Universität Trier

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Ludwig von Auer

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by
Ludwig von Auer*
Universität Trier, Germany

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Abstract: Textbooks of macroeconomics regularly remind their readers that they should not interpret the macroeconomic price variable as some sort of average price. Instead it represents some price index indicating the *average of the individual items' price changes* between the period considered and some base period. The Laspeyres and the Paasche index are cited as the best known examples of this approach.

The present study challenges this tradition. It develops the family of *generalised unit value indices.* They do not average the individual items' price changes but relate the average price of the period considered to that of the base period. It is shown that a wide range of index formulas can be interpreted in this way, including the Laspeyres and the Paasche index. The study also provides an axiomatic comparison between the generalised unit value indices and several traditional price indices (e.g., Laspeyres index, Paasche index, Fisher index).

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1   Introduction

Macroeconomic models usually feature some price variable. Intuitively one would want to interpret this price variable as a sort of *average price level* prevailing during the respective period. However, textbooks of macroeconomics regularly remind their readers that they must not follow their intuition. Instead, the price variable represents some *price index* indicating the average of the price changes of the individual items (goods and services) between the period considered and some base period. Following Fisher's (1911, pp. 198-203) seminal work, it is usually recommended to view the price variable as a Paasche price index. This interpretation is also reflected in the construction of many national accounts. There, the change in nominal GDP is equal to the change in real GDP multiplied by the economy's overall price change. For practical reasons, the change in real GDP is usually measured by a Laspeyres quantity index. As a consequence, the overall price change is measured by a Paasche price index.

For measuring the overall price change of their economy, some national statistical institutes (e.g., Germany's) also compute a Laspeyres index. In the literature, many other price indices have been proposed and a few of them have been adopted by national statistical institutes (e.g., Sweden's computes a Walsh index). Even though the views on the best price index vary, there is widespread agreement that a sensible price index should be such that it can be formulated as a weighted average of the individual items' price changes. This approach to price measurement is denoted here as the *APC-approach* (short for "Average of Price Changes"). Both the Laspeyres and the Paasche index are consistent with this approach.

However, an important alternative to the APC-approach exists. If, against all textbook recommendations, one were to interpret the price variable of macroeconomic models as some sort of average price level, one would want to measure the overall price change as the ratio of the average price level of the period considered and the average price level of some base period. This approach is denoted here as the *CPL-approach* (short for "Change in Price Levels").

This paper develops a new class of price indices that all follow the CPL-approach. This class is denoted as the family of *generalised unit value indices*. It is shown that the Laspeyres and the Paasche index are members of this index family, and therefore, consistent also with the CPL-approach. Furthermore, it is demonstrated that some members of the family of generalised unit value indices can compete with the most highly regarded price indices developed in the context of the APC-approach. As a consequence, macroeconomic textbooks can take a more relaxed view on the interpretation of the price variable in macroeconomic models. There is nothing wrong with considering this variable as some sort of average price level.

The paper proceeds as follows. Section 2 provides a brief and critical discussion of the price statistical objections raised against the notion of price levels, and therefore, against the CPL-approach. Beginning with Section 3, the paper proceeds to develop price index formulae that are based on the change of price levels. The starting point is the *unit value index* proposed by Drobisch (1871a, p. 39). However, for measuring the change in a price level, this simple index formula cannot be directly applied. Therefore, in Section 3 the concept of the *amended unit value index* is introduced. This type of price index can be employed in the context of homogeneous or almost

homogeneous items, though not in the context of heterogeneous items. The latter requires an additional refinement. This refinement leads to the family of *generalised unit value indices.* The basic idea and the definition of this index family is presented in Section 4. Section 5 provides a more detailed discussion of this family of price indices. A comparison to some of the traditional index formulae (e.g., Fisher index) can be found in Section 6. The axiomatic properties of the generalised unit value indices are explored in Section 7 and are compared to those of the most popular traditional price indices. Section 8 provides some remarks on promising areas of future research.

2   Fisher's Reservations against the CPL-Approach

The purpose of a price index is to measure the overall price change between a base period $t = 0$ and a comparison period $t = 1$. Let p_i^t denote the price of item i observed in period t. Similarly, let x_i^t denote the quantity of this item purchased in period t. It is assumed that in both periods the same N items are sold in the marketplace. Following the APC-approach, one measures the overall price change by first computing the N items' individual price changes and then averaging these price changes to obtain some overall price change. As an alternative to the APC-approach, one could compute the price levels of the base period and the comparison period and from the ratio of these two price levels the overall price change. This is the CPL-approach and it necessitates to compute separate price levels for the two periods to be compared. However, in his seminal book on price statistics, Fisher (1922) firmly rejects the notion of a price level. He acknowledges that it can be computed but that it "*... is apt, in general, to prove a delusion and a snare. The reason is that an average of prices of wheat, coal, cloth, lumber, etc. is an average of incommensurables and therefore has no fixed numerical value* (p. 451)."

Of course, Fisher is correct in saying that one cannot assign a meaningful numerical value to a price level when this price level is looked at in isolation. However, this is no justification to discard the notion of a price level. The issue is a familiar one from microeconomic price theory. An item's price looked at in isolation is meaningless. However, defining one item as the numeraire good and measuring the prices of all other items in terms of this numeraire good, gives all these prices a meaningful interpretation. Similarly, one could define some period as the "numeraire period." Usually, this period is the base period. The value of the price level of some comparison period can be interpreted relative to the price level of the base period. Defining a suitable price index formula along these lines, one obtains a measure for the percentage change in the price level between the base period and the comparison period. Such a measure is far from meaningless and it is based on the CPL-approach instead of the APC-approach.

Half a century after Fisher had published his reservations against price level measurement, his view was supported by a formal argument from axiomatic index theory. This formal argument can be found in Eichhorn and Voeller (1976, pp. 75-78). Very similar axiomatic objections are presented in Eichhorn (1978, pp. 144-146), Diewert (1993, pp. 7-9), and ILO *et al.* (2004, p. 292). In Auer (2009a) it is demonstrated that all these axiomatic objections are not convincing.

Therefore, this study takes a fresh look at developing price index formulae that follow the CPL-approach. In this context, it develops the family of generalised unit value indices and it demonstrates that the Laspeyres and the Paasche index are members of this index family. Another member of this family is the Banerjee (1997, p. 27) index:

$$P_B = \frac{V^1}{V^0} \frac{(V^0 + V^{10})}{(V^1 + V^{01})} \ , \tag{1}$$

where $V^t = \sum p_i^t x_i^t$ and $V^{st} = \sum p_i^s x_i^t$. This index is almost identical to the Fisher index, since the latter can be expressed in the form

$$P_F = \frac{V^1}{V^0} \frac{\sqrt{V^0 V^{10}}}{\sqrt{V^1 V^{01}}} \ . \tag{2}$$

The only difference between the Fisher index (2) and the Banerjee index (1) is the method of averaging the values V^t and V^{st}. Where the Fisher index uses a geometric mean, the Banerjee index uses an arithmetic mean. Since the Fisher index is often praised as the best existing price index formula, the Banerjee index (1) and the other members of the family of generalised unit value indices deserve unprejudiced consideration.

3   Amended Unit Value Index

As pointed out above, the current literature favours price indices consistent with the APC-approach. Many of the most popular price indices (e.g., Walsh index) are exclusively embedded in the APC-approach. Some price indices exist that are in line with both the APC- and the CPL-approach. As will be demonstrated in Section 5, the Laspeyres and the Paasche index belong to this class of price indices. There are some price indices which can neither be associated with the APC-approach nor with the CPL-approach (e.g., Stuvel index).

