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Stochastic Discount Factor and Relative Entropy

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Abstract

In this paper, I answer a question what is an SDF in a simple, yet general one-period (i.e., buy-and-hold) setting by linking an SDF to the relative entropy. I show that there is a unique SDF that minimizes the relative entropy, which I call the M-SDF. If the market risk premium is positive, the M-SDF is monotonically decreasing. If the market risk premium is negative, the M-SDF is monotonically increasing. Given the M-SDF, I define an $\widetilde{M}$-SDF which is exponentially-a□ne in the logarithm of the stock price and unique. I apply findings in the paper to financial derivatives and to the stochastic volatility model.

Keywords: SDF, relative entropy, derivatives
JEL: G12, G13

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1 Introduction

A stochastic discount factor (SDF) is a central object in asset pricing. An SDF prices assets and discounts future cashflows by accounting for both risk and time. It integrates both the risk-value of money and the time value of money.

Knowing an SDF yields multiple practical advantages. We can price assets under the physical measure. We can understand the cross-section of returns better. We could estimate parameters, e.g., in the stochastic volatility model in a simpler way.

In this paper, I link an SDF to the relative entropy in a *simple, yet general buy-and-hold (or single period) setting* and answer a question: What is an SDF? To be more precise, I consider a financial market that includes a risky stock and the risk-free bond. Given the financial market, I define the M-SDF as a random variable that minimizes the relative entropy and prices the stock and the bond. The M-SDF turns out to be unique and exponentially-affine in the stock price. Moreover, I show that if the market risk premium is positive, the M-SDF is monotonically decreasing. If the market risk premium is zero, the M-SDF is constant. And if the market risk premium is negative, the M-SDF is monotonically increasing.

Given the M-SDF and since it would be easier to work with an SDF that is exponentially-affine in the logarithm of the stock price, I define the $\widetilde{M}$-SDF to be a random variable that is exponentially-affine in the *logarithm* of the stock prices and that prices the stock and the bond. Similar to the M-SDF, the $\widetilde{M}$-SDF is unique. If the market risk premium is positive, the $\widetilde{M}$-SDF is monotonically decreasing in the stock price. If the market risk premium is zero, the $\widetilde{M}$-SDF is constant. And if the market risk premium is negative, the $\widetilde{M}$-SDF is monotonically increasing.

As a byproduct of the $\widetilde{M}$-SDF, I derive an expression for the S-derivative, which is, by definition, a derivative that maximizes investor's expected utility. The S-derivative is a transformation of an SDF, in particular, of the $\widetilde{M}$-SDF.

When I apply findings to the stochastic volatility model of the Heston type (Heston 1993), I get the expected results. If the market risk premium is positive, the $\widetilde{M}$-SDF is monotonically decreasing. As the market risk premium approaches zero or the relative risk aversion increases, the S-derivative resembles more the risk-free bond. And since the S-derivative does not incorporate the volatility risk premium explicitly, the S-derivative does not outperform the optimal buy-and-hold portfolio significantly, which is in line with the results for the Black-Scholes setting (Hens & Rieger 2014).

This paper contributes to the literature on the kernel puzzle which says that the SDF has a U-shape (Jackwerth 2000, Aıt-Sahalia & Lo 2000, Rosenberg & Engle 2002). Recently, Schneider & Trojani (2019) minimize the L^2-norm of an SDF subject to pricing constraints and show that an SDF may be monotonically decreasing or have a U-shape. On the other hand, Chaudhuri & Schroder (2015) show that the SDFs of individual stocks are usually downward sloping. Linn et al. (2018) propose a new nonparametric estimator of an SDF and argue that the estimated U-shape of an SDF in previous studies is due to SDF estimation flaws.

This paper is organized as follows. In Section 2, I define and solve for the M-SDF, the $\widetilde{M}$-SDF and S-derivative. In Section 3, I apply the results to the stochastic volatility model. Section 4 concludes.

2 M-SDF, $\widetilde{M}$-SDF and S-Derivative

2.1 General Setting and Definition of the Problem

We start by defining a financial market. We assume the financial market consists of a risk free bond B and a risky stock S.

Assumption 1. *Consider a probability space $\left(\Omega, F_T, (F_t)_{t\ [0,T]}, \mathbb{P}\right)$ where $(F_t)_{t\ [0,T]}$ is a filtration. A financial market is given by a semimartingale*

$$\{B_t, S_t\}_{t\ [0,T]}, \tag{1}$$

where B is a risk free bond with $B_0 := 1$ and S is a risky stock. In what follows, let $\overline{S}_t := \dfrac{S_t}{B_t}$, $x_t := \log S_t$, and $\overline{x}_t = \log \overline{S}_t$.

As the next step, we define a stochastic discount factor (SDF).

Definition 1 (SDF). *A stochastic discount factor (SDF) is a strictly positive semimartingale*

π with $\pi_0 := 1$ such that

$$\{\pi_t S_t; \mathcal{F}_t\}_{t \in [0,T]} \text{ and } \{\pi_t B_t; \mathcal{F}_t\}_{t \in [0,T]} \text{ are local martingales.}$$

Next, we assume there is no arbitrage and hence the existence of an SDF π (Delbaen & Schachermayer 1994).

Assumption 2. *There is no free lunch with vanishing risk.*

For our discussion, we need the following definition, which allows us to compare two measures defined on the same measurable space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \in [0,T]})$.

Definition 2. *Given two measures $\mathbb{P}$ and $\mathbb{Q}$ on the measurable space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \in [0,T]})$ and the corresponding measure zero sets $N^{\mathbb{P}}$ and $N^{\mathbb{Q}}$, we say that measure $\mathbb{P}$ dominates measure $\mathbb{Q}$ if $N^{\mathbb{P}} \subseteq N^{\mathbb{Q}}$. Symbolically, $\mathbb{Q} \ll \mathbb{P}$. Similarly, we say that measure $\mathbb{Q}$ is equivalent to measure $\mathbb{P}$ if $N^{\mathbb{P}} = N^{\mathbb{Q}}$, Symbolically, $\mathbb{Q} \sim \mathbb{P}$.*

Since $M := \pi B = \pi e^{rt}$ is a local martingale, we have that

$$\pi_t = e^{-rt} M_t. \tag{2}$$

In our context, M is a change of measure:

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}} := \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} &:= M_T \\ \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} &:= M_t \text{ for } t \in [0,T), \end{aligned} \tag{3}$$

where $\mathbb{Q} \sim \mathbb{P}$.

We are now ready to define the relative entropy or Kullback-Leibler divergence in our setting (Schweizer 2010, Frittelli 2000).

Definition 3 (Relative Entropy). *Let $\mathbb{Q}$ and $\mathbb{P}$ be two measures such that $\mathbb{Q} \ll \mathbb{P}$ ($\mathbb{P}$ dominates $\mathbb{Q}$). Let $K := \left\{Z \in L^1 \;\middle|\; Z \overset{a.s}{\geq} 0\right\}$. The relative entropy or Kullback-Leibler* *divergence from* $\mathbb{Q}$ *to* $\mathbb{P}$ *is a map*[1] $H : K \to [0, \infty]$ *which is defined as*

$$H\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) := \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}\right] \tag{4}$$

By Lemma A.1, the relative entropy $H(Z)$ is non-negative for the case where $Z \overset{\text{a.s.}}{\geq} 0$ and $\mathbb{E}[Z] := 1$[2].

In this paper, I consider the following problem with the **single period** or **buy-and-hold** pricing constraints[3]:

$$\begin{aligned} \underset{\frac{d\mathbb{Q}}{d\mathbb{P}} \overset{\text{a.s.}}{\geq} 0}{\text{minimize}}\ & H\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}\right] \\ \text{subject to}\ & \mathbb{E}\left[\left|\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}\right|\right] < \infty, \\ & \mathbb{E}\left[e^{-rT} \frac{d\mathbb{Q}}{d\mathbb{P}} e^{x_T}\right] = e^{x_0}, \\ & \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = 1. \end{aligned} \tag{5}$$

Given Problem 5, the following assumption is analogous to Assumption 2.

