + \kappa\theta_2) t \right). \] (36)

Now, it’s time to expand our particular solutions \( f^{(1)} \) and \( f^{(2)} \) to a general one. First, we note that operators corresponding to PDEs (26a) - (26c)

\[ L_2 := \eta_S + \frac{\partial}{\partial x} \left( \eta_S - \frac{1}{2} \right) - \kappa \frac{\partial}{\partial V} + \frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial V^2} + \sigma \rho \frac{\partial}{\partial x \partial V} + \frac{\partial}{\partial x} + \sigma \rho \frac{\partial}{\partial V}. \] (37a)

\[ L_3 := (\eta_S - \frac{1}{2}) \frac{\partial}{\partial x} - \kappa \frac{\partial}{\partial V} + \frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial V^2} + \sigma \rho \frac{\partial}{\partial x \partial V} \] (37b)

\[ L_1 := r + \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \kappa \theta \frac{\partial}{\partial V}, \] (37c)

are linear. Therefore,

\[ f(t, x_t, V_t) := \lambda_1 f^{(1)}(t, x_t, V_t) + \lambda_2 f^{(2)}(t, x_t, V_t) \text{ for any } \lambda_1, \lambda_2 \in \mathbb{R} \] (38)

satisfies PDEs (26a) - (26c). Next, if we apply the initial condition (26d) to the general solution (38), we get

\[ 1 = \lambda_1 + \lambda_2. \] (39)

If we define \( \lambda := \lambda_1 \), the general solution (38) to the PDE system (26a) - (26d) becomes

\[ f(t, x_t, V_t) = \lambda f^{(1)}(t, x_t, V_t) + (1 - \lambda) f^{(2)}(t, x_t, V_t) \text{ where } \lambda \in \mathbb{R}. \] (40)

To insure that the general solution (40) is strictly positive for any \( t \in [0, T], x_t \in \mathbb{R} \) and \( V_t \in \mathbb{R}_{>0} \) (i.e., we have no-arbitrage), we impose conditions: \( \lambda \geq 0 \) and \( 1 - \lambda \geq 0 \). Hence, under no-arbitrage, the general solution to the PDE system (26a) - (26d) is a convex combination
of \( f^{(1)} \) and \( f^{(2)} \):

\[
f(t, x_t, V_t) = \lambda f^{(1)}(t, x_t, V_t) + (1 - \lambda)f^{(2)}(t, x_t, V_t) \quad \text{where} \quad \lambda \in [0, 1].
\] (41)

### 3.2 Examples of SV-Local-Projection

In this subsection, we consider examples of SV-local projections \( f(t, \log S_t, V_t; \lambda) \) in equation (14) with the mixture parameter \( \lambda = 0, 0.25, 0.5, 1 \). **For this end and unless otherwise stated**, I assume the following parameters in Assumption 2:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_0 )</td>
<td>2.00</td>
</tr>
<tr>
<td>( V_0 )</td>
<td>0.04</td>
</tr>
<tr>
<td>( T )</td>
<td>1.00</td>
</tr>
<tr>
<td>( r )</td>
<td>0.00</td>
</tr>
<tr>
<td>( \rho )</td>
<td>-0.60</td>
</tr>
<tr>
<td>( \eta_S )</td>
<td>1.50</td>
</tr>
<tr>
<td>( \sigma_V )</td>
<td>0.40</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>3.00</td>
</tr>
<tr>
<td>( \theta )</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Table 1: Parameters for Assumption 2. \( T \) denotes the length of the time interval.

Below, we consider two cases: when the correlation between Brownian motions \( W_S \) and \( W_V \) in Assumption 2 is \( \rho \neq 0 \) and when \( \rho = 0 \).