It is useful to begin the analysis with a particularly simple case of price measurement: All price and quantity observations of the base and comparison period refer to the prices and quantities of homogeneous items. Fisher (1923, p. 743) acknowledges that for this case not only the APC- but also the CPL-approach could be followed. Official price statistics, as documented in ILO *et al.* (2004, p. 164), explicitly recommend the CPL-approach as the best method to aggregate the prices of homogeneous items. More specifically, it is suggested to compute for each period separate unit values P_{UV}^t as defined by Segnitz (1870, p. 184):

$$\begin{aligned} P_{UV}^t &= \left(\sum p_i^t x_i^t\right) \Big/ \left(\sum x_i^t\right) \\ &= V^t \Big/ \left(\sum x_i^t\right) \\ &= \sum \left[x_i^t \Big/ \left(\sum x_j^t\right) \right] p_i^t \ . \end{aligned} \tag{3}$$

To calculate the overall price change between the base period $t = 0$ and the comparison period $t = 1$ one can use Drobisch's (1871a, p. 39; 1871b, p. 149) unit value

index. Using Equation (3), this price index can be expressed as

$$P_{UV} = \frac{P_{UV}^1}{P_{UV}^0} = \frac{V^1}{V^0} \frac{\sum x_i^0}{\sum x_i^1} . \tag{4}$$

An axiomatic justification for the use of the unit value index can be found in Auer (2009b).

Price index (4) was recommended for the case in which the overall price change is calculated from price observations that all relate to homogeneous items. For this case the unit value index can be viewed as a "first best solution". The unit value index and, as a consequence, also the CPL-approach also lends itself for the case of "almost homogeneous items", that is, for items that differ only with respect to the location of purchase and the moment of purchase within period t. Of course, an amendment of the basic unit value index becomes necessary, when the items exhibit more significant heterogeneity. For example, the items could differ with respect to the size of their packages. In order to make the observations comparable, the unit value index must be amended by transformation factors z_i. The value of each item's transformation factor z_i depends on the item's package size.

An example may illustrate the idea and the mechanics of the amended unit value index. Suppose that two items are considered. The first is a chocolate bar of 200g and the other is a chocolate bar of 300g. It is assumed that apart from their weights no differences exist between the two items. Consumers are indifferent between consuming two bars of 300g and three bars of 200g. Also producers are indifferent between producing two bars of 300g and three bars of 200g. The transformation factors are denoted by z_{big} and z_{small}. The observed prices and quantities are stated in Table 1.

Table 1: Numerical Example for the Mechanics of the Amended Unit Value Index.

	base period		comparison period	
	price	quantity	price	quantity
big chocolate bar (300g)	8	2	8	3
small chocolate bar (200g)	4	2	6	2

One can choose either an arbitrary number for the transformation factor z_{big} or for the transformation factor z_{small}. The relationship between z_{big} and z_{small}, however, is determined by the weight ratio of the two types of chocolate bars:

$$\frac{z_{big}}{z_{small}} = \frac{3}{2} . \tag{5}$$

One could interpret the ratio of transformation factors as the items' exchange ratio, since two units of 300g-bars are equivalent to three units of 200g-bars. In the following, the ratios of transformation factors are denoted as "ratios of equivalence."

How are the transformation factors incorporated into the unit value formula? The appropriate definition is

$$P_{AUV}^t = \left[\sum (p_i^t/z_i) x_i^t z_i \right] \Big/ \left(\sum x_i^t z_i \right) \tag{6}$$

$$= V^t \Big/ \left(\sum x_i^t z_i \right) . \tag{7}$$

In the chocolate example, choosing a number for z_{big} automatically determines the corresponding number for z_{small}. For $z_{big} = 3$, and therefore $z_{small} = 2$, formulae (6) and (7) would translate all original price and quantity data (p_i^t and x_i^t) into prices and quantities related to 100g-units of chocolate. The transformed quantity $x_{small}^0 z_{small} = 2 \cdot 2 = 4$ is the number of 100g-units sold during the base period in the form of 200g-bars. Correspondingly, the transformed price $p_{small}^0 / z_{small} = 4/2 = 2$ is the base period price of a 100g-unit purchased in the form of a 200g-bar. In the comparison period, this price would be $p_{small}^1 / z_{small} = 6/2 = 3$. Using the transformation factors, also the amended unit values P_{AUV}^0 and P_{AUV}^1 relate to a 100g-unit. Formula (7) produces the values

$$P_{AUV}^1 = \frac{\sum p_i^1 x_i^1}{\sum x_j^1 z_j} = \frac{36}{2 \cdot 2 + 3 \cdot 3} = 2.769$$

$$P_{AUV}^0 = \frac{\sum p_i^0 x_i^0}{\sum x_j^0 z_j} = \frac{24}{2 \cdot 2 + 2 \cdot 3} = 2.4 \ .$$

The *amended unit value index (AUV)* is defined as

$$P_{AUV} = \frac{P_{AUV}^1}{P_{AUV}^0} \ . \tag{8}$$

Using formula (7), the amended unit value index (8) can be expressed in the form

$$P_{AUV} = \frac{V^1}{V^0} \frac{\sum x_i^0 z_i}{\sum x_i^1 z_i} \ . \tag{9}$$

With identical package sizes, all transformation factors would have the same value: $z_i = z$. In this case the amended unit value index (9) would simplify to the unit value index (4). Formula (9) reveals that the transformation factors affect only the second ratio on the right hand side and that these factors are irrelevant when all quantities remain constant over time.

As previously stated, one transformation factor can be arbitrarily chosen. As long as the correct ratio of transformation factors is used, formula (8) applied to the chocolate example always generates the value

$$P_{AUV} = \frac{P_{AUV}^1}{P_{AUV}^0} = 1.154 \ .$$

This number indicates that the overall price level of chocolate bars has increased by 15.4 per cent.

As an alternative one could have used the transformation factors $z_{big} = 3/10$ and $z_{small} = 2/10$. These factors translate all prices and quantities into 1000g-units of chocolate. This is also the total weight of the base period's basket of items. Another admissable variant one obtains when the purchasing power of a unit of money during the base period is used as a reference. The total monetary value of the base period basket is $V^0 = 24$. Since the total weight of the basket is 1000g, one unit of money buys $1000g/24 \approx 41.66g$. For

$$z_{big} = 24(3/10) = 7.2$$

prices and quantities relate to 41.66g-units. Correspondingly, Equation (5) yields

$$z_{small} = (2/3)24(3/10) = 4.8 \ .$$

The transformed price $p^0_{small}/z_{small} = 4/4.8 = 5/6$ is the base period price of a 41.66g-unit of chocolate purchased in the form of a 200g-bar. Again the amended unit value index generates the value $P_{AUV} = 1.154$.

4 Generalised Unit Value Index

The ratios of equivalence (i.e., the ratio of transformation factors) affect the value of the amended unit value index P_{AUV}. In the chocolate example, the ratios were directly determined by the weight ratio of the two items. How could one proceed, however, when the weight ratio of the small and the large chocolate bars were unknown? Obviously, a "first best solution" no longer exists. One has to pursue a "second best solution".

Is it possible to find such a solution in the context of the CPL-approach? Such a proposal can be found in an early contribution by Davies (1924, pp. 183-185). Unfortunately, his suggestion did not receive the attention it deserved. Instead, price statisticians, following Irving Fisher's lead, discarded the CPL-approach and pursued a "second best solution" in the context of the APC-approach. The reasons for this change in paradigm were outlined in Section 2. There it was argued that the reasons given in the literature are not compelling. Therefore, this study undertakes a new attempt at developing a "second best solution" in the context of the CPL-approach.