Assumption 3. *The solution to Problem 5 is strictly positive almost surely.*

2.2 M-SDF

We call the solution to the Problem (5) the M-SDF.

[1] Note that if $\mathbb{Q}$ is equivalent to $\mathbb{P}$, then $\mathbb{E}^{\mathbb{Q}}\left[\log \frac{d\mathbb{Q}}{d\mathbb{P}}\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}\right]$. Also, for $h(x) := x \log x$, we set $h(0) := 0$ so that h is strictly convex and continuous on $[0, \infty)$.

[2] In what follows, I omit the superscript next to the expectation operator if I mean an expectation under $\mathbb{P}$-measure.

[3] Chen et al. (2021) examine this problem with a different proof and in a different context.

Definition 4 (*M*-SDF)**.** *An M-**SDF** is the random variable* $\pi_T := e^{-rT}\dfrac{d\mathbb{Q}}{d\mathbb{P}}$ *where* $\dfrac{d\mathbb{Q}}{d\mathbb{P}}$ *solves Problem* (5).

Proposition 1 states the expression for the M-SDF.

Proposition 1 (*M*-SDF)**.** *Under Assumptions 1 and 3, the unique M-SDF* π_T *is*

$$\pi_T = \exp\Big(-rT\Big)\frac{\exp\Big(\lambda\Big(\exp\left(\overline{x}_T - \overline{x}_0\right) - 1\Big)\Big)}{\mathbb{E}\Big[\exp\Big(\lambda\Big(\exp\left(\overline{x}_T - \overline{x}_0\right) - 1\Big)\Big)\Big]}, \tag{6}$$

where λ *is the unique solution to*

$$\begin{aligned} &\mathbb{E}\Big[\exp\Big(\lambda\big(\exp\left(\overline{x}_T - \overline{x}_0\right) - 1\big)\Big)\exp\Big(\overline{x}_T - \overline{x}_0\Big)\Big] = \\ &= \mathbb{E}\Big[\exp\Big(\lambda\big(\exp\left(\overline{x}_T - \overline{x}_0\right) - 1\big)\Big)\Big]. \end{aligned} \tag{7}$$

and $\overline{x}_T := \log\left(\frac{S_T}{e^{rT}}\right)$.

Proof of Proposition 1. Assume that we work in the Banach space L^1 and define the following sets:

$$\begin{aligned} K &:= \Big\{Z \quad L^1 \Big|\, Z \overset{\text{a.s.}}{\geq} 0\Big\}, \\ \text{int}K &:= \Big\{Z \quad L^1 \Big|\, Z \overset{\text{a.s.}}{>} 0\Big\}, \\ C &:= \Big\{Z \quad L^1 \Big|\, \mathbb{E}\big[Z\exp\left(\overline{x}_T - \overline{x}_0\right)\big] = 1,\ \mathbb{E}\big[Z\big] = 1\Big\}, \\ D &:= \Big\{Z \quad L^1 | Z \overset{\text{a.s.}}{\geq} 0,\ \mathbb{E}\left[|Z\log Z|\right] < \infty\Big\}, \\ G &:= K \cap C \cap D. \end{aligned}$$

Let $Z := \dfrac{d\mathbb{Q}}{d\mathbb{P}}$. The minimization problem (5) becomes then

$$\inf_{Z \quad G} H(Z) = \mathbb{E}\big[Z\log Z\big]. \tag{8}$$

We note that $G = K \cap C \cap D$ is a *closed* set since K, C and D are closed sets by Lemmas A.2-

A.4. Next, we note that G is *bounded* in L^1 since

$$\mathbb{E}\left[\left|Z - Z\right|\right] \leq \mathbb{E}\left[Z\right] + \mathbb{E}\left[Z\right] = 2 \text{ for all } Z,\ Z \quad G.$$

Also, since $K, C,$ and D are convex, G is a *convex* set.

Since the sublevel set $S_\alpha := \left\{Z \quad L^1 \middle| Z \overset{\text{a.s.}}{\geq} 0,\ \mathbb{E}[Z \log Z] \leq \alpha,\ \mathbb{E}[|Z \log Z|] < \infty\right\}$ is closed by Lemma A.5, the function $H(Z) = \mathbb{E}[Z \log Z]$ is *lower semicontinous*.

Given that G is bounded, convex and closed and that H is strictly convex (since $h(x) := x \log x$ is strictly convex) and lower semicontinuous, the optimization problem (8) has the unique solution by Proposition 1.2 in Ekeland & Temam (1999).

Now let Z be the unique solution to the minimization problem (8), i.e.,

$$\mathbb{E}[Z \log Z] > \mathbb{E}[Z \ \log Z\ ] \text{ for any } Z \quad G \text{ and } Z \overset{\text{a.s.}}{=} Z\ . \tag{9}$$

Due to the strict convexity of $h(x) = x \log x$ and due to $Z \overset{\text{a.s.}}{>} 0$ (Assumption 3), we have that

$$Z \log Z \geq Z \ \log Z\ + (1 + \log Z\ )(Z - Z\ ) \text{ for any } Z \quad G \text{ and } Z \overset{\text{a.s.}}{=} Z\ . \tag{10}$$

Applying an expectation operator to the equation (10) and due to the strict convexity of $h(x) := x \log x$, we get

$$\mathbb{E}[Z \log Z] > \mathbb{E}[Z \ \log Z\ ] + \mathbb{E}[(1 + \log Z\ )(Z - Z\ )] \text{ for any } Z \quad G \text{ and } Z \overset{\text{a.s.}}{=} Z\ . \tag{11}$$

Since $Z \quad intK \cap G$ is the solution of the minimization problem (8), we have

$$\mathbb{E}[(1 + \log Z\ )(Z - Z\ )] = 0 \text{ for any } Z \quad G \text{ and } Z \overset{\text{a.s.}}{=} Z\ . \tag{12}$$

Since both Z and $Z \quad G$, the last condition reduces to

$$\mathbb{E}[\log Z\ (Z - Z\ )] = 0 \text{ for any } Z \quad G \text{ and } Z \overset{\text{a.s.}}{=} Z\ . \tag{13}$$

If we set $\log Z\ = \mu + \lambda \exp\left(\overline{x}_T\right)$ with $\mu, \lambda \quad (-\infty, +\infty)$, we observe that the condition (13) is satisfied. If we solve for μ, we get the desired result. The minimum value attained by the objective function[4] is

$$\mathbb{E}[Z \ \ \log Z \ ] = \mu + \lambda \exp\left(\overline{x}_0\right). \tag{14}$$

□

To discuss whether the M-SDF is decreasing in the stock price S_T, we define the market risk premium.

Definition 5 (Market Risk Premium). *The market risk premium (MRP) is defined as*

$$MRP := \mathbb{E}[\overline{S}_T] - S_0, \tag{15}$$

where $\overline{S}_T := \dfrac{S_T}{e^{rT}}$.

Given Definition 5,

$$\begin{aligned} MRP =& \mathbb{E}[\overline{S}_T] - S_0 = \mathbb{E}[\overline{S}_T] - \mathbb{E}[\pi_T S_T] \\ =& \mathbb{E}[\overline{S}_T] - \frac{\mathbb{E}\left[\exp\left(\lambda \overline{S}_T\right)\overline{S}_T\right]}{\mathbb{E}\left[\exp\left(\lambda \overline{S}_T\right)\right]} \\ =& \frac{\mathbb{E}[\overline{S}_T]\mathbb{E}\left[\exp\left(\lambda \overline{S}_T\right)\right]}{\mathbb{E}\left[\exp\left(\lambda \overline{S}_T\right)\right]} - \frac{\mathbb{E}\left[\exp\left(\lambda \overline{S}_T\right)\overline{S}_T\right]}{\mathbb{E}\left[\exp\left(\lambda \overline{S}_T\right)\right]} \\ =& -\frac{\mathbb{Cov}\left(\overline{S}_T, \exp(\lambda \overline{S}_T)\right)}{\mathbb{E}\left[\exp\left(\lambda \overline{S}_T\right)\right]}, \end{aligned} \tag{16}$$

where π_T is the M-SDF and $\mathbb{Cov}$ is the covariance operator. This observation yields the following proposition.

