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by

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Abstract: The inflation rate is normally computed as a weighted average of individual price changes. Alternatively, this rate could be evaluated by comparing average price levels. Unfortunately, this methodology has received limited attention in past research. This study attempts to remedy this situation by introducing a group of Generalized Unit Value indices that evaluate price level changes. The group includes some well-known (Laspeyres, Paasche, Banerjee), hardly known (Lehr, Davies), and previously unknown price indices. An assessment of their axiomatic properties is presented.

Keywords: Index Theory, Inflation, Unit Value, Measurement

JEL Classification: C43, E31, E52

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1 Introduction

In most countries, the published official inflation rate is usually a weighted average of the price changes that occurred in a set of product groups between a base and a comparative time period. This method of price measurement utilizes a methodology hereinafter called the Average of Price Changes, APC. Alternatively, price inflation could be measured as the ratio of two average price levels. It is the quotient of the price level in the comparative time period divided by its counterpart in the base period. This methodology is hereinafter referred to as the Change in Price Levels, CPL. Relating price levels is a popular methodology in interregional price comparisons (e.g. Geary-Khamis method). In intertemporal price comparisons, however, this straightforward method has only received a scant amount of consideration in the published literature.

1.1 The Literature

Opinions are varied regarding the worthiness of the various price indices that are available. Nevertheless, widespread agreement seems to exist that the price index of choice should compute a weighted average of the individual price changes that were measured in the first stage. This is the APC methodology and both the Laspeyres and Paasche indices are consistent with it.

The APC methodology has been analyzed extensively. On the other hand, the CPL methodology has received less scrutiny. This study attempts to remedy this situation by addressing the fundamental question: Does the CPL methodology offer a suitable alternative for price measurement purposes?

Recently, the CPL methodology has been the subject of several studies, for example, de Haan (2002, 2004, 2007), Dalén (2001), and Silver (2010). These studies, however, are concerned with computing the overall price change of products that differ but at the same time provide the same function, e.g., washing machines. This is the borderline between the elementary level that employs the CPL methodology and the upper level of price measurement that uses the APC methodology. These authors suggest the application of an adapted version of the Unit Value (UV) index that they call the Quality Adjusted UV index. The modification comes in the form of quality adjustment factors that take into account the quality differences that exist between the products. As one possibility, hedonic regression techniques are suggested for the estimation of these quality adjustment factors.

Hedonic regression and some other quality adjustment methods, however, require the availability of external data concerning the inherent qualitative characteristics of the products involved. If this information is unavailable, then these inherent product differences must be handled in a manner that relies solely upon the observable price and quantity data from the marketplace. Many years ago, Lehr (1885, pp. 37-39) and Davies (1924, pp. 182-186) developed methods for accomplishing this task. They regarded their methodology as not only suitable for similar products but also for the heterogeneous ones as well. This noteworthy insight is the point of departure for this present research.

1.2 The Contribution

First, this study elucidates the arguments put forth by Lehr and Davies that underlie the derivation of the indices they proposed. In addition, some further price indices are presented that are consistent with these fundamental ideas and, moreover, an acknowledgement for their inspiring contributions is set forth.

Second, it classifies some well-known, hardly known, and previously unknown price indices into a general framework called the Generalized Unit Value (GUV) index family. It demonstrates that the Laspeyres and Paasche indices, as well as those proposed by Davies (1924, p. 185) and Lehr (1885, p. 39), are members of this GUV index family. Family members differ with respect to the precise manner in which their assessed-value transformation rates, a term more general than quality adjustment factors, are computed. They share, however, a unifying feature. The calculation of these assessed-value transformation rates requires no supplementary sources of information, such as measurements of the innate qualitative characteristics that are responsible for the differences in the products. They are based solely upon the observed price and quantity data from the marketplace.

Third, the GUV indices are extremely useful for aggregating the prices of similar products. Additionally, Lehr (1885, pp. 37-38) and Davies (1924, pp. 182-183) argued that their respective index formulas could be used for the aggregation of heterogeneous products as well. This study supports this notion and extends it to encompass the whole family of GUV indices. This brings credence to the claim that even in the context of heterogeneous products meaningful results can be obtained.

Fourth, the GUV indices presented employ the CPL methodology. Therefore, this research seeks to provide support to those who might want to use this methodology. In order to accomplish this task, a thorough axiomatic analysis was conducted. This analysis confirms that the members of the GUV price index family have a solid axiomatic record.

Fifth, this study demonstrates that both the Laspeyres and Paasche indices are members of the GUV index family. Therefore, they are consistent with the CPL methodology as well.

This paper is organized as follows. Some background material concerning price measurement utilizing the UV index is contained in Section 2. The applicability of this form of price index is demonstrated for the case of identical products. An amended version is presented for use with those products that are defined to be similar. The similar products considered have product differences that are observable and measurable. Section 3 considers the case of heterogeneous products where the auxiliary information concerning product differences is unavailable. The family of GUV indices is introduced in this section as well. In Section 4 the axiomatic properties of the GUV indices are compared side by side with some of the most highly regarded traditional price indices. Concluding remarks together with some suggestions for promising areas of future research are contained in Section 5.

2 Preliminaries

The words of Irving Fisher remain and his remarks continue to exert a strong influence upon those who perform official price measurement. In his seminal book on price statistics (Fisher, 1922, p. 451), the notion of a price level is firmly rejected. He acknowledged that price level calculations can be made, but he cautioned:

> "*... it is apt, in general, to prove a delusion and a snare. The reason is that an average of prices of wheat, coal, cloth, lumber, etc. is an average of incommensurables and therefore has no fixed numerical value ...*"

Simply stated, these very different commodities lack a common identifying unit. Without this common unit, Fisher argued, all attempts at producing a suitable average price level are doomed to failure. The fundamental question is: Do methods even exist that would allow the prices of these incommensurables to be aggregated into a useable price level? Moreover, if such a method does exist: How would this seemingly impossible task be accomplished?

2.1 Background

Fisher was right in emphasizing that a precise numerical interpretation cannot be given to price levels viewed in isolation. This is insufficient justification, however, to discard this valuable concept altogether. The fundamental issue is well known in microeconomic price theory. Considering prices in isolation is of limited value in determining resource allocation. Relative prices, on the other hand, are a valuable source of information. Similarly, the ratios of average price levels can be useful as well.

Half a century later, Fisher's warnings came to life once again. They were resurrected and gained additional credence by formal arguments originating in axiomatic index theory (Eichhorn and Voeller, 1976, pp. 75-78). Likewise, Eichhorn (1978, pp. 144-146) and Diewert (1993, pp. 7-9; 2004, p. 292) put forth analogous objections. All of these studies argued that the CPL methodology is based upon the concept of unilateral price indices. Unilateral price indices measure price levels based only upon the observed prices and quantities in the same time period. In addition, these studies stated that the concept of a unilateral price index is fundamentally flawed because these indices cannot satisfy a set of indispensable axioms including the Commensurability axiom. This axiom postulates that an index must be invariant with respect to the quantity units employed.

There are two basic fallacies in this line of reasoning. First, the complete repudiation of the concept of unilateral price indices is not credible. Even though compliance with the Commensurability axiom is a vital criterion for bilateral price indices, Auer (2009a) demonstrated that it is an inappropriate condition when considering unilateral indices. Therefore, if a unilateral price index satisfies the Commensurability axiom, it should not be used. Second, even if unilateral price indices were considered to be unsuitable, this should not be construed as grounds for rejecting the CPL methodology altogether. Price indices associated with the CPL methodology are usually not simply the ratio of two unilateral price indices. The GUV indices are a good case in point.

2.2 The Unit Value Index

When the products being considered are identical in nature, Irving Fisher (1923, p. 743) acknowledged that either the APC or the CPL methodology could be used for price measurement purposes. Moreover, in the Consumer Price Index Manual (a joint publication of the ILO, IMF, OECD, UNECE, Eurostat, and The World Bank), Boldsen and Hill (2004, p. 164) recommend the UV index for identical products. Further support is expressed by Balk (1998, p. 8) who explored the link between economic theory and the UV index. All of these studies have implicitly endorsed the use of the CPL methodology.