The amended unit value index with its transformation factors, z_i, serves as a natural starting point for such a solution. Unfortunately, having no information about the weight ratios, the ratios of equivalence are also unclear. In order to solve this problem, one could try to obtain reasonable estimates of these ratios. This approach defines a new class of price indices. Accordingly, this class is denoted as the family of *generalised unit value indices (GUV)*:

$$P_{GUV} = \frac{P^1_{GUV}}{P^0_{GUV}} = \frac{\left[\sum(p^1_i/\hat{z}_i)x^1_i\hat{z}_i\right]/\left(\sum x^1_i\hat{z}_i\right)}{\left[\sum(p^0_i/\hat{z}_i)x^0_i\hat{z}_i\right]/\left(\sum x^0_i\hat{z}_i\right)} = \frac{V^1}{V^0}\frac{\sum x^0_i\hat{z}_i}{\sum x^1_i\hat{z}_i} \ . \tag{10}$$

The formal structure of this class of price indices is identical to the amended unit value index P_{AUV} defined in Equation (8) and expressed in even simpler form in Equation (9). This is the reason why the amended unit value index was introduced in the first place. The only difference is the "hat" appearing on all transformation factors. The hat emphasises that in the generalised unit value index the values of the transformation factors are not given by known weight ratios but must be estimated.

Formula (10) is not novel. It can be found also in Haan (2001, p. 24) for two items. Subsequently, Dalén (2001, p. 11) extended it to the multi-item case. In these studies, the formula was motivated by the problem of aggregating the prices of products of similar quality into some average price change. Accordingly, the formula was labeled as "quality-adjusted unit value." Both studies point out that the estimation of the $\hat{z}_i$-values ideally would be based on hedonic regression techniques.

These necessitate additional information such as the qualitative characteristics of the items.

In the present study, however, it is assumed that besides prices p_i^t and quantities x_i^t no additional information is available. Is it nevertheless possible to generate reasonable estimates of $\hat{z}_i$? Following the strategy pursued in the development of the amended unit value index P_{AUV}, one could again try to transform the original price and quantity data (p_i^t and x_i^t) into price and quantity data of an almost homogeneous item. For this purpose one needs a reasonable estimate of the ratio of equivalence of each pair of items. Since the physical weights of the items are no longer known, the only information available are the original price and quantity data. The estimates must be based on these data alone.

How this can be accomplished, will be described in the following section. At the current stage it is instructive to have a closer look at the nature of the informational deficiency of the chocolate example (with unknown weights). The set of information consists of the prices and quantities of two items. One item is called "big chocolate bar" and the other "small chocolate bar". No additional information exists. Because the weights are not known, the ratio of equivalence between the two items (that is, the ratio of their transformation factors) is unclear. However, not knowing the ratio of equivalence between "big chocolate bars" and "small chocolate bars" is in no way different from not knowing the ratio of equivalence of any other pair of items ("fuel" and "small chocolate bars", say). In other words, the two items "big chocolate bars" and "small chocolate bars" can be viewed as two heterogeneous items.

Therefore, the proposed method of overcoming the informational deficiency of the chocolate example (with unknown weights) can be equally applied when dealing with any type of heterogeneous items: To change the heterogeneous items into comparable units one can use estimated transformation factors. The ratio of each pair of estimated transformation factors reflects the heterogeneous items' ratio of equivalence. The estimations of the transformation factors must be exclusively based on the items' prices and quantities. How this can be accomplished is the object of the next section.

5 Estimation of the Ratios of Equivalence

In the chocolate example, the base period price of the large chocolate bar is twice as large as the base period price of the small chocolate bar (the price ratio is 8:4). Therefore, one could suspect that the items' ratio of equivalence is also 8:4 = 2. Following this line of reasoning, the ratio of the transformation factors would be

$$\frac{\hat{z}_{big}}{\hat{z}_{small}} = \frac{p_{big}^0}{p_{small}^0} = 2 \ . \tag{11}$$

Accordingly, one could replace in Equation (10) all transformation factors by the respective base period prices: $\hat{z}_i = p_i^0$. Interestingly, the resulting generalised unit value index is the Paasche index (P_P):

$$P_{GUV} = \frac{V^1}{V^0} \frac{\sum p_i^0 x_i^0}{\sum p_i^0 x_i^1} = \frac{\sum p_i^1 x_i^1}{\sum p_i^0 x_i^1} = P_P \ . \tag{12}$$

In other words, the Paasche index is a generalised unit value index with transformation factors estimated from the prices of the base period only. Therefore, the Paasche index is consistent with the CPL-approach.

In Equation (11), the ratios of equivalence have been derived from the price ratios of the base period only. This variant is closely related to a proposal of Haan (2001, p. 24). He suggests the consideration of a period in which the two items are available on the market and are preferably in equilibrium. The respective price ratio of the two items can be used as an estimator for the ratio of the items' transformation factors.

However, when the items are available in both the base and the comparison period (as is usually assumed in index theory), then, instead of looking at the base period prices, one could also look at the prices prevailing during the comparison period. In the chocolate example, the price ratio is 8:6. The corresponding ratio of equivalence would be

$$\frac{\hat{z}_{big}}{\hat{z}_{small}} = \frac{p^1_{big}}{p^1_{small}} = \frac{4}{3} \ . \tag{13}$$

Accordingly, one could replace in Equation (10) all transformation factors by the respective comparison period prices: $\hat{z}_i = p^1_i$. The resulting generalised unit value index is the Laspeyres index (P_L):

$$P_{GUV} = \frac{V^1}{V^0} \frac{\sum p^1_i x^0_i}{\sum p^1_i x^1_i} = \frac{\sum p^1_i x^0_i}{\sum p^0_i x^0_i} = P_L \ . \tag{14}$$

This demonstrates that also the Laspeyres index is consistent with the CPL-approach. It can be interpreted as the generalised unit value index which estimates its transformation factors exclusively from the prices of the comparison period.

The generalised unit value indices (12) and (14) produce different results. One approach to deal with such ambiguities is to average the two index formulae. Taking the arithmetic mean of estimates (12) and (14) yields the Drobisch index,

$$P_D = (P_L + P_P)/2 \ ,$$

and taking the geometric mean yields the Fisher index,

$$P_F = \sqrt{P_L P_P} \ .$$

The problem of ambiguity is well familiar from the APC-approach. There, the ambiguity exists with respect to the appropriate weights to be used for averaging the individual price ratios. In response to this problem, the weights are usually computed from the data of both periods and not just from one period. The same principle can be used in the context of the CPL-approach.