Proposition 2 (Shape of the M-SDF). *Consider the M-SDF π_T in Proposition 1. Let* $\overline{S}_T := \dfrac{S_T}{B_T} = \dfrac{S_T}{e^{rT}}$.

1. $MRP = \mathbb{E}[\overline{S}_T] - S_0 > 0$ *if and only if the M-SDF is monotonically **decreasing** in the stock price S_T, i.e., $\lambda < 0$.*

2. $MRP = \mathbb{E}[\overline{S}_T] - S_0 < 0$ *if and only if the M-SDF is monotonically **increasing** in the*

[4] I implicitly assume that the objective function is *proper*, i.e., it is not identically equal to ∞.

stock price S_T, i.e., $\lambda > 0$.

3. *$MRP = \mathbb{E}[\overline{S}_T] - S_0 = 0$ if and only if the M-SDF is **constant** in the stock price S_T and equal to regular time discounting e^{-rT}, i.e., $\lambda = 0$.*

2.3 $\widetilde{M}$-SDF

We note that by Proposition 1, we can write out the M-SDF π as follows:

$$\begin{aligned}\pi_T &= e^{-rT}\frac{\exp\Big(\lambda\big(\exp\left(\overline{x}_T-\overline{x}_0\right)-1\big)\Big)}{\mathbb{E}\Big[\exp\Big(\lambda\big(\exp\left(\overline{x}_T-\overline{x}_0\right)-1\big)\Big)\Big]}\\ &= e^{-rT}\frac{\exp\Big(\lambda\overline{r}_T\Big)}{\mathbb{E}\Big[\exp\Big(\lambda\overline{r}_T\Big)\Big]}\\ &= e^{-rT}\frac{\exp\left(\lambda\Big((\overline{x}_T-\overline{x_0})+O\big((\overline{x}_T-\overline{x_0})^2\big)\Big)\right)}{\mathbb{E}\left[\exp\left(\lambda\Big((\overline{x}_T-\overline{x_0})+O\big((\overline{x}_T-\overline{x_0})^2\big)\Big)\right)\right]},\end{aligned} \tag{17}$$

where $\overline{r}_T := \dfrac{\overline{S}_T}{\overline{S}_0} - 1$ is the **discounted net return**, $\overline{x}_T := \overline{S}_T := \dfrac{S_T}{e^{rT}}$, and $\overline{x}_T - \overline{x}_0$ is the **discounted log return**. If we replace the discounted net return with the discounted log return in (17), we get the $\widetilde{M}$-SDF, which we define below.

Definition 6 ($\widetilde{M}$**-SDF**)**.** *Let $\overline{y}_T := \overline{x}_T - \overline{x}_0$. The $\widetilde{M}$**-SDF** is a random variable $\widetilde{\pi}_T$ such that*

$$\widetilde{\pi}_T := e^{-rT}\frac{\exp\left(\lambda(\overline{x}_T-\overline{x}_0)\right)}{\mathbb{E}\left[\exp\left(\lambda(\overline{x}_T-\overline{x}_0)\right)\right]} = e^{-rT}\frac{\exp\left(\lambda\overline{y}_T\right)}{\mathbb{E}\left[\exp\left(\lambda\overline{y}_T\right)\right]}, \tag{18}$$

where λ is the unique solution to

$$\mathbb{E}\Big[\exp\Big((\lambda+1)\overline{y}_T\Big)\Big] = \mathbb{E}\Big[\exp\Big(\lambda\overline{y}_T\Big)\Big]. \tag{19}$$

by Lemma A.6.

Using Definition 5 and similar to the M-SDF case, we get

$$
\begin{aligned}
MRP =& \mathbb{E}[\overline{S}_T] - S_0 = \mathbb{E}[\overline{S}_T] - \mathbb{E}[\widetilde{\pi}_T S_T] \\
=& \mathbb{E}[\overline{S}_T] - \frac{\mathbb{E}\left[\exp\left(\lambda(\overline{x}_T - \overline{x}_0)\right)\overline{S}_T\right]}{\mathbb{E}\left[\exp\left(\lambda(\overline{x}_T - \overline{x}_0)\right)\right]} \\
=& \frac{\mathbb{E}[\overline{S}_T]\mathbb{E}\left[\exp\left(\lambda(\overline{x}_T - \overline{x}_0)\right)\right]}{\mathbb{E}\left[\exp\left(\lambda(\overline{x}_T - \overline{x}_0)\right)\right]} - \frac{\mathbb{E}\left[\exp\left(\lambda(\overline{x}_T - \overline{x}_0)\right)\overline{S}_T\right]}{\mathbb{E}\left[\exp\left(\lambda(\overline{x}_T - \overline{x}_0)\right)\right]} \\
=& -\frac{\mathbb{Cov}\left(\overline{S}_T, \exp(\lambda(\overline{x}_T - \overline{x}_0))\right)}{\mathbb{E}\left[\exp\left(\lambda(\overline{x}_T - \overline{x}_0)\right)\right]},
\end{aligned}
\tag{20}
$$

where $\widetilde{\pi}_T$ is the $\widetilde{M}$-SDF and $\overline{x}_T := \log S_T - rT$.

The observation above yields the following proposition.

Proposition 3 (Shape of the $\widetilde{M}$-SDF). *Consider the $\widetilde{M}$-SDF $\widetilde{\pi}_T$. Let $\overline{S}_T := \dfrac{S_T}{B_T} = \dfrac{S_T}{e^{rT}}$.*

1. *$MRP = \mathbb{E}[\overline{S}_T] - S_0 > 0$ if and only if the $\widetilde{M}$-SDF is monotonically **decreasing** in the stock price S_T, i.e., $\lambda < 0$.*
2. *$MRP = \mathbb{E}[\overline{S}_T] - S_0 < 0$ if and only if the $\widetilde{M}$-SDF is monotonically **increasing** in the stock price S_T, i.e., $\lambda > 0$.*
3. *$MRP = \mathbb{E}[\overline{S}_T] - S_0 = 0$ if and only if the $\widetilde{M}$-SDF is **constant** in the stock price S_T and equal to regular time discounting e^{-rT}, i.e., $\lambda = 0$.*

2.4 S-Derivative

In this subsection, I apply the $\widetilde{M}$-SDF to find an expression for the optimal derivative, which I call the S-derivative.

Definition 7 (S-Derivative). *Let $u : \mathbb{R}_{>0} \to \mathbb{R}$ be a strictly concave, strictly increasing, di erentiable utility function. The **optimal derivative** or the S-**derivative** is a non-negative payo X_T that solves the optimization problem*

$$
\begin{aligned}
&\text{maximize}_{X \quad L^1,\, X \geq 0} \mathbb{E}\left[u(X_T)\right] \\
&\textit{subject to } \mathbb{E}\left[X_T \widetilde{\pi}_T\right] = X_0,
\end{aligned}
\tag{21}
$$

where $\widetilde{\pi}_T$ is the $\widetilde{M}$-SDF and X_0 is the initial funds available.

Next, we state an expression for the optimal derivative.

Proposition 4 (S-Derivative). *Suppose $X_T \overset{a.s.}{>} 0$. The S-derivative X_T is*

$$X_T = (u\ )^{-1}(C\widetilde{\pi}_T), \tag{22}$$

where C solves

$$\mathbb{E}\left[(u\ )^{-1}(C\widetilde{\pi}_T)\widetilde{\pi}_T\right] = X_0 \tag{23}$$

and $\widetilde{\pi}_T$ is the $\widetilde{M}$-SDF, u is the derivative of the utility function u, and $(u\ )^{-1}$ is the inverse of u .