#### 3.2.1 Case: \( \rho \neq 0 \)

Figure 1 shows that, for the mixture parameter \( \lambda = 0 \), the SV-local-projection is *monotonically decreasing* in the stock \( S_T \) for any values of the instantaneous variance \( V_T \). In contrast, Figures 2-3 show, for the mixture parameter \( \lambda = 0.25 \) and \( \lambda = 0.5 \), the SV-local-projection is *U-shaped* in the stock \( S_T \) for higher values of the instantaneous variance \( V_T \). Finally, if
we look at the case when the mixture parameter $\lambda = 1$ in Figure 4, we observe that the $SV$-local-projection is non-decreasing in $S_T$ for all values of the instantaneous variance $V_T$.

Figure 1: $SV$-Local-Projection with $\lambda = 0$ and $\rho = -0.6$.

Figure 2: $SV$-Local-Projection with $\lambda = 0.25$ and $\rho = -0.6$. 
Figure 3: SV-Local-Projection with $\lambda = 0.5$ and $\rho = -0.6$.

Figure 4: SV-Local-Projection with $\lambda = 1$ and $\rho = -0.6$.
3.2.2 Case: \( \rho = 0 \)

While Subsection 3.2.1 shows that \( SV \)-local-projection may be U-shaped in the stock price \( S_T \) when \( \rho \neq 0 \) (as an example, we assumed \( \rho = -0.6 \)), Figures 5 - 8 show that, in the special case of \( \rho = 0 \), the \( SV \)-local-projection is not U-shaped: it is decreasing in the stock price \( S_T \) for various levels of the mixture parameter \( \lambda \) (see Proposition 1) and various values of the instantaneous variance \( V_T \).

![SV-Local-Projection with \( \lambda = 0 \)](image)

Figure 5: \( SV \)-Local-Projection with \( \lambda = 0 \) and \( \rho = 0.0 \).
Figure 6: SV-Local-Projection with $\lambda = 0.25$ and $\rho = 0.0$.

Figure 7: SV-Local-Projection with $\lambda = 0.5$ and $\rho = 0.0$. 
3.3 SV-Projection

Consider an SV-local-projections $f(t, \log S_t, V_t; \lambda)$ where the mixture parameter $\lambda \in [0, 1]$. By construction, $f_B$ and $f_S$ are local martingales. We now check numerically using a Monte-Carlo simulation for which values of $\lambda$ in $f(t, \log S_t, V_t; \lambda)$ the following equations hold:

\[
\mathbb{E}[f(T, \log S_T, V_T; \lambda)B_T] = B_0, \tag{42}
\]
\[
\mathbb{E}[f(T, \log S_T, V_T; \lambda)S_T] = S_0, \tag{43}
\]

where relevant parameters are given in Table 1. The value of the mixture parameter $\lambda$ for which the conditions (42) - (43) do hold numerically will point us, using Fact 2, at the SV-local projection which is, in fact, the SV-projection. That is, conditions (42) - (43) will help us identify numerically the mixture parameter $\lambda^* \in [0, 1]$ for which $f_B$ and $f_S$ are martingales.

Figure 9 shows that, for $\lambda = 0$, conditions (42) - (43) hold and $f_B$ and $f_S$ are martingales. This observation leads to the following proposition (albeit only based on the numerical
result).

**Proposition 2.** Under Assumption 2, the SV-projection $f$ of an SDF $\pi$ is

$$f(t, \log S_t, V_t) = \exp \left( \alpha (\log S_t - \log S_0) + \beta (V_t - V_0) - (r + \alpha r + \kappa \theta \beta) t \right),$$  \hspace{1cm} (44)

$$\alpha = -\eta_S - \sigma \rho \beta$$

$$\beta = -\frac{b - \sqrt{D}}{\sigma^2(1 - \rho^2)},$$

$$b : = \frac{1}{2} \sigma \rho - \kappa - \sigma \rho \eta_S,$$

$$D : = \left( \frac{1}{2} \sigma \rho - \kappa \right)^2 + (\sigma \eta_S + \kappa \rho)^2 - \kappa^2 \rho^2 - \sigma^2 \eta_S.$$