The calculation of the generalised unit value index (10) requires a single number for each ratio of equivalence (i.e., each ratio of transformation factors). If the prices of the chocolate bars were in both periods directly proportional to their physical weights, the values generated by (11) and (13) would both produce the same number. However, in the example considered, there is no such proportionality. One way to obtain a single number for each ratio of equivalence is to average the ratios (11)

and (13) in one way or the other. Taking the arithmetic mean (known as the Carli index) would generate the ratio

$$\frac{\hat{z}_{big}}{\hat{z}_{small}} = \left(\frac{p^0_{big}}{p^0_{small}} + \frac{p^1_{big}}{p^1_{small}} \right) \frac{1}{2} = \frac{5}{3} = 1.667 \ . \tag{15}$$

Therefore, in formula (10), one could use $\hat{z}_{big} = 1.667$ and $\hat{z}_{small} = 1$. Taking the geometric mean (Jevons index) would generate the ratio

$$\frac{\hat{z}_{big}}{\hat{z}_{small}} = \sqrt{\frac{p^0_{big}}{p^0_{small}} \frac{p^1_{big}}{p^1_{small}}} = 1.633 \tag{16}$$

and taking the harmonic mean would generate the ratio

$$\frac{\hat{z}_{big}}{\hat{z}_{small}} = 2 \left(\frac{p^0_{small}}{p^0_{big}} + \frac{p^1_{small}}{p^1_{big}} \right)^{-1} = 1.6 \ .$$

As an alternative to averaging the price ratios (11) and (13), one could also average the prices of one item over time and relate this average value to the corresponding average value of the other item. Using an arithmetic mean would be equivalent to calculating a Dutot index:

$$\frac{\hat{z}_{big}}{\hat{z}_{small}} = \frac{p^0_{big} + p^1_{big}}{p^0_{small} + p^1_{small}} = \frac{8+8}{4+6} = 1.6 \ .$$

Taking the harmonic average would also be possible. Obviously, deciding for a geometric average would again generate the result (16).

One should notice that for the value of the generalised unit value index it is irrelevant whether the transformation factors $\hat{z}_{big} = 1.633$ and $\hat{z}_{small} = 1$ are used or some uniform proportional transformation of these values. For example, one could use $\hat{z}_{big} = 8$ and $\hat{z}_{small} = 4.899$. In this last variant, each item's transformation factor is computed from the geometric mean of the items' prices:

$$\hat{z}_{big} = \sqrt{p^0_{big} \cdot p^1_{big}} = 8 \quad \text{and} \quad \hat{z}_{small} = \sqrt{p^0_{small} \cdot p^1_{small}} = 4.899 \ .$$

Closer inspection of the price index formula proposed by Davies (1924, p. 185) reveals that it boils down to this last variant.

In this section, seven different representatives of the family of generalised unit value indices have been developed. Each of these members generates its own price index value. It is a common feature of the seven members that the estimation of the transformation factors z_i is exclusively based on the items' prices but not on their quantities. The quantities purchased are relevant in formula (10) but not in the estimation of the transformation factors. It is not difficult to develop a family member that takes account of these quantities. For averaging the base and comparison period prices of an item, one could calculate the item's unit value. In the chocolate example, this approach gives the ratio

$$\frac{\hat{z}_{big}}{\hat{z}_{small}} = \frac{\left(p^0_{big} x^0_{big} + p^1_{big} x^1_{big} \right) / \left(x^0_{big} + x^1_{big} \right)}{\left(p^0_{small} x^0_{small} + p^1_{small} x^1_{small} \right) / \left(x^0_{small} + x^1_{small} \right)} = \frac{8}{5} = 1.6 \ . \tag{17}$$

6 Comparison to Some Traditional Price Indices

In this study it has been shown that the Paasche index (P_P) and the Laspeyres index (P_L) can be reformulated as generalised unit value indices that estimate the ratios of equivalence from the prices of one period only. In the chocolate example, the index numbers produced by these traditional price indices are

$$P_P = \frac{\sum p_i^1 x_i^1}{\sum p_i^0 x_i^1} = 1.125$$

$$P_L = \frac{\sum p_i^1 x_i^0}{\sum p_i^0 x_i^0} = 1.167 \ .$$

The more sophisticated members of the family of generalised unit value indices represent alternatives to traditional price indices such as the Fisher (P_F), the Marshall-Edgeworth (P_{ME}), and the Walsh index (P_W). In the chocolate example, the index numbers produced by these traditional price indices are

$$P_F = \sqrt{P_L P_P} = 1.146$$

$$P_{ME} = \frac{\sum p_i^1 \left(x_i^0 + x_i^1\right)/2}{\sum p_i^0 \left(x_i^0 + x_i^1\right)/2} = 1.143$$

$$P_W = \frac{\sum p_i^1 \sqrt{x_i^0 x_i^1}}{\sum p_i^0 \sqrt{x_i^0 x_i^1}} = 1.145 \ .$$

The five listed price index numbers deviate from the result produced by the amended unit value index ($P_{AUV} = 1.154$) which, knowing the chocolate bars' correct weight ratio, could be viewed as a "first best solution" and therefore be used as a reference for evaluating other price indices.

Of course, the generalised unit value indices also produce results that deviate from the result of the amended unit value index. The source of the deviation can be seen more clearly, when one expresses the family of generalised unit value indices defined by Equation (10) in the following form:

$$P_{GUV} = \frac{V^1}{V^0} \sum \frac{x_i^1 \hat{z}_i}{\sum x_j^1 \hat{z}_j} \left(\frac{x_i^0}{x_i^1}\right) \ .$$

The first factor on the right hand side of this equation is the value ratio V^1/V^0. The second factor is a weighted average of the quantity ratios (x_i^0/x_i^1). In the chocolate example, the ratio of equivalence $\hat{z}_{big}/\hat{z}_{small}$ is too large when compared with the ratio obtained from the correct but unknown weight ratio. As a consequence, in the weighted average, the large chocolate bars' weight is too large relative to that of small chocolate bars. Since the quantity of large chocolate bars increased over time and the quantity of small chocolate bars remained constant over time, the generalised unit value index produces a smaller value than the amended unit value index. Using formula (16), the chocolate example yields

$$P_{GUV} = 1.145 \ .$$

Formula (17) produces the value

$$P_{GUV} = 1.147 \,.$$

A striking feature of the listed index numbers is the small deviation between the results produced by the sophisticated members of the family of generalised unit value indices (i.e., excluding Laspeyres and Paasche index) and the Fisher, Marshall-Edgeworth, and Walsh index. All these price index formulae produce numbers that take a middle position between the Laspeyres and the Paasche index. Closer inspection reveals that the small deviation does not come as a surprise. If the transformation factors of the generalised unit value index are computed by

$$\hat{z}_i = \left(p_i^0 + p_i^1 \right)/2 \,,$$

this yields the Banerjee index. This index was defined in Equation (2). As pointed out in Section 2, this index is almost identical to the Fisher index defined in Equation (1).

The price index proposed by Davies (1924, p. 185) follows the CPL-approach. Nevertheless, the traditional price indices took centre stage and the proposal of Davies was never pursued further. What are the reasons for this neglect? Did price statisticians follow a herd instinct initiated by Irving Fisher's (1922, p. 451) rejection of the CPL-approach? It is possible to prove this reproach invalid, if axiomatic arguments could be advanced against the family of generalised unit value indices. Therefore, the following section investigates the axiomatic properties of the family of generalised unit value indices and compares these properties to those of the most highly regarded traditional price indices.