Proof of Proposition 4. Here, we sketch a proof of Proposition 4. For details, see, e.g., Korn & Korn (2001), Karatzas et al. (1991), Cox & Huang (1989). Let

$$G := \{X \overset{\text{a.s.}}{\geq} 0 \mid \mathbb{E}[X_T\widetilde{\pi}_T] = X_0\} \tag{24}$$

X_T G is the S-derivative if

$$\mathbb{E}[u(X\ )] > \mathbb{E}[u(X)] \text{ for any } X \quad G \text{ and } X \overset{\text{a.s.}}{=} X\ . \tag{25}$$

We note that due to the strict concavity of $u(\cdot)$ and due to $X \overset{\text{a.s.}}{>} 0$, we have that

$$\begin{aligned} u(X_T) - \lambda(X_T\widetilde{\pi}_T - X_0) \leq u(X_T) - \lambda(X_T\widetilde{\pi}_T - X_0) + \\ + \left(u\ (X_T) - \lambda\widetilde{\pi}_T\right)(X_T - X_T) \text{ for any } X \quad G \text{ and } X \overset{\text{a.s.}}{=} X \end{aligned} \tag{26}$$

and for some $\lambda \quad (-\infty, \infty)$. Applying the expectation to both sides of the inequality above, we get

$$\mathbb{E}[u(X_T)] < \mathbb{E}[u(X_T)] + \underbrace{\mathbb{E}[u\ (X_T)(X_T - X_T)]}_{=0 \text{ if and only if } X_T \text{ is optimal}}\ . \tag{27}$$

Therefore, X_T is optimal (i.e., solves the optimization problem) if and only if

$$\mathbb{E}[u\ (X_T)(X_T - X_T)] = 0. \tag{28}$$

Equation (28) implies that

$$u'(X_T) = C\widetilde{\pi}_T, \tag{29}$$

where C is some positive constant. We pin down C using the budget constraint:

$$\mathbb{E}\left[\left(u'\right)^{-1}(C\widetilde{\pi}_T)\widetilde{\pi}_T\right] = X_0.$$

The result follows. □

3 Application: Stochastic Volatility Model

In this section, we specify the dynamics for the stock S and compute the $\widetilde{M}$-SDF and the S-derivative.

First, we assume that the stock has the following dynamics:

Assumption 4. *The dynamics for the stock S and its variance V under $\mathbb{P}$-measure is given by*

$$\begin{aligned} \frac{dS_t}{S_t} &= (r + \eta_S V_t)\, dt + \sqrt{V_t} dW_{S,t}, \\ dV_t &= \kappa\,(\theta - V_t)\, dt + \sigma_V \sqrt{V_t}\left(\rho dW_{S,t} + \sqrt{1-\rho^2} dW_{V,t}\right), \end{aligned} \tag{30}$$

where W_S and W_V are two independent Brownian motions, ρ is the correlation between W_S and W_V, η_S is the market risk premium parameter, θ is the long run level of the variance V, κ is the speed of reversion to the long run variance level θ, σ_V is the volatility of the variance, and r is the risk free interest rate.

Second, we assume the following parameters for the stochastic volatility model:

Assumption 5. *We assume the following parameters for the model in Assumption 4:*

Parameter	Value
S_0	1.00
V_0	0.04
T	1.00
r	0.00
ρ	-0.60
η_S	1.50
σ_V	0.40
κ	3.00
θ	0.04

Table 1: Parameters for model in Assumption 4. T denotes the length of the investment period.

Third, we assume the utility function.

Assumption 6. *An investor has a CRRA utility function:*

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma}, \tag{31}$$

where γ is the coe cient of the risk aversion.

Next, we define two benchmark portfolios: the optimal constant risky fraction portfolio and the optimal buy-and-hold portfolio.

Definition 8 (Optimal Buy-and-Hold Portfolio). *The payo of the optimal buy-and-hold portfolio X_T^{bh} is defined as*

$$X_T^{bh} := \underset{X \quad A^{bh}(x)}{\arg\max}\, \mathbb{E}\Big[u(X)\Big], \tag{32}$$

where

$$A^{bh}(x) := \left\{ X \overset{a.s.}{>} 0 : X = x\left(w\frac{S_T}{S_0} + (1-w)\frac{B_T}{B_0}\right), w \quad [0,1] \right\}. \tag{33}$$

and x is the initial wealth.

Definition 9 (Optimal Constant Risky Fraction Portfolio). *Given Assumption 4, the payoff of the optimal constant risky fraction portfolio X_T^{frac} is defined as*

$$X_T^{frac} := \underset{X \in A^{frac}(x)}{\arg\max}\ \mathbb{E}\Big[u(X)\Big], \tag{34}$$

where

$$A^{frac}(x) := \Big\{X \overset{a.s.}{>} 0 : X_0 = x,\ dX_t = X_t r dt + w X_t \Big[\eta_S V_t dt + \sqrt{V_t} dW_{S,t}\Big], w \in [0,1]\Big\}. \tag{35}$$

Finally, to measure the performance of the S-derivative and the benchmark portfolios, we define the certainty equivalent return.

Definition 10 (Certainty Equivalent Return). *Let $u : \mathbb{R}_{>0} \to \mathbb{R}$ be a utility function. The certainty equivalent return CER of an F_T-measurable payoff X for the investment period $[0, T]$ is the risk free interest rate r such that*

$$u\big(x \exp(rT)\big) = \mathbb{E}\big[u(X)\big], \tag{36}$$

where x is the $t = 0$ price of the payoff X.

Figure 1 shows that if $\eta_S > 0$ then the $\widetilde{M}$-SDF is monotonically decreasing. The higher value of the market risk premium parameter η_S is, the higher the slope of the $\widetilde{M}$-SDF in its absolute value is.

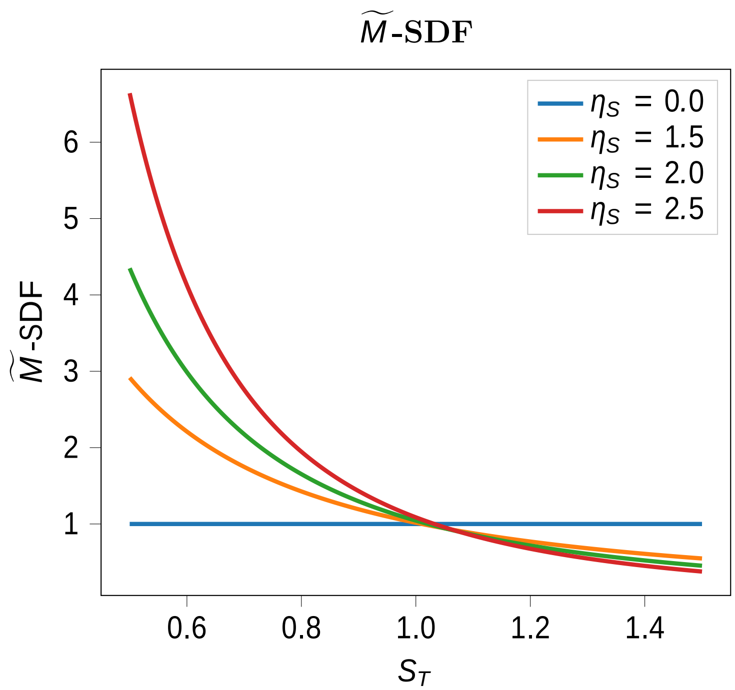

Figure 1: $\widetilde{M}$-SDF. The initial stock price is $S_0 = 1$.

Figure 2 shows the payoff of the S-derivative for different values of the market risk premium parameter η_S. As the market risk premium parameter η_S decreases, the slope of the payoff decreases. When the market risk premium parameter is equal to zero ($\eta_S = 0$), the payoff of the optimal derivative becomes equal to the payoff of the risk free bond.