Consider N identical products that are sold in the marketplace in both the base time period, $t = 0$, and a comparison period, $t = 1$. Furthermore, assume that the prevailing market conditions permitted these products to sell for different prices. Let p_i^t ($i = 1,\ldots,N$) denote the unit price of product i in time period t. Similarly, let x_i^t denote the number of units transacted. Consequently, the value aggregates, $V^t = \sum p_i^t\, x_i^t$, are the total expenditure on the goods traded. The unit value (Segnitz, 1870, p. 184), P_{UV}^t, in time period t is:

$$P_{UV}^t = \frac{\sum p_i^t x_i^t}{\sum x_i^t} = \frac{V^t}{\sum x_i^t}.$$

The UV index (Drobisch, 1871a, p. 39; 1871b, p. 149), P_{UV}, is a ratio of unit values and it is used to measure the average price level change between the base and comparison time periods:

$$P_{UV} = \frac{P_{UV}^1}{P_{UV}^0} = \frac{V^1}{V^0}\frac{\sum x_i^0}{\sum x_i^1}. \tag{1}$$

The product quantity summations, $\sum x_i^t$, yield accurate results because the product-identifying units being summed are identical. An axiomatic justification for the use of the UV index (1) can be found in Auer (2009b). If the products considered are classified as almost identical, then the UV index, i.e., the CPL methodology, continues to be the appropriate choice. This situation occurs when the products differ only with respect to the location and/or moment of purchase within a given time period.

2.3 The Amended Unit Value Index

When products are similar, but not identical, an amended version of the UV index is required. Similar products are defined as having innate differences that are observable and measurable. Such product differences occur frequently and stem from such things as quality levels, operating features, or simply the size of the packaging. These products have dissimilar product-identifying units and, consequently, they are unsuitable for the quantity summations in the UV index (1). The situation is correctable, however, by the inclusion of N product transformation rates, z_i (i = 1,…,N). They are defined to be an appropriate number of common units per product-identifying unit. Accordingly, the transformed prices, p_i^t/z_i, become monetary units per common unit of product i and the transformed quantities, $x_i^t\, z_i$, are the number of common units transacted in the form of product i. By definition, these common units are identical and, for that reason, reliable results are now obtained from the quantity summations, $\sum x_i^t\, z_i$. The value aggregates, V^t, remain unaffected by this transformation.

The functioning of these transformation rates, z_i, can best be illustrated by an example. Two similar products are presented in Table 1. They are gift boxes that contain the same assorted chocolates and differ only with respect to their net weight. Product B contains 300 grams while the smaller box, Product S, contains only 200 grams. Assume that if producers were called upon to produce 600 grams of candy, they would be indifferent between producing two of the larger 300-gram boxes or three of the smaller 200-gram boxes. Moreover, consumers are indifferent in their consuming preferences. Finally, both types of packaging are not equally accessible to all consumers.

Table 1: Example – Similar Products

	t = 0		t = 1	
	Price	Quantity	Price	Quantity
Product B	12	2	12	4
Product S	6	4	9	4

The product-identifying units, the big and small boxes, are not identical; therefore, the price and quantity data of product B and/or product S must be transformed. A convenient set of transformation rates are z_B = 1.5 and z_S = 1.0. These rates transform the product-identifying units into a common identical unit, the 200-gram portion. The transformed prices and quantities are presented in Table 2. For example, the transformed quantity, $x_B^1\, z_B = 4 \cdot 1.5 = 6$, is the total quantity of 200-gram portions transacted during the comparison period in the form of 300-gram boxes. Each of these 6 common units are sold at the price $p_B^1 / z_B = 12 / 1.5 = 8$.

Table 2: Prices and Quantities Relating to Common Units

	t = 0		t = 1	
	Price	Quantity	Price	Quantity
Product B	8	3	8	6
Product S	6	4	9	4

Applying the unit value formula to the transformed data yields the amended unit value in time period t:

$$P_{AUV}^t = \frac{\sum(p_i^t/z_i)(x_i^t z_i)}{\sum x_i^t z_i} = \frac{V^t}{\sum x_i^t z_i}.$$

Consequently, the Amended Unit Value (AUV) index is:

$$P_{AUV} = \frac{P_{AUV}^1}{P_{AUV}^0} = \frac{V^1}{V^0}\,\frac{\sum x_i^0 z_i}{\sum x_i^1\, z_i}. \qquad (2)$$

This index measures the change in the unit value of a common unit.

The numerical example yields P_{AUV} = 1.225. This indicates a 22.5 percent increase in the price level of assorted chocolates. This index is invariant to multiples of the transformation rates. For example, multiplying the transformation rates by 200 simply reduces the common unit into the one-gram portion. Nevertheless, the numerical value of the AUV index (2) remains unaltered. In the case of identical products, $z_i = z$, the AUV index (2) simplifies to the UV index (1). If the transacted product quantities remain constant over time, then the AUV index (2) simplifies to the ratio of value aggregates, V^1/V^0.

2.4 Discussion

The amended unit value index, using the CPL methodology, is not original to this study. It stems from the work of Dalén (2001, p. 11) and de Haan (2002, p. 81-82). Furthermore, additional elucidations, together with some empirical applications, can be found in de Haan (2004, pp. 6-7). A related proposal is provided by Silver (2010 p. S220). In all of these publications the price indices derived were concerned with the problem of aggregating the price changes of similar products into some average price change. By similar products these authors envisioned those products that serve the same purpose in consumption yet have distinct quality differences. Accordingly, the formulas were labeled the Quality Adjusted UV index. The authors point out that if the data required for the quality adjustment factors is not directly observable, then some form of estimation will be required. Hedonic regressions or similar techniques are recommended for the job. These estimation techniques, however, require the availability of auxiliary information concerning product quality. Can anything be accomplished if this information is unavailable? Moreover, when products serve completely different purposes, i.e., they are heterogeneous, is it still possible to use some amended version of the UV index for price measurement purposes?

3 The Generalized Unit Value Index Family

If two products are identical, then a unit of either product will provide the same amount of intrinsic worth to the consumer. It is not the equivalence in the tangible makeup of these products (e.g., their chemical, material, or technological characteristics) but this identical unit worth that permits the quantity summations in the UV index (1) to produce reliable results. In the context of similar products, the use of the quality adjustment factors is an attempt to emulate the case of the identical products. The quality adjustment factors convert the price and quantity data into numbers that all relate to the same common unit. The common unit is identical for all products and, therefore, the quantity summations can take place.

Accordingly, the essential prerequisite for reliable price measurements does not depend upon the products in question having the same tangible makeup. What is essential, however, is the presence of an identical worth unit. When an equivalence of worth is present, then a sufficient condition exists for appropriate price measurement and even incommensurables can be added in a price level calculation. Once the diverse product-identifying units have been transformed into a suitable number of intrinsic-worth units, then a meaningful quantity summation is possible. This is the first of two essential messages in the aforementioned studies by Lehr (1885, pp. 37-38) and Davies (1924, pp. 183-184).

3.1 Assessed-Value Transformation Rates

In the example presented in Section 2.3, two similar products with different product-identifying units were transformed into a common unit. This common unit was the 200-gram portion and it provided an identical worth to consumers. After transforming the data, the AUV index (2) provided a proper price measurement. A precondition for using the AUV index (2), however, is that all of the necessary information for determining the values of the transformation rates, z_i, is readily available. Unfortunately, in practice this is rarely the case.

When the distinguishing characteristics defining the product differences are unavailable, de Haan (2002, p.82) recommended using a time period in which the products were being sold in the marketplace and they were preferably in a state of equilibrium. In this case, the observed unit prices could be used to assess the implicit worth of the products.

Lehr (1885, pp. 37-39), publishing in the German language, discussed the implications of this proposition many years earlier. Unaware of this research, however, Davies (1924, pp. 183-185) some forty years later independently took a similar position. Going far beyond the proposal of de Haan, who was concerned only with similar products, these authors claimed and justified that the transformation rates calculated from available price data could be applied to the case of heterogeneous products as well. This is the second essential message contained in the studies authored by Lehr and Davies.