7   Axiomatic Analysis

In the previous sections, the family of generalised unit value indices P_{GUV} was developed. It was defined in Equation (10):

$$P_{GUV} = \frac{V^1}{V^0} \frac{\sum x_i^0 \hat{z}_i}{\sum x_i^1 \hat{z}_i} \,. \tag{18}$$

For $\hat{z}_i = p_i^1$ one obtains the Laspeyres index and for $\hat{z}_i = p_i^0$ the Paasche index. In both variants the transformation factors are estimated from the prices of only one period. Therefore both index formulae represent two rather crude members of the family of generalised unit value indices. The sophisticated members described in

Section 5 utilise the available information of both periods:

$$\hat{z}_i = \left(p_i^0 x_i^0 + p_i^1 x_i^1\right) / \left(x_i^0 + x_i^1\right) \tag{19}$$

$$\text{or} \quad \hat{z}_i = \sqrt{p_i^0 p_i^1} \tag{20}$$

$$\text{or} \quad \hat{z}_i = \left(p_i^0 + p_i^1\right)/2 \tag{21}$$

$$\text{or} \quad \hat{z}_i = 2 \cdot \left[(1/p_i^0) + (1/p_i^1)\right]^{-1} \tag{22}$$

$$\text{or} \quad \hat{z}_i = \left(p_i^0 + \frac{p_1^0}{p_1^1}p_i^1\right)\frac{1}{2} \tag{23}$$

$$\text{or} \quad \hat{z}_i = 2\left(\frac{1}{p_i^0} + \frac{p_1^1}{p_1^0}\frac{1}{p_i^1}\right)^{-1} . \tag{24}$$

As Equation (18) reveals, it would be admissable to multiply all transformation factors z_i with some arbitrary positive constant k. This holds true for all six variants (19) to (24). Multiplying variants (23) and (24) with the constant $1/p_1^0$ yields the formulae

$$\hat{z}_i = \left(\frac{p_i^0}{p_1^0} + \frac{p_i^1}{p_1^1}\right)\frac{1}{2}$$

$$\hat{z}_i = 2\left(\frac{p_1^0}{p_i^0} + \frac{p_1^1}{p_i^1}\right)^{-1} .$$

From these formulations it can be more easily seen that variants (23) and (24) represent the arithmetic and the harmonic averages of the price ratios.

Are the sophisticated members of the family of generalised unit value indices as attractive as the most highly regarded traditional price indices (e.g., Fisher index)? Axiomatic index theory can contribute to answering this question. It analyses whether a proposed price index formula satisfies a list of postulates (called axioms or tests) that are regarded as indispensable for a meaningful price index formula. However, there is some discussion as to which postulates are convincing axioms and which are not. Therefore, in this study a broad range of axioms is included. The axiomatic properties derived for the family of generalised unit value indices are compared to those of the Fisher, Marshall-Edgeworth, and Walsh index.

It turns out that the sophisticated members of the family of generalised unit value indices defined by Equations (19) to (24) have different axiomatic properties. Table 2 provides an overview of these properties. The postulates (axioms) listed in the first column are formally defined in Appendix A. Proofs of the results of Table 2 are given in Appendix B.

All listed price indices violate the permutation axiom and the circularity axiom. Almost all sophisticated members of the general unit value index violate the strict monotonicity axiom, the Banerjee index defined by variant (21) being a notable exception. Even though a violation can occur only with extreme intertemporal price and quantity changes, this aspect represents a deficiency of the respective price index formulae. The weak monotonicity axiom is satisfied by all members of the generalised unit value index.

The member defined by variant (19) violates the proportionality axiom and therefore also the strict mean value axiom. Both axioms represent tightenings of the

13

Table 2: Overview of the Axiomatic Properties of the Price Index Formulas (A Filled Triangle Indicates *Test Satisfied* and an Empty Triangle Indicates *Test Violated*).

	P_F	P_{ME}	P_W	P_{GUV}						P_L	P_P
				(19)	(20)	(21)	(22)	(23)	(24)		
A1 Strict Mean Value	▲	▲	▲	▽	▲	▲	▲	▲	▲	▲	▲
A2 Proportionality	▲	▲	▲	▽	▲	▲	▲	▲	▲	▲	▲
A3 Identity	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A4 Inv. to Re-Ordering	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A5 Permutation	▽	▽	▽	▽	▽	▽	▽	▽	▽	▽	▽
A6 Inversion	▲	▲	▲	▲	▲	▲	▲	▲	▲	▽	▽
A7 Strict Commens.	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A8 Weak Commens.	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A9 Price Dimension.	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A10 Quantity Dimension.	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A11 Strict Quant. Prop.	▲	▽	▲	▽	▲	▲	▲	▲	▲	▲	▲
A12 Weak Quant. Prop.	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A13 Lin. Homogeneity	▲	▲	▲	▽	▲	▽	▽	▲	▲	▲	▲
A14 Strict Monotonicity	▲	▲	▲	▽	▽	▲	▽	▽	▽	▲	▲
A15 Weak Monotonicity	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A16 Price Ratio	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A17 Constant Quant.	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲	▲
A18 Time Reversal	▲	▲	▲	▲	▲	▲	▲	▲	▲	▽	▽
A19 Circularity	▽	▽	▽	▽	▽	▽	▽	▽	▽	▽	▽

identity axiom. This is true also for the linear homogeneity axiom. This axiom is violated by variants (19), (21), and (22). In Auer (2009b) it is argued, however, that the identity axiom and its tightenings do not represent compelling postulates. If one accepts the reasoning of that study, then, looking at the remaining axioms, it is variant (21) of the generalised unit value index (the Banerjee index) that exhibits a particularly attractive axiomatic profile. It is equivalent to the profiles of the Fisher index and the Walsh index. A particularly simple formulation of variant (21) was given in Equation (1).

8 Concluding Remarks

The existing price statistical literature discourages the computation of a period's average price level. If one accepted this position, an economy's overall price change should not be calculated by a price index consistent with the CPL-approach. In other words, one should not compute the ratio of the comparison period's average price level and the base period's price level. This study has demonstrated, however, that such a computation is not only feasible and meaningful, but also simple and easy to interpret. The key concept for such a computation is a group of price indices labeled as the family of generalised unit value indices. This family of price indices incorporates into the (simple) unit value index so called transformation factors. These are necessary to take account of the heterogeneous nature of the items included

in the computation. It was shown that the Laspeyres and the Paasche index are members of this family and therefore consistent with the CPL-approach.

The members of the family of generalised unit value indices differ only with respect to the method applied to compute the values of their transformation factors. In this study, each member has been examined with respect to its axiomatic properties. One of these members is known in the literature as the Banerjee index. In spite of its simplicity, this index has an axiomatic record that stands up to the standards set by the Fisher, the Marshall-Edgeworth, and the Walsh index, that is, to the three traditional price indices that are widely regarded as the ones with the best axiomatic profiles.

Besides the axiomatic approach to index theory, the economic and the stochastic approach also exist. In future research one should investigate how the family of generalised unit value indices relates to economic theory. In addition, one could pursue a stochastic approach to evaluating the family of generalised unit value indices. The stochastic approach to index theory usually assumes that all observed price ratios are realisations of the same random variable with an expected value equal to the "common inflation." The CPL-approach suggests to pursue a stochastic analysis that is based on a less controversial assumption: For a given pair of items the observed price ratios of the base period and the comparison period represent realisations of a random variable with an expected value given by the items' ratio of equivalence. Based on this assumption, one could compare the statistical properties of the estimators of the ratios of equivalence used by the various members of the family of generalised unit value indices.

The family of generalised unit value indices can also be used for measuring the average price change of similar items with different qualitative characteristics, where hedonic analysis and other sophisticated quality adjustment methods are either not possible or too demanding. Furthermore, this index family can be applied to other areas of measurement that have not been mentioned in this study. An obvious area is interregional price comparisons. In the context of such comparisons, the transformation factors of the generalised unit value index provide additional flexibility to adjust for regional particularities.