S-Derivative in the Stochastic Volatility Model, γ = 5

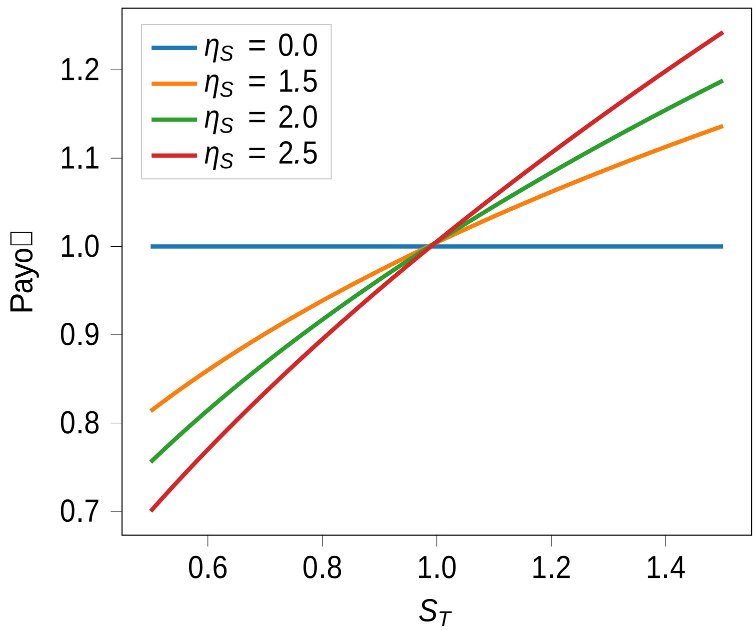

Figure 2: S-derivative for dierent levels of the market risk premium parameter η_S

Figure 3 shows the payo of the optimal derivative for dierent values of the risk aversion. As the risk aversion γ increases, the payo of the S-derivative converges to the payo of the risk-free bond.

S-Derivative in the Stochastic Volatility Model, η_S = 1.5

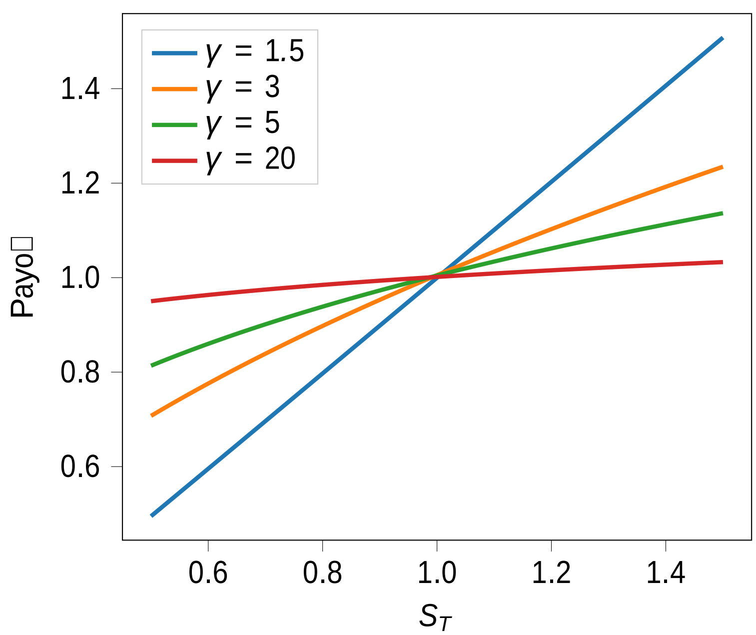

Figure 3: S-derivative for dierent levels of the risk aversion γ

We now compare the S-derivative to the two benchmarks: the optimal constant risky fraction portfolio and the optimal buy-and-hold portfolio. Figures 4-5 show that the S-derivative does not outperform significantly the optimal constant risky fraction portfolio and the optimal buy-and-hold portfolio, respectively. We observe this result because the S-derivative incorporates the market risk premium and does not incorporate (at least explicitly) the volatility risk premium.

Certainty Equivalent Return and Risk Aversion: Constant Risky Fraction

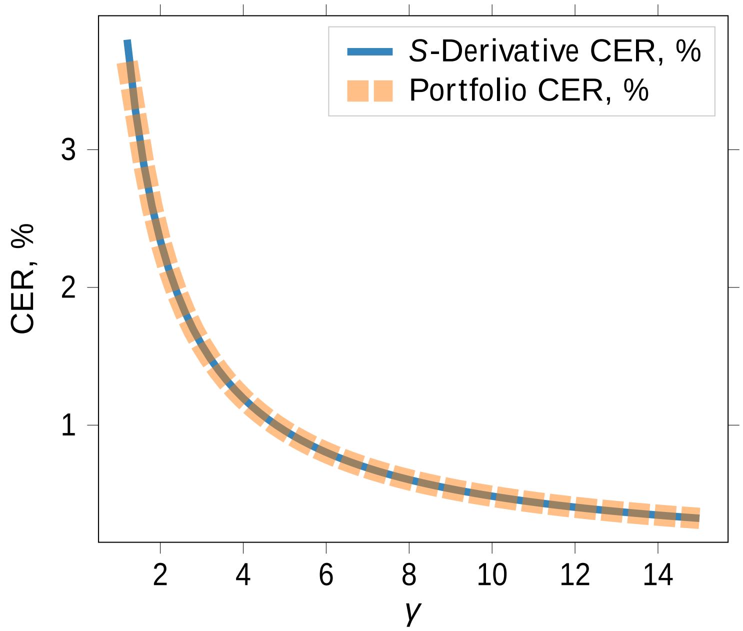

Figure 4: Certainty equivalent return for the optimal derivative and the optimal constant risky fraction portfolio

Certainty Equivalent Return and Risk Aversion: Buy-And-Hold Portfolio

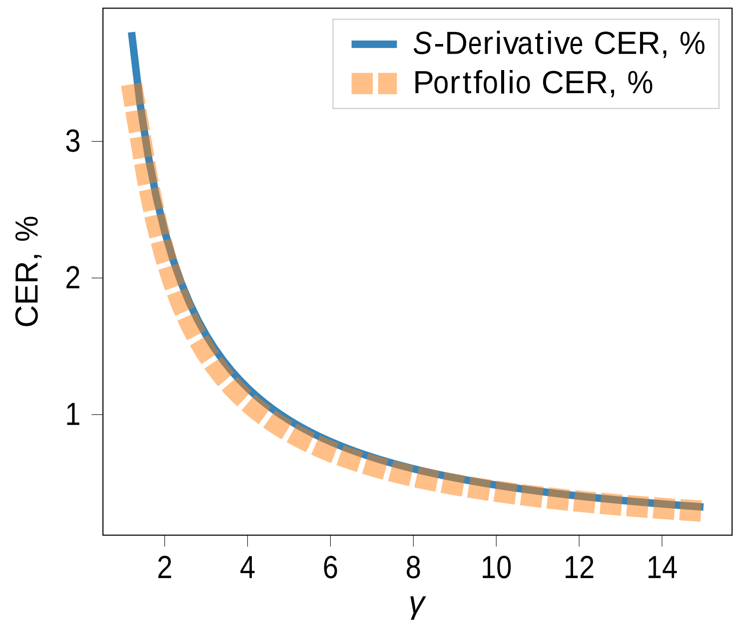

Figure 5: Certainty equivalent return for the optimal derivative and the optimal buy-and-hold portfolio.

4 Conclusion

The M-SDF is exponentially-affine in the stock price and unique. If the market risk premium is positive, then the M-SDF is monotonically decreasing. If the market risk premium is zero, then the M-SDF is constant. In case where the market risk premium is negative, the M-SDF is increasing. Given the expression for the M-SDF, we *define* the $\widetilde{M}$-SDF to be exponentially-affine in the logarithm of the stock price. Similar to the M-SDF, $\widetilde{M}$-SDF is unique and can be monotonically decreasing, increasing or constant, depending on the value of the market risk premium. Given the definition of the $\widetilde{M}$-SDF, we get the expression for the S-derivative, i.e., the derivative that maximizes investor's expected utility. Finally, when we apply our results to the stochastic volatility model, we observe that the S-derivative does not outperform (significantly) the optimal buy-and-hold portfolio.