Regrettably, these studies did not receive the attention they deserved. Perhaps, these thoughts were considered to be too unorthodox at the time. It would be an unwarranted mistake, however, to dismiss this inspiration prematurely. Whether the approach is reasonable or not should be judged on the basis of the results obtained. If the resulting price indices correspond to some existing and highly respected price index formulas and if they produce reasonable results, then sufficient justification for the underlying approach should have been demonstrated.

3.2 The Definition

The N assessed-value transformation rates, $\hat{z}_i$ (i = 1,…,N), are a certain number of intrinsic-worth units per product-identifying unit. The numerical magnitudes of these rates are determined by the appraisal of the intrinsic worth of the products that was made. Some straightforward appraisal methods will be introduced in Section 3.3.

Replacing the hereinbefore-defined transformation rates, z_i, in the AUV index (2) by these (assessed-value) transformation rates, $\hat{z}_i$, yields the basic formula for the Generalized Unit Value (GUV) index:

$$P_{GUV} = \frac{P_{GUV}^1}{P_{GUV}^0} = \frac{[\sum(p_i^1/\hat{z}_i)x_i^1\hat{z}_i]/(\sum x_i^1\hat{z}_i)}{[\sum(p_i^0/\hat{z}_i)x_i^0\hat{z}_i]/(\sum x_i^0\hat{z}_i)} = \frac{V^1}{V^0}\frac{\sum x_i^0\hat{z}_i}{\sum x_i^1\hat{z}_i}. \tag{3}$$

This index measures the change in the unit value of an intrinsic-worth unit. Multiplying all of the transformation rates, $\hat{z}_i$, by an arbitrary constant does not alter the value of the GUV index (3). In other words, the value of the index is not dependent upon the absolute values of these rates, $\hat{z}_i$, but rather their ratios, $\hat{z}_i/\hat{z}_j$ (i,j = 1,…,N). Analogous to the UV index (1) and the AUV index (2), the GUV index (3) also uses the CPL methodology.

The basic GUV index formula (3) produces many different price indices depending upon how the transformation rates, $\hat{z}_i$, are computed. In order to qualify as a legitimate member of the GUV index family, however, the selected definition of the transformation rates must conform to four common characteristics that can be stated in the form of formal axioms. Hereinafter, as a matter of convenience, bold print will signify a column vector and the subscript "-*i*" will indicate a column vector that contains all elements except for the i^{th} one.

Z1 *The* ***Base axiom*** *postulates that the method used for computing the transformation rates,* $\hat{z}_i$ *(i* = 1*,…,N), must be the same for all products and must utilize only observed price and quantity data:*

$$\hat{z}_i = z(p_i^0, x_i^0, p_i^1, x_i^1, \mathbf{p}_{-i}^0, \mathbf{x}_{-i}^0, \mathbf{p}_{-i}^1, \mathbf{x}_{-i}^1).$$

Z2 *The* ***Weak Monotonicity axiom*** *postulates that the values of the transformation rates,* $\hat{z}_i$ *(i* = 1*,…,N), are weakly monotonically increasing with the observed market prices:*

$$\frac{\partial \hat{z}_i}{\partial p_i^0} \geq 0 \quad and \quad \frac{\partial \hat{z}_i}{\partial p_i^1} \geq 0.$$

Z3 *The* ***Price Dimensionality axiom*** *postulates that the ratio of the transformation rates,* $\hat{z}_i / \hat{z}_j$ *(i,j* = 1*,…,N), should not be affected by a change in the currency:*

$$\frac{z(p_i^0\eta, x_i^0, p_i^1\eta, x_i^1, \boldsymbol{p}_{-i}^0\eta, \boldsymbol{x}_{-i}^0, \boldsymbol{p}_{-i}^1\eta, \boldsymbol{x}_{-i}^1)}{z\left(p_j^0\eta, x_j^0, p_j^1\eta, x_j^1, \boldsymbol{p}_{-j}^0\eta, \boldsymbol{x}_{-j}^0, \boldsymbol{p}_{-j}^1\eta, \boldsymbol{x}_{-j}^1\right)} = \frac{z(p_i^0, x_i^0, p_i^1, x_i^1, \boldsymbol{p}_{-i}^0, \boldsymbol{x}_{-i}^0, \boldsymbol{p}_{-i}^1, \boldsymbol{x}_{-i}^1)}{z\left(p_j^0, x_j^0, p_j^1, x_j^1, \boldsymbol{p}_{-j}^0, \boldsymbol{x}_{-j}^0, \boldsymbol{p}_{-j}^1, \boldsymbol{x}_{-j}^1\right)}.$$

Z4 *The* ***Commensurability axiom*** *postulates that a change in the units of measure,* $\lambda_i > 0$ *(i* = 1*,…,N), of a product should change the value of the transformation rate,* $\hat{z}_i$*, by the same proportion:*

$$z(p_i^0\lambda_i, x_i^0/\lambda_i, p_i^1\lambda_i, x_i^1/\lambda_i, \boldsymbol{p}_{-i}^0, \boldsymbol{x}_{-i}^0, \boldsymbol{p}_{-i}^1, \boldsymbol{x}_{-i}^1) = \lambda_i z(p_i^0, x_i^0, p_i^1, x_i^1, \boldsymbol{p}_{-i}^0, \boldsymbol{x}_{-i}^0, \boldsymbol{p}_{-i}^1, \boldsymbol{x}_{-i}^1).$$

Utilizing axioms Z1 through Z4, the GUV index family can be defined as follows:

Definition: *The basic GUV index formula (3) defines a family of price indices that differ from one another by the selected definition of the (assessed-value) transformation rates,* $\hat{z}_i$ *(i* = 1*,…,N). Moreover, the selected definition must conform to axioms Z1 through Z4.*

3.3 Some Members of the GUV Index Family

The GUV indices differ from one another depending upon the precise manner in which the transformation rates, $\hat{z}_i$, are computed. Their values depend upon an assessment of intrinsic product worth. A straightforward appraisal of this worth is found by setting the number of intrinsic-worth units equal to the number of monetary units required to purchase the product. Using the observed prices in the base time period, p_i^0, yields the transformation rates:

$$\hat{z}_i = z(p_i^0) = p_i^0 \,. \tag{GUV-1}$$

Substituting the expression (GUV-1) into the basic GUV index formula (3) yields quite a surprising result, the Paasche index, P_P:

$$P_{GUV} = \frac{V^1}{V^0}\frac{\sum p_i^0 x_i^0}{\sum p_i^0 x_i^1} = \frac{\sum p_i^1 x_i^1}{\sum p_i^0 x_i^1} = P_P \,. \tag{4}$$

Similarly, using the observed prices in the comparison period, p_i^1, to numerically measure intrinsic worth, the transformation rates are:

$$\hat{z}_i = z(p_i^1) = p_i^1 \,. \tag{GUV-2}$$

Substituting the rates defined by (GUV-2) into the basic GUV index formula (3) and simplifying, the Laspeyres index, P_L, is obtained:

$$P_{GUV} = \frac{V^1}{V^0}\frac{\sum p_i^1 x_i^0}{\sum p_i^1 x_i^1} = \frac{\sum p_i^1 x_i^0}{\sum p_i^0 x_i^0} = P_L \,. \tag{5}$$

The Paasche and Laspeyres indices are bona fide members of the GUV index family. These two indices are usually described as tracking the change in the cost of some fixed group of products or as a weighted average of the individual price changes that occur in the products involved. A very different interpretation of these two indices, however, is provided by the specification of the basic GUV index formula (3). There, they measure the change in the unit value of an intrinsic-worth unit.