Appendix A

A price index is a function P that maps all prices and quantities of some base period $t = 0$ and comparison period $t = 1$ into a positive index number:

$$P : \mathbb{R}_{++}^{4N} \to \mathbb{R}_{++} , \quad (\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \to P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) ,$$

where $\mathbf{p^t} = (p_1^t, ..., p_N^t)$ and $\mathbf{x^t} = (x_1^t, ..., x_N^t)$.

A 1 *The **strict mean value axiom** (Olt, 1996, p. 26) postulates that*

$$\min_i \{p_i^1/p_i^0\} < P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) < \max_i \{p_i^1/p_i^0\} ,$$

where for $\mathbf{p^1} = \lambda \mathbf{p^0}$ the relation "$<$" is to be replaced by the relation "$=$".

A 2 *The **proportionality axiom** (Walsh, 1901, p. 115) postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \lambda\mathbf{p^0}, \mathbf{x^1}) = \lambda \,, \quad \text{for all } \lambda > 0 \,.$$

A 3 *The **identity axiom** (Laspeyres, 1871, p. 308) postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}) = 1 \,.$$

A 4 *The vectors $\widetilde{\mathbf{p}}^0$, $\widetilde{\mathbf{x}}^0$, $\widetilde{\mathbf{p}}^1$, and $\widetilde{\mathbf{x}}^1$ are arbitrary uniform permutations of the vectors $\mathbf{p^0}$, $\mathbf{x^0}$, $\mathbf{p^1}$, and $\mathbf{x^1}$. The **invariance to re-ordering axiom** (Fisher, 1922, p. 63) postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = P(\widetilde{\mathbf{p}}^0, \widetilde{\mathbf{x}}^0, \widetilde{\mathbf{p}}^1, \widetilde{\mathbf{x}}^1) \ .$$

A 5 *The vectors $\widetilde{\mathbf{p}}^0$ and $\widetilde{\mathbf{x}}^0$ are arbitrary uniform permutations of the vectors $\mathbf{p^0}$ and $\mathbf{x^0}$. The **permutation axiom** (Auer, 2002, p. 534) postulates, that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \widetilde{\mathbf{p}}^0, \widetilde{\mathbf{x}}^0) = 1 \,.$$

A 6 *The vectors $\widetilde{\mathbf{p}}^0$ and $\widetilde{\mathbf{x}}^0$ are special permutations of the vectors $\mathbf{p^0}$ and $\mathbf{x^0}$, such that $p_j^0 = \widetilde{p}_k^0$, $p_k^0 = \widetilde{p}_j^0$, $x_j^0 = \widetilde{x}_k^0$, $x_k^0 = \widetilde{x}_j^0$, and for all items $i \neq j, k$, $p_i^0 = \widetilde{p}_i^0$ and $x_i^0 = \widetilde{x}_i^0$. The **inversion axiom** (Auer, 2002, p. 534) postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \widetilde{\mathbf{p}}^0, \widetilde{\mathbf{x}}^0) = 1 \,.$$

A 7 *The **strict commensurability axiom** (Pierson, 1896, p. 131) postulates that*

$$P(\mathbf{p^0}\boldsymbol{\Lambda}, \mathbf{x^0}\boldsymbol{\Lambda}^{-1}, \mathbf{p^1}\boldsymbol{\Lambda}, \mathbf{x^1}\boldsymbol{\Lambda}^{-1}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \,,$$

where $\boldsymbol{\Lambda}$ is a $N \times N$ diagonal matrix with positive elements λ_i.

A 8 *The **weak commensurability axiom** (Swamy, 1965, p. 620) postulates that*

$$P(\mathbf{p^0}\lambda, \mathbf{x^0}\lambda^{-1}, \mathbf{p^1}\lambda, \mathbf{x^1}\lambda^{-1}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \,, \quad \text{for all } \lambda > 0 \,.$$

A 9 *The **price dimensionality axiom** (Eichhorn and Voeller, 1976, p. 24) postulates that*

$$P(\lambda\mathbf{p^0}, \mathbf{x^0}, \lambda\mathbf{p^1}, \mathbf{x^1}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \,, \quad \text{for all } \lambda > 0 \,.$$

A 10 *The **quantity dimensionality axiom** (Funke et al., 1979, p. 680) postulates that*

$$P(\mathbf{p^0}, \lambda\mathbf{x^0}, \mathbf{p^1}, \lambda\mathbf{x^1}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \,, \quad \text{for all } \lambda > 0 \,.$$

A 11 *The **strict quantity proportionality axiom** (Vogt, 1980, p. 70, and Diewert, 1992, p. 216) postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \lambda\mathbf{x^1}) = P(\mathbf{p^0}, \delta\mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \,, \quad \text{for all } \lambda, \delta > 0 \,.$$

A 12 *The **weak quantity proportionality axiom** (Auer, 2001, p. 6) postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \lambda\mathbf{x^0}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^0}) \,, \quad \text{for all } \lambda > 0 \,.$$

A 13 *The **linear homogeneity axiom** (Walsh, 1901, p. 385, and Eichhorn and Voeller, 1976, p. 28) postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \lambda\mathbf{p^1}, \mathbf{x^1}) = \lambda P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = P((1/\lambda)\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}), \text{ for all } \lambda > 0 \ .$$

A 14 *Consider two different scenarios for the comparison period ($t = 1$ and $t = 1^*$) and the base period ($t = 0$ and $t = 0^*$). If for all items $p_i^{1^*} \geq p_i^1$ and for at least one item i the strict relation holds, then the **strict monotonicity axiom** (Eichhorn and Voeller, 1976, p. 23) postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^{1^*}}, \mathbf{x^1}) > P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \ ,$$

and if for all items $p_i^{0^*} \geq p_i^0$ and for at least one item i the strict relation holds, then the strict monotonicity axiom postulates that

$$P(\mathbf{p^{0^*}}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) < P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \ .$$

A 15 *If for all items $p_i^1 \geq p_i^0$ and for at least one item i the strict relation holds, then the **weak monotonicity axiom** (Olt, 1996, p. 37) postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) > P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}) \ .$$

If for all items $p_i^1 \leq p_i^0$ and for at least one item i the strict relation holds, then the axiom postulates that

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) < P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}) \ .$$

A 16 *The **price ratio axiom** (Eichhorn and Voeller, 1990, p. 326) postulates that for $N = 1$*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = \frac{p_1^1}{p_1^0} \ .$$

A 17 *The **constant quantities axiom** (Lowe, 1822, Appendix, p. 95) postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^0}) = \frac{\sum p_i^1 x_i^0}{\sum p_i^0 x_i^0} \ .$$

A 18 *The **time reversal axiom** (Pierson, 1896, p. 128, and Walsh, 1901, p. 368) postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = \frac{1}{P(\mathbf{p^1}, \mathbf{x^1}, \mathbf{p^0}, \mathbf{x^0})} \ .$$

A 19 *The **circularity axiom** (Westergaard, 1890, p. 218) postulates that*

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^2}, \mathbf{x^2}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \cdot P(\mathbf{p^1}, \mathbf{x^1}, \mathbf{p^2}, \mathbf{x^2}) \ .$$

Appendix B

This appendix sketches out the proofs of the results listed in Table 2. The proofs associated with the Laspeyres (P_L), Paasche (P_P), Fisher (P_F), Marshall-Edgeworth (P_{ME}), and Walsh index (P_W) are either documented in Auer (2001) or they are trivial. The following proofs relate to the sophisticated members of the family of generalised unit value indices (P_{GUV}) defined in Equations (18) to (24). In order to simplify the notation, the hat on the transformation factors is omitted. Furthermore, the term "all sophisticated variants of P_{GUV}" stands for variants (19) to (24).