**SV-Local-Projection $f$: Honest Martingale vs Strict Local Martingale.** $\rho = -0.6$

![Graph](image)

Figure 9: The initial stock price is 2. The initial bond price is 1. When $\lambda > 0$, $fB$ and $fS$ are strict local martingales. Here, $T = 1$. On the $y$-axis, we have $E[fS_T]$ and $E[fB_T]$

**Remark 1.** For the case when $\rho = 0$ in Assumption 2, the results shown in Figure 9 still hold.
4  S-(Local)-Projection

In this section, we plot examples of $S$-local-projection when the correlation between Brownian motions $W_S$ and $W_V$ in Assumption 2 is not equal to zero, $\rho \neq 0$, and when $\rho = 0$. Finally, we write down an expression for the $S$-projection.

4.1  Examples of $S$-Local-Projection

4.1.1  Case: $\rho \neq 0$

For the case when $\rho = -0.6$, I compute the $S$-local projection using a non-parametric kernel regression, given the simulated data for the stock $S_T$ and its instantaneous variance $V_T$. Figure 10 shows the computed $S$-local-projections for the mixture parameter $\lambda = 0, 0.5, 0.75$. As expected and in accordance with Figures 1 - 4, for $\lambda = 0$, the $S$-local projection is downward sloping whereas, for $\lambda = 0.5$ and $\lambda = 0.75$, the $S$-local projection is U-shaped.

![Figure 10: The initial stock price is 2. The initial bond price is 1. When $\lambda > 0$, $f_B$ and $f_S$ are strict local martingales. Here, $T = 1$.](image-url)
4.1.2 Case: $\rho = 0$

Figure 11 shows $S$-local-projections for the case when $\rho = 0$. In this special case, the $S$-local-projection is decreasing not only for the mixture parameter $\lambda = 0$, but also for $\lambda = 0.25, 0.5, 1$, which is in line with Figures 5 - 8.

![Figure 11: The initial stock price is 2. The initial bond price is 1. When $\lambda > 0$, $f_B$ and $f_S$ are strict local martingales. Here, $T = 1$.](image)

4.2 $S$-Projection

In this subsection we state an expression for the $S$-projection in Definition 6.

**Corollary 1.** Under Assumption 2 and given Proposition 2, the $S$-projection $f$ of an SDF $\pi$ is

$$f(t, \log S_t) = \exp \left( \alpha (\log S_t - \log S_0) - (r + \alpha r + \kappa \theta \beta) t \right) \mathbb{E} \left[ \exp \left( \beta (V_t - V_0) \right) \Big| S_t \right], \quad (45)$$
where

\[ \alpha = -\eta S - \sigma \rho \beta \]
\[ \beta = \frac{-b - \sqrt{D}}{\sigma^2(1 - \rho^2)}, \]
\[ b : = \frac{1}{2} \sigma \rho - \kappa - \sigma \rho \eta S, \]
\[ D : = \left( \frac{1}{2} \sigma \rho - \kappa \right)^2 + (\sigma \eta S + \kappa \rho)^2 - \kappa^2 \rho^2 - \sigma^2 \eta S. \]

Figures 10 - 11 show that the S-projection (in blue) is monotonically decreasing.

5 Discussion and Further Implications

5.1 Bubbles

Definition 7. Let \( \pi \) be an SDF. A financial market has a \textbf{bubble} if \( \pi S \) or \( \pi B \) is a \textbf{strict} local martingale (Herdegen & Schweizer 2016).