The Laspeyres and Paasche indices produce different numerical results because their definition of the applicable intrinsic-worth unit is based upon different time periods. This ambiguity problem is a familiar one in the APC methodology where a choice of the appropriate weights to be used in the averaging of the individual price ratios is required. As a solution to the problem, the weights are normally computed using data from both time periods. An analogous procedure can be employed here as well. Utilizing the arithmetic, geometric, or harmonic mean, the observed prices from both time periods can be used to form the required appraisals of product worth:

$$\hat{z}_i = z(p_i^0, p_i^1) = (p_i^0 + p_i^1)/2 \tag{GUV-3}$$

$$\hat{z}_i = z(p_i^0, p_i^1) = (p_i^0 p_i^1)^{1/2} \quad \text{(GUV-4)}$$

$$\hat{z}_i = z(p_i^0, p_i^1) = 2(1/p_i^0 + 1/p_i^1)^{-1}. \quad \text{(GUV-5)}$$

Substituting the transformation rates defined by (GUV-3) into the basic GUV index formula (3) and simplifying, yields the Banerjee (1977, p. 27) index, P_B:

$$P_{GUV} = \frac{V^1}{V^0} \frac{\sum x_i^0 (p_i^0 + p_i^1)/2}{\sum x_i^1 \left(p_i^0 + p_i^1\right)/2} = \frac{V^1}{V^0} \frac{(V^0 + V^{10})}{(V^1 + V^{01})} = P_B \,, \quad (6)$$

where $V^{st} = \sum p_i^s x_i^t$. Inserting the rates defined by (GUV-4) into the basic GUV index formula (3) and simplifying, yields the Davies (1924, p. 185) index, P_D. Therefore, the Banerjee and the Davies indices are also members of the GUV index family.

A common feature among the transformation rate definitions (GUV-1) to (GUV-5) is the fact that they are all exclusively based upon un-weighted market prices. Other GUV indices are possible that attach weights to these prices. For example, the expenditure shares,

$$w_i = \frac{p_i^0 x_i^0}{p_i^0 x_i^0 + p_i^1 x_i^1} \,,$$

could be used to weight the observed prices from the two time periods in a geometric mean:

$$\hat{z}_i = z(p_i^0, x_i^0, p_i^1, x_i^1) = (p_i^0)^{w_i} (p_i^1)^{1-w_i} \,. \quad \text{(GUV-6)}$$

An interesting discovery is made when the transformation rates are defined as follows:

$$\hat{z}_i = z(p_i^0, x_i^0, p_i^1, x_i^1) = \frac{p_i^0 x_i^0 + p_i^1 x_i^1}{x_i^0 + x_i^1} \,. \quad \text{(GUV-7)}$$

Inserting the rates defined by (GUV-7) into the basic GUV index formula (3) and simplifying, yields the Lehr (1885, p. 39) index, P_{Le}. Consequently, the Lehr index is also a bona fide member of the GUV index family. For an alternative interpretation of the Lehr index see Balk (2008, p. 8).

3.4 Numerical Example

For expository purposes, reconsider the example presented in Table 1 with two very different products. Product B is a package of twelve AA batteries and product S is a box of table salt (net weight: 1.2 kilogram). Utilizing expression (GUV-1) yields $\hat{z}_B = 12$.

This number says that the original unit of measurement (a package of twelve batteries) is equivalent to twelve intrinsic-worth units. Therefore, an intrinsic-worth unit is commensurate to one battery. Analogously, (GUV-1) yields $\hat{z}_S = 6$ which says that the original unit of measurement (1.2 kilogram of salt) is equivalent to six intrinsic-worth units. Therefore, an intrinsic-worth unit is not only commensurate to one battery but also to a 200-gram portion of salt. It is this equivalence that in the basic GUV index formula (3) allows for a meaningful summation over the transformed quantities, $\sum x_i^t \hat{z}_i$. It should be noted that, as an implication of utilizing definition (GUV-1), the base period price of an intrinsic-worth unit is 1, because this is the price of both, one battery and 200 gram of salt.

The price level change between the base and comparative time period using the various GUV indices defined hereinbefore is presented in Table 3.

Table 3: Example – GUV Index Family

GUV Index	Type/Name	Index Number
	$\hat{z}_i = z(p_i^{\mathrm{t}})$	
GUV-1	Paasche: P_P	1.167
GUV-2	Laspeyres: P_L	1.250
	$\hat{z}_i = z(p_i^0, p_i^1)$	
GUV-3	Banerjee: P_B	1.212
GUV-4	Davies: P_D	1.207
GUV-5	−	1.203
	$\hat{z}_i = z(p_i^0, x_i^0, p_i^1, x_i^1)$	
GUV-6	−	1.219
GUV-7	Lehr: P_{Le}	1.212

3.5 Further Z-Axioms

The transformation rates defined by definitions (GUV-1) through (GUV-7) satisfy all of the axioms Z1 through Z4. These axioms are probably not contentious. Additional axioms can help to differentiate further between better or worse definitions of the transformation rates, $\hat{z}_i$, and, thus, between the more or less reasonable GUV indices. Inspiration for the construction of these axioms can be obtained from traditional axiomatic index theory (see e.g., Section 4 and the Appendix). A selection of some possible additional axioms is listed below.

Z5 *The* ***Strict Monotonicity axiom*** *postulates that the values of the transformation rates, $\hat{z}_i$ (i = 1,…,N), are strictly monotonically increasing with the observed market prices:*

$$\frac{\partial \hat{z}_i}{\partial p_i^0} > 0 \quad and \quad \frac{\partial \hat{z}_i}{\partial p_i^1} > 0\,.$$

Obviously, any definition of the transformation rates, $\hat{z}_i$, that satisfies axiom Z5 will also satisfy axiom Z2 as well. Therefore, axiom Z5 can be viewed as a tightening of the condition posited by axiom Z2. The rates defined by definitions (GUV-1) and (GUV-2) both violate axiom Z5.

Z6 *The* ***Proportionality axiom*** *postulates that:*

$$if\;\; \frac{p_i^0}{p_j^0} = \frac{p_i^1}{p_j^1} = \lambda\;, then\;\; \frac{\hat{z}_i}{\hat{z}_j} = \lambda\;, \qquad with\; i, j\; =\; 1, \dots, N.$$

Z7 *The* ***Weak Mean Value axiom*** *postulates that:*

$$min\left(\frac{p_i^0}{p_j^0}, \frac{p_i^1}{p_j^1}\right) \leq \frac{\hat{z}_i}{\hat{z}_j} \leq max\left(\frac{p_i^0}{p_j^0}, \frac{p_i^1}{p_j^1}\right)\;, \qquad with\; i, j\; =\; 1, \dots, N.$$

It is clearly seen that the definitions (GUV-6) and (GUV-7) violate the axioms Z6, and, consequently, Z7.

Z8 *The* ***Independence axiom*** *postulates that the values of the ratios formed by the transformation rates, $\hat{z}_i/\hat{z}_j$, are independent from all products k ≠ i,j (i,j,k = 1,…,N).*

This tightening of the condition posited by axiom Z1 is satisfied by definitions (GUV-1) to (GUV-7). Additional axioms could easily be added to this list.

3.6 Discussion

The GUV index variants GUV-3 to GUV-7 represent viable alternatives to some of the most highly respected traditional price indices, for example, the Fisher index:

$$P_F = \sqrt{P_L P_P} = \frac{V^1}{V^0} \frac{\sqrt{V^0 V^{10}}}{\sqrt{V^1 V^{01}}}.$$

The only difference between the Fisher index and the GUV-3 index, that is, the Banerjee index (6), is the manner in which the value aggregates V^t and V^{st} are combined. The Fisher index utilizes a geometric mean while the Banerjee index employs the arithmetic mean. As Table 4 shows, the index numbers produced by the Fisher index, the Marshall-Edgeworth index,

$$P_{ME} = \frac{\sum p_i^1 (x_i^0 + x_i^1)/2}{\sum p_i^0 \left(x_i^0 + x_i^1\right)/2},$$

and the Walsh index,

$$P_W = \frac{\sum p_i^1 \sqrt{x_i^0 x_i^1}}{\sum p_i^0 \sqrt{x_i^0 x_i^1}},$$

are very similar to those produced by the GUV-3 through GUV-7 indices documented in Table 3.