A Proofs

Lemma A.1. *Let H be a relative entropy as in Definition 3. Then $H(Z) \geq 0$ where $Z \overset{a.s.}{\geq} 0$ and $\mathbb{E}[Z] := 1$.*

Proof. Let $h(x) := x \log x$ with $h(0) := 0$. Since h is strictly convex, we have

$$\begin{cases} x \log x > (x-1) & \text{for all } x \quad [0, \infty),\ x = 1, \\ x \log x = x - 1 & \text{for } x = 1. \end{cases}$$

Therefore, for any $Z \overset{\text{a.s.}}{\geq} 0$, we have

$$\begin{cases} \mathbb{E}[Z \log Z] > \mathbb{E}[Z] - 1 = 0 & \text{for all } Z \overset{\text{a.s.}}{\geq} 0,\ Z \overset{\text{a.s.}}{=} 1 \text{ and } \mathbb{E}[Z] = 1, \\ \mathbb{E}[Z \log Z] = 0 & \text{for } Z \overset{\text{a.s.}}{=} 1 \text{ and } \mathbb{E}[Z] = 1. \end{cases} \tag{37}$$

□

Lemma A.2. *The closed half-space $K := \left\{Z \quad L^1 \,\middle|\, Z \overset{a.s}{\geq} 0\right\}$ is a closed set in L^1.*

Proof. Let $\{Z_n\}$ be a sequence in K such that $Z_n \xrightarrow[n\to\infty]{L^1} Z$. Since $\{Z_n\}$ is convergent, it is also Cauchy. Since L^1 is complete, we have that $Z \quad L^1$. Suppose $\mathbb{P}\left[\omega \middle| Z(w) < 0\right] > 0$ and suppose $\mathbb{E}\left[|Z_n - Z|\right] \xrightarrow[n\to\infty]{} 0$ where $Z_n \quad K$ for $n \geq 1$. Since $\mathbb{E}\left[|Z_n - Z|\right] \xrightarrow[n\to\infty]{} 0$, we should have $\mathbb{E}\left[|Z_n - Z|\mathbf{1}(Z \geq 0)\right] \xrightarrow[n\to\infty]{} 0$ and $\mathbb{E}\left[|Z_n - Z|\mathbf{1}(Z < 0)\right] \xrightarrow[n\to\infty]{} 0$. However, since $Z_n \overset{\text{a.s.}}{\geq} 0$ for $n \geq 1$, we have

$$\mathbb{E}\left[|Z_n - Z|\mathbf{1}(Z < 0)\right] \geq \mathbb{E}\left[|-Z|\mathbf{1}(Z < 0)\right] = c > 0 \text{ for some } c \quad (0, \infty).$$

That is, $\mathbb{E}\left[|Z_n - Z|\mathbf{1}(Z < 0)\right] \geq c > 0$ for some $c \quad (0, \infty)$ for all $n \geq 1$ and, consequently, $\mathbb{E}\left[|Z_n - Z|\mathbf{1}(Z < 0)\right] \not\to 0$ as $n \to \infty$. Hence, $Z \quad K$. □

Lemma A.3. *Let $C := \left\{Z \quad L^1 \middle| \mathbb{E}\left[Z \exp\left(\overline{x}_T - \overline{x}_0\right)\right] = 1,\ \mathbb{E}\left[Z\right] = 1\right\}$. The set $C \quad L^1$ is closed.*

Proof. Let $\{Z_n\}$ be a sequence in C such that $Z_n \xrightarrow[n\to\infty]{L^1} Z$. Since $\{Z_n\}$ is convergent, it is also Cauchy. Since L^1 is complete, we have that $Z \quad L^1$.

We note that $Z_n \xrightarrow[n\to\infty]{L^1} Z$ is equivalent to

$$0 \leq \left|\mathbb{E}[Z_n] - \mathbb{E}[Z]\right| \leq \mathbb{E}\left[\left|Z_n - Z\right|\right] \xrightarrow[n\to\infty]{} 0.$$

Hence,

$$\mathbb{E}[Z] = \lim_{n\to\infty} \mathbb{E}[Z_n] = 1.$$

Since $Z_n \xrightarrow[n\to\infty]{L^1} Z$, we have that Z_n converges to Z in probability

$$Z_n \xrightarrow[n\to\infty]{p} Z$$

and, as a consequence, using the continuous mapping theorem,

$$\exp\left(\overline{x}_T - \overline{x}_0\right)Z_n \xrightarrow[n\to\infty]{p} \exp\left(\overline{x}_T - \overline{x}_0\right)Z.$$

Since $\mathbb{E}\left[\exp\left(\overline{x}_T - \overline{x}_0\right)Z_n\right] = 1$ for $n \geq 1$, we note that the sequence $\left\{\exp\left(\overline{x}_T - \overline{x}_0\right)Z_n\right\}$ is uniformly convergent. The convergence in probability and the uniform convergence of the sequence $\left\{\exp\left(\overline{x}_T - \overline{x}_0\right)Z_n\right\}$ are equivalent to the convergence in L^1

$$\exp\left(\overline{x}_T - \overline{x}_0\right)Z_n \xrightarrow[n\to\infty]{L^1} \exp\left(\overline{x}_T - \overline{x}_0\right)Z.$$

The convergence in L^1 above implies that

$$0 \leq \left|\mathbb{E}\left[\exp\left(\overline{x}_T - \overline{x}_0\right)Z_n\right] - \mathbb{E}\left[\exp\left(\overline{x}_T - \overline{x}_0\right)Z\right]\right| \leq \mathbb{E}\left[\left|\exp\left(\overline{x}_T - \overline{x}_0\right)Z_n - \exp\left(\overline{x}_T - \overline{x}_0\right)Z\right|\right] \xrightarrow[n\to\infty]{} 0.$$

Hence,

$$\mathbb{E}\left[Z\exp\left(\overline{x}_T - \overline{x}_0\right)\right] = \lim_{n\to\infty} \mathbb{E}\left[\exp\left(\overline{x}_T - \overline{x}_0\right)Z_n\right] = 1. \tag{38}$$

Equations (A) - (38) imply that $Z \ \ C$, and C is closed. ◻

Lemma A.4. *Let* $D := \left\{Z \ \ L^1 \middle| Z \overset{a.s.}{\geq} 0,\ \mathbb{E}[|Z\log Z|] < \infty\right\}$. *The set* D *is closed.*

Proof. Let $\{Z_n\}$ be a sequence in $D \ \ L^1$ such that $Z_n \xrightarrow[n\to\infty]{L^1} Z$. Since $\{Z_n\}$ is convergent, it also Cauchy. Since L^1 is complete, we have that $Z \ \ L^1$. Also, since $D \ \ K$, we have that $Z \overset{\text{a.s.}}{\geq} 0$ by Lemma A.2.

Next, since $Z_n \in D$ for $n \geq 1$, the sequence $\{Z_n \log Z_n\}$ is uniformly integrable.

Further, since $Z_n \xrightarrow[n\to\infty]{L^1} Z$, we have that $Z_n \xrightarrow[n\to\infty]{p} Z$ (in probability). By the continuous mapping theorem, we then have $Z_n \log Z_n \xrightarrow[n\to\infty]{p} Z \log Z$. By Vitali convergence theorem, we have that $Z_n \log Z_n \xrightarrow[n\to\infty]{L^1} Z \log Z$. Since $Z \log Z \in L^1$, we have that $\mathbb{E}[|Z \log Z|] < \infty$ and $Z \in D$. ◻

Lemma A.5. *The sublevel set* $S_\alpha := \left\{Z \in L^1 \middle| Z \overset{a.s.}{\geq} 0,\ \mathbb{E}\left[Z \log Z\right] \leq \alpha,\ \mathbb{E}[|Z \log Z|] < \infty\right\}$ *for some* $\alpha \in (0, \infty)$ *is closed.*

Proof. The proof is similar to the proof of Lemma A.4. Let $\{Z_n\}$ be a sequence in $S_\alpha \subseteq L^1$ such that $Z_n \xrightarrow[n\to\infty]{L^1} Z$. Since $\{Z_n\}$ is convergent, it also Cauchy. Since L^1 is complete, we have that $Z \in L^1$. Also, since $S_\alpha \subseteq K$ and K is closed by Lemma A.2, we have that $Z \overset{a.s.}{\geq} 0$.