In the paper, we have shown numerically that, under Assumption 2 with \( \rho \neq 0 \), the bubble parameter \( \lambda > 0 \) is equivalent to an SV-local projection \( f(t, S_t, V_t) \) and the corresponding S-local projection \( f(t, S_t) \) being U-shaped. Also, if the bubble parameter \( \lambda > 0 \), we have that \( f(t, S_t, V_t)B \) and \( f(t, S_t, V_t)S \) are strict local martingales, i.e., we have a bubble. In other words, a U-shaped SDF implies a bubble. Below, we set forth a conjecture an existence of a bubble in the market is equivalent to a U-shaped SDF:

Conjecture 1. Suppose Assumption 2 holds and suppose \( \rho \neq 0 \). Let \( f \) be an S-local-projection. \( f \) is U-shaped if and only if we have a bubble in the market. \( f \) is monotonically decreasing in the stock price if and only if there is no bubble in the market.

The conjecture above is in line with recent findings that, depending on the market conditions, an S-local-projections may be downward sloping or U-shaped (Schneider & Trojani 2019).
While our findings tell us that a U-shaped SDF implies a bubble, our conjecture specifies that an existence of a bubble implies a U-shaped SDF.

## 5.2 Pricing

Given the $SV$-projection in Proposition 2, we can state a no-arbitrage price $h_0$ at time $t = 0$ of an arbitrary derivative with a payoff $h(S_T)$ for a no-bubble situation as follows:

**Proposition 3.** Let $h$ be a derivative with an $\sigma(S_T)$-measurable payoff $h(S_T)$. Under Assumptions 1 - 2, the no-arbitrage price $h_0$ at time $t = 0$ of the derivative $h$ is

\[
h_0 = \mathbb{E} \left[ h(S_T) \exp \left( \alpha (\log S_t - \log S_0) + \beta (V_t - V_0) - (r + \alpha r + \kappa \theta \beta) t \right) \right]
\]

where

\[
\alpha = -\eta_s - \sigma \rho \beta \\
\beta = \frac{-b - \sqrt{D}}{\sigma^2 (1 - \rho^2)}, \\
b = \frac{1}{2} \sigma \rho - \kappa - \sigma \rho \eta_s, \\
D = (\frac{1}{2} \sigma \rho - \kappa)^2 + (\sigma \eta_s + \kappa \rho)^2 - \kappa^2 \rho^2 - \sigma^2 \eta_s.
\]

The upshot of Proposition 3 is that it provides an expression for a no-arbitrage price of a derivative under $\mathbb{P}$-measure without any assumption about the stochastic discount factor or the volatility risk premium.

## 5.3 Volatility Risk Premium

We define first the market risk premium and the volatility risk premium.

**Definition 8** (Market Risk Premium). Let $\pi$ be an SDF, $\tilde{\pi}_t := \frac{d\pi_t}{\pi_t}$, and $\tilde{S}_t := \frac{dS_t}{S_t}$ The
market risk premium MRP is defined as

\[ \text{MRP}_t := -d\langle \tilde{\pi}, \tilde{S} \rangle_t, \] (47)

where \( \langle \cdot \rangle \) stands for the quadratic covariation.

**Definition 9** (Volatility Risk Premium). Let \( \pi \) be an SDF, \( \tilde{\pi}_t := \frac{d\pi_t}{\pi_t} \) and \( V \) is the variance of the stock \( S \). The volatility risk premium VRP is defined as

\[ \text{VRP}_t := -d\langle \tilde{\pi}, V \rangle_t, \] (48)

where \( \langle \cdot \rangle \) stands for the quadratic covariation.

Next, we note that the SV-projection in Proposition 2 corresponds to a particular SDF. The next proposition shows to which exactly SDF the SV-projection corresponds.