Table 4: Example – Traditional Indices

Name	Index Number
Fisher: P_F	1.208
Marshall-Edgeworth: P_{ME}	1.200
Walsh: P_W	1.207

Moreover, the basic GUV index formula (3) factors into a very useful form. When deriving price indices, most price statisticians seek to decompose the value aggregate ratio, V^1/V^0, into an overall price change, P, and an overall quantity change, Q:

$$V^1/V^0 = P \cdot Q.$$

Substituting the basic GUV index formula (3) for P and solving for Q yields:

$$Q = \frac{\sum x_i^1 \hat{z}_i}{\sum x_i^0 \hat{z}_i}.$$

This is a very appealing quantity index because it is simply the ratio of the sum of transformed quantities in the comparison period divided by those in the base time period.

Lehr (1885, p. 39) and Davies (1924, p. 185) proposed their price indices many years ago. Nevertheless, presently the traditional price indices, Laspeyres, Paasche, Fisher, and Walsh, occupy center stage. The proposals of Lehr and Davies have never been actively pursued. What is behind this neglect? Did price statisticians succumb to a "herd instinct" mentality as they followed the lead of Irving Fisher (1922, p. 451) in rejecting the CPL methodology? In an attempt to highlight this as a possibility and to remove any existing prejudice that might exist, an axiomatic based investigation of the GUV index family members was conducted. The following section reports the results of this analysis and compares them with some of the most highly respected traditional price index formulas.

4 The Axiomatic Analysis

Axiomatic index theory can be used to determine if the GUV indices presented hereinbefore are appropriate for price measurement purposes. It is used to analyze whether certain proposed price indices satisfy a list of postulates called axioms that are deemed indispensable for the proper functioning of a meaningful price index. There exists some controversy, however, regarding which of the postulates provide the most compelling verification and which fail to do so. For that reason, a broad range of axioms is considered in this study. The axiomatic results that were derived for the indices GUV-1 to GUV-5 as well as the index GUV-7 are put side by side with those of the Fisher, Marshall-Edgeworth, and Walsh indices in Table 5. Proofs together with formal definitions of the axioms considered are available in the Appendix.

All of the price indices considered violate the A6 Permutation and the A18 Circularity axioms. Moreover, the Davies (GUV- 4), the GUV-5, and the Lehr (GUV-7) indices violate the A15 Strict Monotonicity axiom. Even though, violations of the A15 Strict Monotonicity axiom occur only in cases of extreme intertemporal price and quantity changes, this does represent a deficiency for the price indices involved. The A16 Weak Monotonicity axiom, however, is satisfied by all of the indices considered. The Lehr (GUV-7) index violates the A2 Proportionality axiom and, therefore, the A19 Strict Mean Value axiom as well. This axiom represents a tightening of the conditions posited by the A1 Identity axiom. This is also true for the A14 Linear Homogeneity axiom, which is violated by three of the GUV indices, namely the Banerjee (GUV-3), the GUV-5, and the Lehr (GUV-7) indices.

Table 5: Axiomatic Comparisons

	P_F	P_{ME}	P_W	P_{GUV}					
				1	2	3	4	5	7
				P_P	P_L	P_B	P_D		P_{Le}
A1 Identity	▲	▲	▲	▲	▲	▲	▲	▲	▲
A2 Proportionality	▲	▲	▲	▲	▲	▲	▲	▲	▽
A3 Inv. to Re-Ordering	▲	▲	▲	▲	▲	▲	▲	▲	▲
A4 Constant Quant.	▲	▲	▲	▲	▲	▲	▲	▲	▲
A5 Price Ratio	▲	▲	▲	▲	▲	▲	▲	▲	▲
A6 Permutation	▽	▽	▽	▽	▽	▽	▽	▽	▽
A7 Inversion	▲	▲	▲	▽	▽	▲	▲	▲	▲
A8 Strict Commens.	▲	▲	▲	▲	▲	▲	▲	▲	▲
A9 Weak Commens.	▲	▲	▲	▲	▲	▲	▲	▲	▲
A10 Price Dimension.	▲	▲	▲	▲	▲	▲	▲	▲	▲
A11 Quant. Dimension.	▲	▲	▲	▲	▲	▲	▲	▲	▲
A12 Strict Quant. Prop.	▲	▽	▲	▲	▲	▲	▲	▲	▽
A13 Weak Quant. Prop.	▲	▲	▲	▲	▲	▲	▲	▲	▲
A14 Lin. Homogeneity	▲	▲	▲	▲	▲	▽	▲	▽	▽
A15 Strict Monotonicity	▲	▲	▲	▲	▲	▲	▽	▽	▽
A16 Weak Monotonicity	▲	▲	▲	▲	▲	▲	▲	▲	▲
A17 Time Reversal	▲	▲	▲	▽	▽	▲	▲	▲	▲
A18 Circularity	▽	▽	▽	▽	▽	▽	▽	▽	▽
A19 Strict Mean Value	▲	▲	▲	▲	▲	▲	▲	▲	▽

Note: A Filled Triangle Indicates Test Satisfied and an Empty Triangle Indicates Test Violated.

The relevance of the axiomatic approach and the axioms that are listed continues to be a subject of controversy (see, e.g., Auer, 2009b). It is probably fair to infer, however, that a sufficient number of the GUV indices possess an axiomatic profile that is good enough to accept the proposition that the CPL methodology is a rational approach for the generation of reliable price index formulas.

5 Concluding Remarks

Price inflation can be computed as the quotient of the average price level in a comparative time period divided by the average price level in the base period. Lehr (1885) and Davies (1924) advocated this CPL methodology for price change measurements not only for homogeneous products but also for heterogeneous ones as well. Irving Fisher (1922), however, warned that the CPL methodology was inappropriate for heterogeneous products and recommended instead the use of the APC methodology, which computes the weighted average of the products' individual price changes. Consequently, current practice stipulates that price measurement using the CPL methodology should be limited to the case of homogeneous or very similar products that share a common identifying unit.

Contrary to that point of view, this study introduces a group of Generalized Unit Value (GUV) indices utilizing the CPL methodology. The study asserts that these indices produce reliable results in the case of heterogeneous products. It demonstrates that the GUV index (3) family includes the well-known Paasche (GUV-1), Laspeyres (GUV-2), and Banerjee (GUV-3) as well as the hardly known Davies (GUV-4) and Lehr (GUV-7) indices.

Moreover, the GUV indices could be used for the price aggregation of similar products with different qualitative characteristics, where hedonic analysis and other sophisticated quality adjustment methods are either impossible to conduct or are deemed excessively extravagant in terms of the resources they require.

Several of the GUV indices have been examined with respect to their axiomatic properties. Overall, they exhibited a solid axiomatic record, lending additional support to the claim that in the context of heterogeneous products the CPL methodology offers a reliable alternative to the currently employed APC methodology.

Some promising areas for future research exist. In addition to the axiomatic approach, the economic and stochastic approaches to price index theory are also available. In future research a systematic investigation into how these GUV indices relate to economic theory could prove to be a fruitful endeavor. The stochastic approach to index theory usually assumes that all observed price ratios are realizations of some random variable with an expected value equal to the "common inflation". The CPL methodology suggests the pursuit of a stochastic analysis that is based upon a less contentious assumption. It assumes that for each pair of products the price ratio observed during the base period and the price ratio observed during the comparison period represent realizations of a random variable with an expected value equal to the ratio of the product's transformation rates. Based upon this assumption, one could compare the statistical properties of the estimators of the ratios of transformation rates used by the various GUV indices. The GUV indices could also be applied in other measurement situations not specifically referred to in this study. An obvious area could involve interregional price comparisons.

Perhaps the time has come to reconsider the words of Irving Fisher in light of the arguments presented many years before by Julius Lehr and George R. Davies. The notion of average price levels deserves a fresh new look in conjunction with the CPL methodology of inflation measurement.

Appendix

The appendix contains proofs of the axiomatic results in Table 5. Results for the GUV-1 (Paasche), GUV-2 (Laspeyres), Fisher, Marshall-Edgeworth, and Walsh indices are found in Auer (2001) or they are trivial. Hereinafter, four of the GUV indices are considered, the GUV-3 (Banerjee), GUV-4 (Davies), GUV-5 and GUV-7 (Lehr) indices. The (assessed-value) transformation rates are denoted simply as z_i.