Since $\{Z_n\} \subseteq S_\alpha \subseteq D$, we have that $\{Z_n \log Z_n\}$ is uniformly integrable.

On the other hand, since $Z_n \xrightarrow[n\to\infty]{L^1} Z$, we have that $Z_n \xrightarrow[n\to\infty]{p} Z$ by Vitali convergence theorem and $Z_n \log Z_n \xrightarrow[n\to\infty]{p} Z \log Z$ by the continuous mapping theorem.

Given that $\{Z_n \log Z_n\}$ is uniformly integrable and $Z_n \log Z_n \xrightarrow[n\to\infty]{p} Z \log Z$, we have that $Z_n \log Z_n \xrightarrow[n\to\infty]{L^1} Z \log Z$ by Vitali convergence theorem so that

$$0 \leq \left|\mathbb{E}\left[Z_n \log Z_n\right] - \mathbb{E}\left[Z \log Z\right]\right| \leq \mathbb{E}\left[\left|Z_n \log Z_n - Z \log Z\right|\right] \xrightarrow[n\to\infty]{L^1} 0$$

and

$$\mathbb{E}[Z_n \log Z_n] \xrightarrow[n\to\infty]{} \mathbb{E}[Z \log Z].$$

Since $\mathbb{E}[Z_n \log Z_n] \in [0, \alpha]$ for $n \geq 1$ and $[0, \alpha]$ is a closed set in $\mathbb{R}$ under usual topology, we have that $\mathbb{E}[Z \log Z] \in [0, \alpha]$:

$$\mathbb{E}[Z \log Z] = \lim_{n\to\infty} \mathbb{E}[Z_n \log Z_n] \leq \alpha.$$

Hence, $Z \in S_\alpha$. ◻

Lemma A.6. *Let* $f(\lambda) := \mathbb{E}[\exp(\lambda X)] < \infty$ *where* $\lambda \in (-\infty, +\infty)$ *and* X *is a random variable such that* $\mathbb{P}[X > 0] > 0$, $\mathbb{P}[X \leq 0] > 0$, *and* $\mathbb{P}[\omega \in \Omega \mid X(\omega) = 0] = 0$. *Then the* *equation*

$$f(\lambda + 1) = f(\lambda) \tag{39}$$

has the unique solution.

Proof. First we note that $f(\lambda) = \mathbb{E}[\exp(\lambda X)]$ is *strictly convex* in λ since $\exp(\lambda x)$ is strictly convex in λ for some fixed x.

Next we note that $f(\lambda)$ is *coercive*, i.e.,

$$\lim f(\lambda) \xrightarrow[|\lambda| \to \infty]{} \infty.$$

To see that $f(\lambda)$ is coercive, consider a sequence of $\lambda_n \xrightarrow[n \to \infty]{} \infty$. Using the convention $0 \cdot \infty = 0$, observe that

$$\exp(\lambda_n X(\omega))\mathbb{1}(X(\omega) > 0) \xrightarrow[n \to \infty]{\text{a.s.}} \begin{cases} \infty, & \text{if } \omega \in \{X > 0\}, \\ 0, & \text{otherwise}. \end{cases} \tag{40}$$

and that

$$\exp(\lambda_n X)\mathbb{1}(X \leq 0) \xrightarrow[n \to \infty]{\text{a.s.}} 0. \tag{41}$$

If we apply Fatou's lemma, we get

$$\mathbb{E}[\exp(\lambda_n X)] = \underbrace{\mathbb{E}[\exp(\lambda_n X)\mathbb{1}(X > 0)]}_{\to \infty \text{ as } n \to \infty} + \underbrace{\mathbb{E}[\exp(\lambda_n X)\mathbb{1}(X \leq 0)]}_{\to \geq 0 \text{ as } n \to \infty}. \tag{42}$$

If we consider a sequence $\lambda_n \xrightarrow[n \to \infty]{} -\infty$, the symmetric argument applies. Now, since $f(\lambda)$ is coercive, it has *bounded sublevel sets*.

Finally, $f(\lambda)$ is *lower semicontinuous* since its sublevel sets are closed. To see this, consider a sublevel set $S_\eta := \{\lambda \mid \mathbb{E}[\exp(\lambda X)] \leq \eta < \infty\}$. Also, consider a sequence $\lambda_n \xrightarrow[n \to \infty]{} \lambda$ such that $\lambda_n \in S_\eta$ for $n = 1, 2, 3, \ldots$. By continuous mapping theorem, $\exp(\lambda_n X) \xrightarrow[n \to \infty]{p} \exp(\lambda X)$. By Vitali convergence theorem, we have that $\exp(\lambda_n X) \xrightarrow[n \to \infty]{L^1} \exp(\lambda X)$:

$$0 \leq \left|\mathbb{E}[\exp(\lambda_n X)] - \mathbb{E}[\exp(\lambda X)]\right| \leq \mathbb{E}\left[\left|\exp(\lambda_n X) - \exp(\lambda X)\right|\right] \xrightarrow[n \to \infty]{} 0.$$

Since $f(\lambda)$ is lower semicontinuous, coercive and strictly convex, it has the unique global minimum. Since $f(\lambda)$ has the unique global minimum and all sublevel sets of $f(\lambda)$ are bounded, $f(\lambda)$ has the unique sublevel set of length equal to 1. ◻

We use the proposition below to compute the $\widetilde{M}$-SDF and the S-derivative in the stochastic volatility model in Section 3.

Proposition A.1. *Let $x_t := \log S_t$ and $f_t := f(t, x_t, V_t) = \mathbb{E}_t[\exp(u \log S_T)]$ where $u \in \mathbb{R}$. Then, under technical conditions*[5],

$$f_t = \exp(\alpha(\tau; u) + \beta(\tau; u)V_t + u x_t), \tag{43}$$

where $\tau := T - t$ and $\alpha(\tau)$ and $\beta(\tau)$ satisfy the following system of ordinary differential equations:

$$\begin{aligned} \frac{d\alpha}{d\tau} &= ru + \kappa\theta\beta(\tau), \\ \frac{d\beta}{d\tau} &= \tfrac{1}{2}\sigma^2\beta^2(\tau) + (\sigma\rho u - \kappa)\beta(\tau) + \left(\tfrac{1}{2}u^2 + \left(\eta_S - \tfrac{1}{2}\right)u\right), \\ s.t.\ \ \alpha(0) &= 0 \text{ and } \beta(0) = 0. \end{aligned} \tag{44}$$

Moreover, let

$$\begin{aligned} a(u) &:= -u^2 - 2u\left(\eta_S - \tfrac{1}{2}\right), \\ b(u) &:= u\sigma\rho - \kappa, \\ d(u) &:= \sqrt{b(u)^2 + a(u)\sigma^2}. \end{aligned} \tag{45}$$

[5] The technical conditions are:

(i) $\mathbb{E}\left[\int_0^T (u \exp(\alpha(\tau;u) + \beta(\tau;u)V_t + ux_t))^2 V_t dt\right] < \infty$,

(ii) $\mathbb{E}\left[\int_0^T (\beta(\tau;u) \exp(\alpha(\tau;u) + \beta(\tau;u)V_t + ux_t))^2 V_t dt\right] < \infty$,

(iii) $\mathbb{E}[\exp(\alpha(\tau;u) + \beta(\tau;u)V_t + ux_t)] < \infty$ for all $t \in [0, T]$.