From Equation (18) it can be directly seen that all sophisticated variants of P_{GUV} satisfy the **invariance to re-ordering axiom** and the **time reversal axiom**. The **circularity axiom** is violated by all sophisticated variants of P_{GUV}. Only for the special case $z_i = z$, it would be satisfied.

In the scenario specified by the **identity axiom** ($p_i^0 = p_i^1 = p_i$) all sophisticated variants of P_{GUV} produce $z_i = p_i$ and therefore

$$P_{GUV} = \frac{V^1}{V^0} \frac{\sum p_i x_i^0}{\sum p_i x_i^1} = \frac{V^1}{V^0} \frac{V^0}{V^1} = 1 \; .$$

As a consequence, all these variants satisfy the identity axiom.

In the scenario of the **proportionality axiom** one gets

$$P_{GUV} = \frac{\lambda V^{01}}{V^0} \frac{\sum \tilde{z}_i x_i^0}{\sum \tilde{z}_i x_i^1} \; ,$$

where $\tilde{z}_i$ is the transformation factor associated with the value of λ. The satisfaction of the proportionality axiom requires that

$$\frac{V^{01}}{V^0} = \frac{\sum \tilde{z}_i x_i^1}{\sum \tilde{z}_i x_i^0}$$

and therefore

$$\frac{\sum p_i^0 x_i^1}{\sum p_i^0 x_i^0} = \frac{\sum \tilde{z}_i x_i^1}{\sum \tilde{z}_i x_i^0} \; .$$

This condition is satisfied, if and only if in the scenario specified by the proportionality axiom the ratios of equivalence ($\tilde{z}_i/\tilde{z}_j$) are independent from λ and simultaneously coincide with the price ratios of the base period. Except for variant (19), all sophisticated variants of P_{GUV} satisfy these requirements.

If $N = 1$, then (18) yields

$$P_{GUV} = \frac{p^1 x^1}{p^0 x^0} \frac{x^0 z}{x^1 z} = \frac{p^1}{p^0} \; .$$

This implies that all sophisticated variants of P_{GUV} satisfy the **price ratio axiom**.

Let $\tilde{z}_i$ indicate the transformation factors resulting from the λ_i-values of the scenario specified by the **strict commensurability axiom**. According to (18), this axiom is satisfied, if and only if

$$\frac{\sum x_i^0 z_i}{\sum x_i^1 z_i} = \frac{\sum (x_i^0/\lambda_i)\, \tilde{z}_i}{\sum (x_i^1/\lambda_i)\, \tilde{z}_i} \; .$$

Variants (19) to (24) give $\tilde{z}_i = z_i \lambda_i$. As a consequence, all sophisticated variants of P_{GUV} satisfy the strict commensurability axiom and therefore also the **weak commensurability axiom**.

The **price dimensionality axiom** is satisfied, if the ratios of equivalence are not affected by a uniform proportional change of all prices. Variants (19) to (24) of P_{GUV} satisfy this requirement. All these variants also satisfy the **quantity dimensionality axiom**, because a price index that satisfies the price dimensionality axiom and the weak commensurability axiom automatically satisfies the quantity dimensionality axiom.

The numerator and the denominator in the second ratio of the right hand side of formula (18) are equal, if and only if

$$\sum z_i(x_i^0 - x_i^1) = 0 . \tag{25}$$

This is satisfied, if the quantities do not change over time. Therefore, all sophisticated variants satisfy the **constant quantities axiom**.

The scenario specified by the **inversion axiom** yields $V^0 = V^1$. Therefore, the satisfaction of the inversion axiom requires that condition (25) is satisfied. This condition is satisfied, if and only if

$$z_j(x_j^0 - x_j^1) + z_k(x_k^0 - x_k^1) = 0 .$$

In the scenario specified by the inversion axiom, this condition is equivalent with the condition

$$z_j(x_j^0 - x_k^0) + z_k(x_k^0 - x_j^0) = 0$$

and therefore with the condition

$$\left(z_j - z_k\right)\left(x_j^0 - x_k^0\right) = 0 .$$

Since $x_j^0 \neq x_k^0$, this condition is satisfied, if and only if

$$z_j = z_k .$$

In the scenario specified by the inversion axiom, all sophisticated variants of P_{GUV} satisfy this condition.

The scenario specified by the **permutation axiom** also yields $V^0 = V^1$. However, no variant of P_{GUV} satisfies condition (25).

The **linear homogeneity axiom** is satisfied, if and only if in formula (18) the ratio $\left[\sum z_i x_i^0\right] / \left[\sum z_i x_i^1\right]$ is not affected by the factor λ, that is, if and only if the ratios of equivalence are independent from the factor λ. Only variants (20), (23), and (24) satisfy this condition.

The **strict quantity proportionality axiom** is satisfied, if and only if

$$\frac{\sum \tilde{z}_i x_i^1}{\sum \tilde{z}_i x_i^0} = \frac{\sum z_i x_i^1}{\sum z_i x_i^0} , \tag{26}$$

where $\tilde{z}_i$ indicates the transformation factors associated with the scenario specified by the strict quantity proportionality axiom. Since variants (20) to (24) yield $\tilde{z}_i = z_i$, these variants also satisfy condition (26). Variant (19) does not satisfy this condition.

In the scenario specified by the **weak quantity proportionality axiom** one gets $x_i^1 = \lambda x_i^0$. Therefore, expression (18) becomes

$$P_{GUV} = \frac{\lambda V^{10}}{V^0} \frac{\sum z_i x_i^0}{\sum z_i \lambda x_i^0} = \frac{V^{10}}{V^0} \ .$$

Since V^{10} and V^0 are independent from λ, all sophisticated variants of P_{GUV} satisfy the weak quantity proportionality axiom.

For the **strict monotonicity axiom** and the **weak monotonicity axiom** one has to consider the case that for all items the relation $dp_k^1 \geq 0$ holds and for at least one item k the strict relation holds. If for this case $dP_{GUV} = \sum (\partial P_{GUV} / \partial p_k^1) dp_k^1 > 0$, then the strict monotonicity axiom is satisfied. From variants (19) to (24) one obtains

$$
\begin{aligned}
(19) \quad & \partial z_k / \partial p_k^1 = x_k^1 / (x_k^0 + x_k^1) \\
(20) \quad & \partial z_k / \partial p_k^1 = 0.5 \sqrt{p_k^0 / p_k^1} \\
(21) \quad & \partial z_k / \partial p_k^1 = 0.5 \\
(22) \quad & \partial z_k / \partial p_k^1 = 2 \left(p_k^1 / p_k^0 + 1 \right)^{-2} \\
(23) \quad & \partial z_k / \partial p_k^1 = 0.5 \left(p_1^0 / p_1^1 \right) \\
(24) \quad & \partial z_k / \partial p_k^1 = 2 \left(p_1^1 / p_1^0 \right) \left[p_1^1 / p_1^0 + p_k^1 / p_k^0 \right]^{-2} \ .
\end{aligned}
$$

From formula (18) it follows that

$$
\begin{aligned}
\frac{\partial P_{GUV}}{\partial p_k^1} &= \frac{1}{V^0} \frac{[x_k^1 \sum x_i^0 z_i + V^1 x_k^0 (\partial z_k / \partial p_k^1)] \sum x_i^1 z_i - V^1 (\sum x_i^0 z_i) x_k^1 (\partial z_k / \partial p_k^1)}{(\sum x_i^1 z_i)^2} \\
&= \frac{x_k^1 \sum x_i^0 z_i + V^1 (\partial z_k / \partial p_k^1) x_k^0 - V^1 (\partial z_k / \partial p_k^1) x_k^1 (\sum x_i^0 z_i) / (\sum x_i^1 z_i)}{V^0 \sum x_i^1 z_i} \\
&= \frac{x_k^1 \sum x_i^0 z_i \left[1 - V^1 (\partial z_k / \partial p_k^1) / (\sum x_i^1 z_i) \right] + V^1 (\partial z_k / \partial p_k^1) x_k^0}{V^0 \sum x_i^1 z_i} \ . \quad (27)
\end{aligned}
$$

The denominator of (27) is positive. For variant (21), the term in squared brackets simplifies to $[1 - V^1 / (V^{01} + V^1)] > 0$. Therefore, variant (21) satisfies the strict monotonicity axiom. However, the other sophisticated variants of P_{GUV} violate this axiom. For sufficiently large values of p_i^1 and x_i^0 ($i \neq k$), the numerator becomes negative.