A price index is a function P that maps all N strictly positive prices, $\mathbf{p}^{\mathrm{t}} = (p_1^t, \dots, p_N^t)$, and quantities, $\mathbf{x}^{\mathrm{t}} = (x_1^t, \dots, x_N^t)$, in the base time period, $t = 0$, as well as a comparison time period, $t = 1$, into a single positive index number:

$$P : \mathbb{R}_{++}^{4N} \to \mathbb{R}_{++}, \quad (\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1) \to P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1) .$$

A1 *The* ***Identity axiom*** *(Laspeyres, 1871, p. 308) postulates that*

$$P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^0, \mathbf{x}^1) = 1\,.$$

In the scenario specified by this axiom, $p_i^0 = p_i^1 = p_i$, leading to $z_i = p_i$ for all four GUV indices (GUV-3, GUV-4, GUV-5, and GUV-7). Therefore, the GUV formula (3) is:

$$P_{GUV} = \frac{V^1}{V^0}\frac{\sum p_i\, x_i^0}{\sum p_i\, x_i^1} = \frac{V^1}{V^0}\frac{V^0}{V^1} = 1\,.$$

***A2** The **Proportionality axiom** (Walsh, 1901, p. 115) postulates that*

$$P(\mathbf{p}^0, \mathbf{x}^0, \lambda\mathbf{p}^0, \mathbf{x}^1) = \lambda, \;\; for\ all\ \lambda > 0.$$

In the scenario specified by this axiom, $p_i^1 = \lambda p_i^0$. Therefore, the GUV index (3) yields:

$$P_{GUV} = \frac{\lambda V^{01}}{V^0}\frac{\sum \tilde{z}_i\, x_i^0}{\sum \tilde{z}_i\, x_i^1}\,,$$

where $\tilde{z}_i$ indicates the transformation rates associated with the value of λ. The satisfaction of this axiom requires that

$$\frac{V^{01}}{V^0} = \frac{\sum \tilde{z}_i x_i^1}{\sum \tilde{z}_i x_i^0}\,,$$

and therefore,

$$\frac{\sum p_i^0\, x_i^1}{\sum p_i^0\, x_i^0} = \frac{\sum \tilde{z}_i\, x_i^1}{\sum \tilde{z}_i\, x_i^0}\,.$$

This requirement is satisfied, if and only if for all values of λ, $p_i^0 = \eta\tilde{z}_i$, with η being some constant. This condition is satisfied by the GUV-3, GUV-4, and GUV-5 indices, but not by the GUV-7 index.

***A3** The **Invariance to Re-Ordering axiom** (Fisher, 1922, p. 63) postulates that*

$$P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1) = P(\tilde{\mathbf{p}}^0, \tilde{\mathbf{x}}^0, \tilde{\mathbf{p}}^0, \tilde{\mathbf{x}}^1)\,,$$

where the vectors $\tilde{\mathbf{p}}^0$, $\tilde{\mathbf{x}}^0, \tilde{\mathbf{p}}^1$, and $\tilde{\mathbf{x}}^1$ *are arbitrary uniform permutations of the original vectors*.

A reordering of the elements in the summations of the GUV index (3) does not alter the value of the index.

***A4** The **Constant Quantities axiom** (Lowe, 1822, Appendix, p. 95) postulates that*

$$P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^0) = \frac{\sum p_i^1\, x_i^0}{\sum p_i^0\, x_i^0}$$

In the scenario specified by this axiom, the GUV index (3) is:

$$P_{GUV} = \frac{V^{10}}{V^0}\,\frac{\sum z_i x_i^0}{\sum z_i x_i^0} = \frac{V^{10}}{V^0} = \frac{\sum p_i^1\, x_i^0}{\sum p_i^0\, x_i^0}\ .$$

***A5** The **Price Ratio axiom** (Eichhorn and Voeller, 1990, p. 326) postulates that*

$$P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1) = p_1^1/p_1^0, \;\; for\ N = 1.$$

If $N = 1$, then the GUV index (3) is:

$$P_{GUV} = (p_1^1 x_1^1/p_1^0 x_1^0)(x_1^0 z_1/x_1^1 z_1) = p_1^1/p_1^0\ \ .$$

***A6** The **Permutation axiom** (Auer, 2002, p. 534) postulates that*

$$P(\mathbf{p}^0, \mathbf{x}^0, \tilde{\mathbf{p}}^0, \tilde{\mathbf{x}}^0) = 1\,,$$

where the vectors $\tilde{\mathbf{p}}^0$ and $\tilde{\mathbf{x}}^0$ are arbitrary uniform permutations of the original vectors.

The scenario specified by this axiom yields $V^0 = V^1$. Accordingly, the GUV index (3) is:

$$P_{GUV} = \frac{\sum x_i^0 \tilde{z}_i}{\sum x_i^1 \tilde{z}_i},$$

where $\tilde{z}_i$ indicates the transformation rates resulting from the scenario specified by this axiom. The axiom is satisfied, if and only if,

$$\sum \tilde{z}_i (x_i^0 - x_i^1) = 0\,. \tag{7}$$

All of the listed GUV indices violate this condition.

A7 *The **Inversion axiom** (Auer, 2002, p. 534) postulates that*

$$P(\mathbf{p}^0, \mathbf{x}^0, \tilde{\mathbf{p}}^0, \tilde{\mathbf{x}}^0) = 1 ,$$

where the vectors , $\tilde{\mathbf{p}}^0$ *and* $\tilde{\mathbf{x}}^0$ *are special permutations of the original vectors, such that* $p_j^0 = \tilde{p}_k^0$ *,* $p_k^0 = \tilde{p}_j^0, x_j^0 = \tilde{x}_k^0, x_k^0 = \tilde{x}_j^0$ *and for all* $i \neq j, k, p_i^0 = \tilde{p}_i^0$ *and* $x_i^0 = \tilde{x}_i^0$.

The scenario specified by this axiom yields $V^0 = V^1$. Therefore, the satisfaction of this axiom requires that condition (7) be satisfied. This condition is satisfied, if and only if,

$$\tilde{z}_j\left(x_j^0 - x_j^1\right) + \tilde{z}_k(x_k^0 - x_k^1) = 0 .$$

This condition is equivalent to

$$\tilde{z}_j\left(x_j^0 - x_k^0\right) + \tilde{z}_k\left(x_k^0 - x_j^0\right) = 0 ,$$

and therefore to

$$\left(\tilde{z}_j - \tilde{z}_k\right)\left(x_j^0 - x_k^0\right) = 0 .$$

Due to the fact that, $x_j^0 \neq x_k^0$, this condition is satisfied, if and only if,

$$\tilde{z}_j = \tilde{z}_k .$$

In the scenario specified by the inversion axiom, the latter condition is satisfied by all of the four GUV indices.

A8 *The **Strict Commensurability axiom** (Pierson, 1896, p. 131) postulates that*

$$P\left(\mathbf{p}^0\boldsymbol{\Lambda}, \mathbf{x}^0\boldsymbol{\Lambda}^{-1}, \mathbf{p}^1\boldsymbol{\Lambda}, \mathbf{x}^1\boldsymbol{\Lambda}^{-1}\right) = P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1),$$

where $\boldsymbol{\Lambda}$ *is a* $(N \times N)$ *diagonal matrix with positive elements* λ_i.

Let $\tilde{z}_i$ indicate the transformation rates resulting from the λ_i values. According to the GUV index (3), this axiom is satisfied, if and only if,

$$\frac{\sum x_i^0 z_i}{\sum x_i^1 z_i} = \frac{\sum(x_i^0/\lambda_i)\tilde{z}_i}{\sum(x_i^1/\lambda_i)\tilde{z}_i} .$$

The four GUV indices have $\tilde{z}_i = z_i\lambda_i$, and therefore, satisfy this axiom.

A9 *The **Weak Commensurability axiom** (Swamy, 1965, p. 620) postulates that*

$$P(\mathbf{p}^0\lambda, \mathbf{x}^0\lambda^{-1}, \mathbf{p}^1\lambda, \mathbf{x}^1\lambda^{-1}) = P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1) , \qquad for\ all\ \lambda > 0.$$

With the same argumentation as axiom A7, this axiom is satisfied by the four GUV indices.