If $b(u)^2 + a(u)\sigma^2 > 0$, *then*

$$\alpha(\tau,u) = ur\tau - \tfrac{\kappa\theta}{\sigma^2}\left[\tau\left(d(u)+b(u)\right) + 2\log\left[1 - \tfrac{d(u)+b(u)}{2d(u)}\left(1-\exp\left(-d(u)\tau\right)\right)\right]\right],$$
$$\beta(\tau,u) = -\frac{a(u)\left[1-\exp\left(-d(u)\tau\right)\right]}{2d(u) - \left(d(u)+b(u)\right)\left[1-\exp\left(-d(u)\tau\right)\right]}. \tag{46}$$

Du□e et al. (2000) provide the proof of Proposition A.1 in a more general setting.

Proof. Using Ito's lemma, we can write $f_t = f(t, x_t, V_t) = \mathbb{E}_t\left[\exp\left(u \log S_T\right)\right]$ as

$$\begin{aligned} f(T,x_T,V_T) - f(0,x_0,V_0) = & \int_0^T \frac{\partial f}{\partial t}dt + \int_0^T \frac{\partial f}{\partial x}\left[\left(r + \eta_S V_t - \tfrac{1}{2}V_t\right)dt + \sqrt{V_t}dW_{S,t}\right] + \\ & + \int_0^T \frac{\partial f}{\partial V}\left[\kappa\left(\theta - V_t\right)dt + \sigma\sqrt{V_t}\left(\rho dW_{S,t} + \sqrt{1-\rho^2}dW_{V,t}\right)\right] + \\ & + \int_0^T \frac{1}{2}\frac{\partial^2 f}{\partial x^2}V_t dt + \int_0^T \frac{1}{2}\frac{\partial^2 f}{\partial V^2}\sigma^2 V_t dt + \int_0^T \frac{\partial^2 f}{\partial x \partial V}V_t\sigma\rho dt. \end{aligned} \tag{47}$$

If we separate regular integrals from stochastic ones in equation (47), we get

$$\begin{aligned} & f(T,x_T,V_T) - f(0,x_0,V_0) = \\ & \int_0^T \underbrace{\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\left(r + \eta_S V_t - \tfrac{1}{2}V_t\right) + \frac{\partial f}{\partial V}\kappa\left(\theta - V_t\right) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}V_t + \frac{1}{2}\frac{\partial^2 f}{\partial V^2}\sigma^2 V_t + \frac{\partial^2 f}{\partial x \partial V}V_t\sigma\rho\right]}_{\text{f is a local martingale} \Leftrightarrow\ =0} dt \\ & + \int_0^T \frac{\partial f}{\partial x}\sqrt{V_t}dW_{S,t} + \int_0^T \frac{\partial f}{\partial V}\sigma\sqrt{V_t}\left(\rho dW_{S,t} + \sqrt{1-\rho^2}dW_{V,t}\right). \end{aligned} \tag{48}$$

Since f is a local martingale, we get the following PDE:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\left(r + \eta_S V - \tfrac{1}{2}V\right) + \frac{\partial f}{\partial V}\kappa\left(\theta - V\right) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}V + \frac{1}{2}\frac{\partial^2 f}{\partial V^2}\sigma^2 V + \frac{\partial^2 f}{\partial x \partial V}V\sigma\rho = 0, \tag{49}$$

$$\text{s.t. boundary condition } f(T,x,V) = \exp\left(ux\right), \tag{50}$$

$$t \quad [0,T],\ x \quad (-\infty,\infty) \text{ and } V \quad [0,\infty). \tag{51}$$

Our guess for the solution of the PDE in equation (49) is

$$f(t,x,V) = \exp\left(A(t) + B(t)V + ux\right), \tag{52}$$

where $A(\cdot)$ and $B(\cdot)$ are functions of $t \quad [0,T]$.

To transform the boundary condition (50) into the initial condition, we note that $t = T - T + t = T - (T - t) = T - \tau$ where $\tau := T - t$. Therefore, given $f(t,x,V)$ in equation (52), there exists

$$\hat{f}(\tau,x,V) = \exp\left(\alpha(\tau) + \beta(\tau)V + ux\right) \tag{53}$$

such that

$$\begin{gathered}\hat{f}(\tau,x,V) = f(t,x,V) \text{ for all } t \quad [0,T] \text{ and the corresponding } \tau := T - t,\\ x \quad (-\infty,\infty), \text{ and } V \quad [0,\infty),\end{gathered} \tag{54}$$

i.e.,

$$\alpha(\tau) = A(t),\ \beta(\tau) = B(t) \text{ for all } t \quad [0,T] \text{ and the corresponding } \tau := T - t. \tag{55}$$

Note that, since $\hat{f} = f$, we have

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial \hat{f}}{\partial t} = \frac{\partial \hat{f}}{\partial \tau}(-1),\\ \frac{\partial f}{\partial x} &= \frac{\partial \hat{f}}{\partial x},\\ \frac{\partial f}{\partial V} &= \frac{\partial \hat{f}}{\partial V}.\end{aligned} \tag{56}$$

Since $\hat{f}(\tau,x,V) = f(t,x,V)$, the PDE in equation (49) becomes

$$\frac{\partial \hat{f}}{\partial t} + \frac{\partial \hat{f}}{\partial x}\left(r + \eta_S V - \tfrac{1}{2}V\right) + \frac{\partial \hat{f}}{\partial V}\kappa(\theta - V) + \frac{1}{2}\frac{\partial^2 \hat{f}}{\partial x^2}V + \frac{1}{2}\frac{\partial^2 \hat{f}}{\partial V^2}\sigma^2 V + \frac{\partial^2 \hat{f}}{\partial x \partial V}V\sigma\rho = 0, \tag{57}$$

$$\text{s.t. initial condition } \hat{f}(0,x,V) = \exp\left(ux\right), \tag{58}$$

$$t \quad [0,T],\ x \quad (-\infty,\infty) \text{ and } V \quad [0,\infty). \tag{59}$$

Using the definition of $\hat{f}$ in equation (53), the PDE (58) simplifies to

$$\underbrace{\Big[-\alpha\,(\tau) + ur + \beta(\tau)\kappa\theta \Big]}_{=0} +$$
$$\underbrace{\left[-\beta\,(\tau) + u\left(\eta_S - \tfrac{1}{2}\right) - \kappa\beta(\tau) + \tfrac{1}{2}u^2 + \tfrac{1}{2}\beta(\tau)^2\sigma^2 + u\beta(\tau)\sigma\rho\right]}_{=0} V = 0 \tag{60}$$
$$\text{s.t. the initial conditions } \alpha(0) = 0 \text{ and } \beta(0) = 0 \text{ and } V \quad [0, \infty).$$

The simplified PDE in equation (60) implies the system of ODEs:

$$\alpha\,(\tau) = ur + \kappa\theta\beta(\tau), \tag{61}$$
$$\beta\,(\tau) = \tfrac{1}{2}\sigma^2\beta^2(\tau) + (u\sigma\rho - \kappa)\,\beta(\tau) + \left[u\left(\eta_S - \tfrac{1}{2}\right) + \tfrac{1}{2}u^2\right], \tag{62}$$
$$\text{s.t. } \alpha(0) = 0 \text{ and } \beta(0) = 0. \tag{63}$$

Above, equation (62) is a Riccati equation, and we can find a solution (see equation (46) in Proposition A.1) for it if we proceed with the guess:

$$\beta(\tau) = k\frac{g\,(\tau)}{g(\tau)}, \text{ where } k \quad (-\infty, \infty). \tag{64}$$

Finally, since $f(t, x_t, V_t) = \hat{f}(\tau, x_t, V_t) = \mathbb{E}_t\left[\exp\left(u \log S_T\right)\right]$, conditions

(i) $\mathbb{E}\left[\int_0^T \left(u \exp\left(\alpha(\tau; u) + \beta(\tau; u)V_t + ux_t\right)\right)^2 V_t dt\right] < \infty$,

(ii) $\mathbb{E}\left[\int_0^T \left(\beta(\tau; u) \exp\left(\alpha(\tau; u) + \beta(\tau; u)V_t + ux_t\right)\right)^2 V_t dt\right] < \infty$,

insure that $\hat{f}(\tau, x_t, V_t) = f(t, x_t, V_t)$, being a local martingale by construction, is in fact a martingale.

□

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