**Proposition 4.** Let \( \tilde{f} \) be the SV-projection in Proposition 2. Then

\[ \frac{d\tilde{f}_t}{\tilde{f}_t} = -rdt - \eta_S \sqrt{V} dW_{S,t} - \tilde{\eta}_V \sqrt{V} dW_{V,t}, \] (49)

where

\[ \tilde{\eta}_V := -\sigma \sqrt{1 - \rho^2} \tilde{\beta} \] (50)

and

\[ \tilde{\beta} = \frac{-b - \sqrt{D}}{\sigma^2(1 - \rho^2)} \] (51)

and \( b, D \) are as in Propositions 2 - 3 defined. Moreover, let \( \hat{f} \) be a counterpart of \( \tilde{f} \): an SV-local-projection with the mixture parameter \( \lambda = 1 \) (see Proposition 1). Then

\[ \frac{d\hat{f}_t}{\hat{f}_t} = -rdt - \eta_S \sqrt{V} dW_{S,t} - \hat{\eta}_V \sqrt{V} dW_{V,t}, \] (52)

where

\[ \hat{\eta}_V := -\sigma \sqrt{1 - \rho^2} \hat{\beta} \] (53)
and

\[ \beta = -b + \sqrt{D} \sigma^2 (1 - \rho^2) \] (54)

and \(b, D\) are as in Propositions 2 and 3 defined.

**Proof.** Since \(\tilde{f}(t, \log S_t)\) is the SV-projection, \(\tilde{f} B\) is a local martingale. The fact that \(\tilde{f} B\) is a local martingale implies, in turn, that

\[ \alpha (\eta - \frac{1}{2}) + \beta (-\kappa) + \frac{1}{2} \alpha^2 + \frac{1}{2} \beta^2 \sigma^2 + \alpha \beta \rho \sigma = 0. \] (55)

Application of Ito’s lemma to \(\tilde{f}\) and Equation (55) yield the desired result. Derivation of the dynamics for \(\hat{f}\) is analogous to derivation of the dynamics for \(\tilde{f}\). \(\square\)

In light of Proposition 4, we have an expression for a SDF implied by Assumptions 1 - 2.

**Corollary 2.** Given Assumptions 1 - 2, a stochastic process \(\pi\) with the dynamics

\[
\frac{d\pi_t}{\pi_t} = -rdt - \eta S \sqrt{V_t} dW_{S,t} - \left( \zeta \eta_V + (1 - \zeta) \eta_V \right) \sqrt{V_t} dW_{V,t} \\
= -rdt - \eta S \sqrt{V_t} dW_{S,t} - \left( \eta_V \sqrt{V_t} \right) dW_{V,t}
\]

\[
- \left( (\zeta - 1) \eta_V + (1 - \zeta) \eta_V \right) \sqrt{V_t} dW_{V,t}
\]

is an SDF, where \(\zeta \in \mathbb{R}\) and \(\eta_V\) is the volatility risk premium parameter. \(\eta_V\) and \(\tilde{\eta}_V\) are defined in Proposition 4.

Corollary 2 tells us that the SV-projection \(f\) in Definition 4 encapsulates the market risk premium. However, the SV-projection may capture only partially the market price of the volatility risk and, hence, the volatility risk premium. There are several reasons for this:

- **Financial Market.** In Assumption 1, I assume that only the stock \(S\) and the bond \(B\) are traded in the financial market.
• **Definition of SV-projection.** In this paper, I define the SV-projection $f$ as a stochastic process such that $fB$ and $fS$ are honest martingales. I do not define the SV-projection as the conditional expectation directly and do not relate the SV-projection explicitly to the instantaneous variance $V$.

• **SDF Assumption.** Although I define an SDF in Definition 1, I do not make any assumption about the SDF. In particular, I make no assumption about the market price of pure volatility risk and the volatility risk premium.

6 Conclusion

I derive projections of the stochastic discount factor (SDF) in the continuous-time stochastic volatility model. If the projection of an SDF is U-shaped or increasing, we have a bubble. On the other hand, if the projection of an SDF is monotonically decreasing, we do not have a bubble. In view of the SDF projection which leads to a no-bubble situation, I propose a no-arbitrage formula for pricing assets under $\mathbb{P}$-measure. In this paper, I have touched upon the relationship between SDF projections and bubbles. A more elaborate investigation of this relationship is an area for future research.

References