In the reference scenario specified by the weak monotonicity axiom ($p_i^0 = p_i^1 = p_i$) all variants of P_{GUV} give $z_i = p_i$ and $0 < \partial z_k / \partial p_k^1 < 1$. In (27), the term in squared brackets simplifies to $[1 - (\partial z_k / \partial p_k^1)] > 0$. Therefore, the partial derivatives (27), and therefore, also the total differential $dP_{GUV} = \sum (\partial P_{GUV} / \partial p_k^1) dp_k^1$ are positive. As a consequence, all sophisticated variants of P_{GUV} satisfy the weak monotonicity axiom.

Since variant (19) violates the proportionality axiom, it also violates the **strict mean value axiom**. A price index that satisfies the linear homogeneity axiom, the weak monotonicity axiom, and the identity axiom, always satisfies the strict mean value axiom. In order to show this, let $p_i^{1*} = p_i^1 / \min_j \{ p_j^1 / p_j^0 \}$, and therefore,

$$\frac{p_i^{1*}}{p_i^0} = \frac{p_i^1}{p_i^0} \cdot \frac{1}{\min_j \{ p_j^1 / p_j^0 \}} \ .$$

As a consequence, one gets $\min_i\{p_i^{1*}/p_i^0\} = 1$. Therefore, for all commodities one obtains $p_i^{1*} \geq p_i^0$. This is a scenario specified by the weak monotonicity axiom. If a price index satisfies this axiom and the identity axiom, then

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^{1*}}, \mathbf{x^1}) > 1$$
$$\Rightarrow \quad \min_j\{p_j^1/p_j^0\} P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^{1*}}, \mathbf{x^1}) > \min_j\{p_j^1/p_j^0\} \ .$$

Due to the satisfaction of the linear homogeneity axiom, this inequality becomes

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) > \min_j\{p_j^1/p_j^0\} \ .$$

Furthermore, let $p_i^{1**} = p_i^1 / \max_j\{p_j^1/p_j^0\}$, and therefore,

$$\frac{p_i^{1**}}{p_i^0} = \frac{p_i^1}{p_i^0} \cdot \frac{1}{\max_j\{p_j^1/p_j^0\}} \ .$$

Therefore, for all commodities one obtains $p_i^{1**} \leq p_i^0$. This is also a scenario specified by the weak monotonicity axiom. If a price index satisfies this axiom and the identity axiom, then

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^{1**}}, \mathbf{x^1}) < 1$$
$$\Rightarrow \quad \max_j\{p_j^1/p_j^0\} P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^{1**}}, \mathbf{x^1}) < \max_j\{p_j^1/p_j^0\} \ .$$

Due to the satisfaction of the linear homogeneity axiom, this inequality becomes

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) < \max_j\{p_j^1/p_j^0\} \ .$$

Since variants (20), (23), and (24) satisfy the linear homogeneity axiom, the weak monotonicity axiom, and the identity axiom, they also satisfy the strict mean value axiom.

A price index that satisfies the proportionality axiom and the strict monotonicity axiom also satisfies the strict mean value axiom (earliest proof in Eichhorn and Voeller, 1990, p. 332), because from the proportionality axiom it follows that

$$P(\mathbf{p^0}, \mathbf{x^0}, \min\{p_i^1/p_i^0\}\cdot\mathbf{p^0}, \mathbf{x^1}) = \min\{p_i^1/p_i^0\}$$
$$P(\mathbf{p^0}, \mathbf{x^0}, \max\{p_i^1/p_i^0\}\cdot\mathbf{p^0}, \mathbf{x^1}) = \max\{p_i^1/p_i^0\}$$

and from the strict monotonicity axiom it follows that

$$P(\mathbf{p^0}, \mathbf{x^0}, \min\{p_i^1/p_i^0\}\cdot\mathbf{p^0}, \mathbf{x^1}) < P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1})$$
$$P(\mathbf{p^0}, \mathbf{x^0}, \max\{p_i^1/p_i^0\}\cdot\mathbf{p^0}, \mathbf{x^1}) > P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \ .$$

Taken together, one obtains

$$\min\{p_i^1/p_i^0\} < P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) < \max\{p_i^1/p_i^0\} \ .$$

Therefore, variant (21) also satisfies the strict mean value axiom.

From formula (18), one obtains for variant (22) the expression

$$P_{GUV} = \frac{\sum \left(v_i^0 / \sum v_j^0\right) \left[1 + (p_i^1/p_i^0)^{-1}\right]^{-1}}{\sum \left(v_i^1 / \sum v_j^1\right) \left[1 + p_i^1/p_i^0\right]^{-1}} \ . \tag{28}$$

If in (28) the price ratios p_i^1/p_i^0 are replaced by the price ratio $\min_j\{p_j^1/p_j^0\}$ (the weights $v_i^0 / \sum v_j^0$ remain unchanged), one obtains the expression

$$\frac{\sum \left(v_i^0 / \sum v_j^0\right) \left[1 + \left(\min_j\{p_j^1/p_j^0\}\right)^{-1}\right]^{-1}}{\sum \left(v_i^1 / \sum v_j^1\right) \left[1 + \min_j\{p_j^1/p_j^0\}\right]^{-1}} = \min_j\{p_j^1/p_j^0\} \ .$$

Replacing in the numerator and denominator the price ratio $\min_j\{p_j^0/p_j^1\}$ by the actual price ratios p_i^1/p_i^0, the value of the numerator increases and the value of the denominator falls, leading to

$$P_{GUV} > \min_j\{p_j^1/p_j^0\} \ .$$

If in (28) the price ratios p_i^1/p_i^0 are replaced by the price ratio $\max_j\{p_j^1/p_j^0\}$ (the weights $v_i^0 / \sum v_j^0$ remain unchanged), one obtains the expression

$$\frac{\sum \left(v_i^0 / \sum v_j^0\right) \left[1 + \left(\max_j\{p_j^1/p_j^0\}\right)^{-1}\right]^{-1}}{\sum \left(v_i^1 / \sum v_j^1\right) \left[1 + \max_j\{p_j^1/p_j^0\}\right]^{-1}} = \max_j\{p_j^1/p_j^0\} \ .$$

Replacing in the numerator and denominator the price ratio $\max_j\{p_j^1/p_j^0\}$ by the actual price ratios p_i^1/p_i^0, the value of the numerator falls and the value of the denominator increases, leading to

$$P_{GUV} < \max_j\{p_j^1/p_j^0\} \ .$$

As a consequence, variant (22) also satisfies the strict mean value axiom.

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