A10 *The* ***Price Dimensionality axiom*** *(Eichhorn and Voeller, 1976, p. 24) postulates that*

$$P(\lambda\mathbf{p}^0,\mathbf{x}^0,\lambda\mathbf{p}^1,\mathbf{x}^1) = P(\mathbf{p}^0,\mathbf{x}^0,\mathbf{p}^1,\mathbf{x}^1)\,, \qquad for\ all\ \lambda > 0.$$

The satisfaction of this axiom requires that

$$\frac{\sum x_i^0 z_i}{\sum x_i^1 z_i} = \frac{\sum x_i^0 \tilde{z}_i}{\sum x_i^1 \tilde{z}_i}, \tag{8}$$

where $\tilde{z}_i$ indicates the transformation rates associated with the value of λ. The four GUV indices have $\tilde{z}_i = z_i\lambda$, and therefore, satisfy this axiom.

A11 *The* ***Quantity Dimensionality axiom*** *(Funke et al., 1979, p. 680) postulates that*

$$P(\mathbf{p}^0,\lambda\mathbf{x}^0,\mathbf{p}^1,\lambda\mathbf{x}^1) = P(\mathbf{p}^0,\mathbf{x}^0,\mathbf{p}^1,\mathbf{x}^1)\,, \qquad for\ all\ \lambda > 0.$$

Any price index that satisfies axioms A8 and A9, automatically satisfies this axiom as well. Therefore, this axiom is satisfied by the four GUV indices.

A12 *The* ***Strict Quantity Proportionality axiom*** *(Vogt, 1980, p. 70, and Diewert, 1992, p. 216) postulates that*

$$P(\mathbf{p}^0,\mathbf{x}^0,\mathbf{p}^1,\lambda\mathbf{x}^1) = P(\mathbf{p}^0,\delta\mathbf{x}^0,\mathbf{p}^1,\mathbf{x}^1) = P(\mathbf{p}^0,\mathbf{x}^0,\mathbf{p}^1,\mathbf{x}^1), \quad for\ all\ \lambda,\delta > 0.$$

This axiom is satisfied, if and only if,

$$\frac{\lambda V^1}{V^0}\frac{\sum x_i^0 \tilde{z}_i}{\sum \lambda x_i^1 \tilde{z}_i} = \frac{V^1}{\delta V^0}\frac{\sum \delta x_i^0 \tilde{z}_i'}{\sum x_i^1 \tilde{z}_i'} = \frac{V^1}{V^0}\frac{\sum x_i^0 z_i}{\sum x_i^1 z_i}\,,$$

and therefore, if

$$\frac{\sum x_i^0 \tilde{z}_i}{\sum x_i^1 \tilde{z}_i} = \frac{\sum x_i^0 \tilde{z}_i'}{\sum x_i^1 \tilde{z}_i'} = \frac{\sum x_i^0 z_i}{\sum x_i^1 z_i}\,, \tag{9}$$

where $\tilde{z}_i$ and $\tilde{z}_i'$ indicate the transformation rates associated with the scenarios specified by this axiom. Condition (9) is satisfied by the GUV-3, GUV-4, and GUV-5 indices, but not by the GUV-7 index.

A13 *The* ***Weak Quantity Proportionality axiom*** *(Auer, 2001, p. 6) postulates that*

$$P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \lambda\mathbf{x}^0) = P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^0), \quad for\ all\ \lambda > 0.$$

In the scenario specified by this axiom, $x_i^1 = \lambda x_i^0$. As a consequence, the GUV index (3) is:

$$P_{GUV} = \frac{\lambda V^{10}}{V^0} \frac{\sum z_i x_i^0}{\sum z_i \lambda x_i^0} = \frac{V^{10}}{V^0}.$$

Therefore, this axiom is satisfied by the four GUV indices.

A14 *The* ***Linear Homogeneity axiom*** *(Walsh, 1901, p. 385, and Eichhorn and Voeller, 1976, p. 28) postulates that*

$$P(\mathbf{p}^0, \mathbf{x}^0, \lambda\mathbf{p}^1, \mathbf{x}^1) = \lambda P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1) = P[(1/\lambda)\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1], \ for\ all\ \lambda > 0.$$

This axiom is satisfied, if and only if, Equation (8) is satisfied, where $\tilde{z}_i$ indicates the transformation rates associated with λ. This requires that for each scenario given by this axiom, $\tilde{z}_i = z_i \eta$, where η is some constant. This requirement is satisfied by the GUV-4 index, but violated by the GUV-3, GUV-5, and GUV-7 indices.

A15 *the* ***Strict Monotonicity axiom*** *(Eichhorn and Voeller, 1976, p. 23) considers two different scenarios for the comparison* ($t = 1$ and $t = 1^*$) *and base time periods* ($t = 0$ and $t = 0^*$). *If for all products $p_i^{1^*} \geq p_i^1$ and for at least one product i the strict relation holds, then the axiom postulates that*

$$P\left(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^{1^*}, \mathbf{x}^1\right) > P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1)\,, \tag{10}$$

and if for all products $p_i^{0^*} \geq p_i^0$ and for at least one product i the strict relation holds, then the axiom postulates that

$$P\left(\mathbf{p}^{0^*}, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1\right) < P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1)\,. \tag{11}$$

According to the scenario specified in (10), for all products the relation $\mathrm{d}p_k^1 \geq 0$ holds and for at least one product k the strict relation holds. It is to be shown that for such a scenario,

$$\mathrm{d}P_{GUV} = \sum(\partial P_{GUV}/\partial p_k^1)\mathrm{d}p_k^1 > 0\,. \tag{12}$$

The variants of the GUV index family yield,

	$\partial z_k/\partial p_k^1$	$\partial z_k/\partial p_k^0$
(GUV-3):	0.5	0.5
(GUV-4):	$0.5(p_k^0/p_k^1)^{1/2}$	$0.5(p_k^1/p_k^0)^{1/2}$
(GUV-5):	$2(p_k^1/p_k^0+1)^{-2}$	$2(p_k^0/p_k^1+1)^{-2}$
(GUV-7):	$x_k^1/(x_k^0+x_k^1)$	$x_k^0/(x_k^0+x_k^1)$

From the GUV index (3) it follows that

$$\begin{aligned}\frac{\partial P_{GUV}}{\partial p_k^1} &= \frac{1}{V^0}\frac{[x_k^1\sum x_i^0 z_i + V^1 x_k^0(\partial z_k/\partial p_k^1)]\sum x_i^1 z_i - V^1(\sum x_i^0 z_i)x_k^1\,(\partial z_k/\partial p_k^1)}{(\sum x_i^1 z_i)^2}\\ &= \frac{x_k^1\sum x_i^0 z_i + V^1(\partial z_k/\partial p_k^1)x_k^0 - V^1(\partial z_k/\partial p_k^1)x_k^1\,(\sum x_i^0 z_i)/(\sum x_i^1 z_i)}{V^0\sum x_i^1\, z_i}\\ &= \frac{x_k^1(\sum x_i^0 z_i)[1 - V^1(\partial z_k/\partial p_k^1)/(\sum x_i^1 z_i)] + V^1(\partial z_k/\partial p_k^1)x_k^0}{V^0\sum x_i^1\, z_i}\,.\end{aligned} \tag{13}$$

The denominator of Equation (13) is positive. For the GUV-3 index, the term in squared brackets simplifies to, $[1 - V^1/(V^{01} + V^1)] > 0$. Therefore, the partial derivative (13) is positive. Consequently, condition (12) is satisfied by the GUV-3 index. Analogous reasoning applies to the scenario specified by inequality (11).

For the GUV-4, GUV-5, and GUV-7 indices, with sufficiently large values of p_i^1 and $x_i^0 (i \neq k)$, the numerator in (13) becomes negative. Therefore, these indices violate the axiom.

A16 *The* ***Weak Monotonicity axiom*** *(Olt, 1996, p. 37) considers two different situations. If for all products $p_i^1 \geq p_i^0$ and for at least one product i the strict relation holds, then the axiom postulates that*

$$P(\mathbf{p}^0,\mathbf{x}^0,\mathbf{p}^1,\mathbf{x}^1) > P(\mathbf{p}^0,\mathbf{x}^0,\mathbf{p}^0,\mathbf{x}^1)\,, \tag{14}$$

and if for all products $p_i^l \leq p_i^0$ *and for at least one product i the strict relation holds, then the axiom postulates that*

$$P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1) < P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^0, \mathbf{x}^1)\ . \tag{15}$$

The reference scenario $P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^0, \mathbf{x}^1)$ implies that $p_i^0 = p_i^1 = p_i$. For such a scenario, the GUV-3, GUV-4, GUV-5, and GUV-7 indices have $z_i = p_i$ and $0 < \partial \mathrm{z_k} / \partial \mathrm{p_k^1} < 1$. In the numerator of Equation (13), the term in squared brackets simplifies to $[1 - (\partial z_k / \partial p_k^1)] > 0$. Therefore, also the total differential in (12) is positive. Analogous reasoning applies to the scenario specified by inequality (15). As a consequence, this axiom is satisfied by the four GUV indices.

A17 *The* ***Time Reversal axiom*** *(Pierson, 1896, p. 128, and Walsh, 1901, p. 368) postulates that*

$$P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1) = 1/P(\mathbf{p}^1, \mathbf{x}^1, \mathbf{p}^0, \mathbf{x}^0)\ .$$

Since the z_i-values in the four GUV indices are invariant with respect to swaps of p_i^0 and p_i^1, the GUV index (3) implies that these indices satisfy the axiom.

A18 *The* ***Circularity axiom*** *(Westergaard, 1890, p. 218) postulates that*

$$P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^2, \mathbf{x}^2) = P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1) \cdot P(\mathbf{p}^1, \mathbf{x}^1, \mathbf{p}^2, \mathbf{x}^2)\ .$$

The four GUV indices violate this axiom, because the z_i-values differ between the three bilateral indices.

A19 *The* ***Strict Mean Value axiom*** *(Olt, 1996, p. 26) postulates that*

$$\min_i\{p_i^1/p_i^0\} < P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1) < \max_i\{p_i^1/p_i^0\}\ ,$$

and for $\mathbf{p}^1 = \lambda \mathbf{p}^0$ *the relation* "<" *becomes* "=".

The fact that the GUV-7 index does not satisfy axiom A2, renders it unable to satisfy this axiom as well. The axiom A19 is satisfied by any price index that can satisfy axioms A1 (Identity), A14 (Linear Homogeneity), and A16 (Weak Monotonicity). In order to demonstrate this, let $p_i^{1^*} = p_i^1/\min_j\{p_j^1/p_j^0\}$. Therefore,

$$\frac{p_i^{1^*}}{p_i^0} = \frac{p_i^1/p_i^0}{\min_j\{p_j^1/p_j^0\}}.$$

Consequently, $\min_i\{p_i^{1^*}/p_i^0\} = 1$, and, therefore, all products i have $p_i^{1^*} \geq p_i^0$. This is the scenario specified in Equation (14) of axiom A16. If a price index satisfies axioms A16 and A1, then,

$$\mathrm{P}(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^{1^*}, \mathbf{x}^1) > 1$$

$$\Rightarrow \quad \min_j\{p_j^1/p_j^0\} P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^{1^*}, \mathbf{x}^1) > \min_j\{p_j^1/p_j^0\} \ .$$

Due to the satisfaction of the axiom A14, this inequality becomes

$$P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1) > \min_j\{p_j^1/p_j^0\}.$$

Furthermore, let $p_i^{1^{**}} = p_i^1/\max_j\{p_j^1/p_j^0\}$. As a consequence,

$$\frac{p_i^{1^{**}}}{p_i^0} = \frac{p_i^1/p_i^0}{\max_j\{p_j^1/p_j^0\}}.$$

Consequently, $\max_i\{p_i^{1^*}/p_i^0\} = 1$, and, therefore, all products i have $p_i^{1^{**}} \leq p_i^0$. This is the scenario specified by Equation (15) of axiom A16. If a price index satisfies axioms A1 and A16, then:

$$P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^{1^{**}}, \mathbf{x}^1) < 1$$

$$\max_j\{p_j^1/p_j^0\} \ P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^{1^{**}}, \mathbf{x}^1) < \max_j\{p_j^1/p_j^0\}.$$

Due to the satisfaction of axiom A14, this inequality becomes

$$P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1) < \max_j\{p_j^1/p_j^0\} \ .$$

Since the GUV-4 index satisfies axioms A1, A14, and A16, it satisfies the axiom A19.

A price index that satisfies axioms A2 (Proportionality) and A15 (Strict Monotonicity) also satisfies this axiom. From axiom A2,

$$P(\mathbf{p}^0, \mathbf{x}^0, \min_i\{p_i^1/p_i^0\} \cdot \mathbf{p}^0, \mathbf{x}^1) = \min_i\{p_i^1/p_i^0\}$$

$$P(\mathbf{p}^0, \mathbf{x}^0, \max_i\{p_i^1/p_i^0\} \cdot \mathbf{p}^0, \mathbf{x}^1) = \max_i\{p_i^1/p_i^0\},$$

and from axiom A15,

$$P(\mathbf{p}^0, \mathbf{x}^0, \min_i\{p_i^1/p_i^0\} \cdot \mathbf{p}^0, \mathbf{x}^1) < P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1)$$
$$P(\mathbf{p}^0, \mathbf{x}^0, \max_i\{p_i^1/p_i^0\} \cdot \mathbf{p}^0, \mathbf{x}^1) > P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1)\,.$$

Taken together,

$$\min_i\{p_i^1/p_i^0\} < P(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1) < \max_i\{p_i^1/p_i^0\}\,.$$

An earlier proof is found in Eichhorn and Voeller, (1990, p. 332). The proofs imply that axiom A19 is satisfied by the GUV-3 index.

From Equation (3), one obtains for the GUV-5 index the expression

$$P_{GUV} = \frac{\sum\left(v_i^0/\sum v_j^0\right)[1 + (p_i^1/p_i^0)^{-1}]^{-1}}{\sum\left(v_i^1/\sum v_j^1\right)[1 + p_i^1/p_i^0]^{-1}}\,. \qquad (16)$$

If in Equation (16) the price ratios p_i^1/p_i^0 are replaced by $\min_j\{p_j^1/p_j^0\}$ (the weights $v_i^0/\sum v_j^0$ remaining unchanged), the right hand side of that equation becomes

$$\frac{\sum\left(v_i^0/\sum v_j^0\right)\left[1 + \left(\min_j\{p_j^1/p_j^0\}\right)^{-1}\right]^{-1}}{\sum\left(v_i^1/\sum v_j^1\right)\left[1 + \min_j\{p_j^1/p_j^0\}\right]^{-1}} = \min_j\{p_j^1/p_j^0\}\,.$$

Replacing in the numerator and denominator $\min_j\{p_j^0/p_j^1\}$ by the actual price ratios p_i^1/p_i^0, the value of the numerator increases and the value of the denominator falls, yielding,

$$P_{GUV} > \min_j\{p_j^1/p_j^0\}\ .$$

If in Equation (16) the price ratios p_i^1/p_i^0 are replaced by $\max_j\{p_j^1/p_j^0\}$ (the weights $v_i^0/\sum v_j^0$ remaining unchanged), the right hand side of that equation becomes

$$\frac{\sum\left(v_i^0/\sum v_j^0\right)\left[1 + \left(\max_j\{p_j^1/p_j^0\}\right)^{-1}\right]^{-1}}{\sum\left(v_i^1/\sum v_j^1\right)\left[1 + \max_j\{p_j^1/p_j^0\}\right]^{-1}} = \max_j\{p_j^1/p_j^0\}\,.$$

In the numerator as well as in the denominator replacing $\max_j\{p_j^1/p_j^0\}$ by the actual price ratios p_i^1/p_i^0, the value of the numerator falls and the value of the denominator increases, yielding

$$P_{GUV} < \max_j\{p_j^1/p_j^0\}\ .$$

As a consequence, the GUV-5 index satisfies axiom A19.

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