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The Dynamics of Currency Crises—Results from Intertemporal Optimization and Viscosity Solutions

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We apply an infinite horizon intertemporal optimization model to a simple speculative attack framework. Thereby, the central bank faces a one control two-state variables optimization problem with endogenuous exit. By setting the interest rate the central bank can stimulate the economy or fend off speculators. We show that two focal points emerge. Depending on the time preference and the state, cycles can improve utility. A regime change is associated with costs and can be forced by the state of the economy or induced by choice. In the latter case the costs for defending outweigh the costs of an immediate opt-out. During the existence of the regime the highest growth is reached through convergence to a no stress steady state, but is only optimal for a central bank with low time preference. Therefore, we propose to take measures assuring a lower time preference like independence, long-term mandates, and long-term policy goals.

Keywords: intertemporal optimization; currency crises; policy design
JEL Codes: C61; E58; E61; F3

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1 Introduction

Previous literature modelling financial crises and speculative attacks highlighted particularly the aspects of speculators attacking a currency. However, it did not incorporate the main role of the central bank adequately. In fact, setting the interest rate influences the fundamentals and the costs of speculators. Thus, the behavior of the central bank is neither a passive reaction due to speculative pressure nor sole signalling—it changes the state of the economy.

If the central bank chooses to defend a fixed exchange rate regime by raising the interest rate, it accepts that fundamentals decline and furthermore accepts that the declining fundamentals reinforce the future attack and thus worsen its future position. Hence, the behavior of the central bank is crucial for both, the evolution of the economy and for its own future position. On the other hand, the speculators know that attacking weakens the position of the central bank and that the attack is successful if the central bank is weak enough. Though, they also have to consider their costs if the central bank decides to defend as a reaction on the attack.

The trade-off for the central bank is that one control influences the possibility to benefit from the regime as well as the probability to bear the costs of a regime change, which occurs if the attack strength exceeds the defensive measure of the central bank.

To incorporate the trade-off, induced by the impact of the interest rate, we apply an infinite horizon intertemporal optimization framework. The time, when the central bank is forced or chooses to abandon the peg, is endogenously determined. Thus, the time horizon exceeds the duration of the regime. After briefly summarizing the literature, we first describe the general framework where we introduce the objective function and two state processes for the fundamentals and the attack. Second, we offer a solution for a simple case of the model where states are just linearly dependent on the interest rate. Third, we describe an extended linear model with fundamental feedback and herding effects.

We find that two focal points emerge, which attract the state space trajectories. A low time preference central bank will bear current costs, caused through defending, to steer the economy to the good focal point. However, a high time preference central bank avoids current losses and steers the economy to the bad focal point. Moreover, in good fundamental states with high pressure it can be optimal for the central bank to abandon the regime immediately, thereby preventing a long-term costly defense.

2 Literature

In the early models of currency crises, termed “first generation”, monetizing a fiscal deficit leads to a steady decline in the reserve stock. Rational speculators anticipating the imminent exhaustion of reserves instantly withdraw their money, causing the actual crisis (cf. Krugman 1979). Flood and Garber (1984) gave an analytical solution of a Krugman type model, where arbitrary speculation can lead to a crisis. The “second generation” models speculation as a coordination problem between investors and implicitly assumes that the underlying fundamental state of the economy is common knowledge. The central bank strategically weighs the costs and benefits of a potential defense of the fixed exchange rate. Thereby, the fundamental state as well as the private expectations about a depreciation play the main role. Since private expectations alter the costs of the central bank, expectations can become self-fulfilling (cf. Obstfeld 1994 and 1996). Speculators face strategic complementarities, so that their payoffs depend on the action of others. High degrees of coordination, e.g. complete information, may result in multiple equilibria. Morris and Shin (1998) showed that if every speculator gets sufficiently precise private information, a unique equilibrium can be determined. Bauer and Herz (2013) explicitly model the strategic options of a central bank in a two stage global game. The central bank chooses its defensive measure after it observes a noisy signal about the attack strength. Thereby, it has to acknowledge the costs of defense as well as the costs for a possible devaluation. Angeletos et al. (2006) investigate the informational effects of central bank actions. Policy decisions convey information regarding the central bank's knowledge about the underlying state. This additional information allows a better coordination of speculators and produces multiple equilibria. Heinemann et al. (2004) find in experiments that global games give a good description of actual behavior. The effects of the information structure and the signals show signs in accordance with theory, but are mostly insignificant in size. This suggests that the main focus on modelling information might not be the most constructive way in approaching a better understanding of currency crises.

Morris and Shin (1999) take an approach to analyze the evolution of beliefs in a dynamic context. They investigate the changes of sentiment based on changes in the underlying fundamentals, which are assumed to follow a stochastic process. Basically, they model a sequence of repeated one shot global games, where the previous realization of the fundamentals is common knowledge. Chamley (2003) examines a dynamic global game, in which speculators utilize the movement of the exchange rate in a band as a proxy for the mass of attackers, so that it suffices as a coordination device. Predictable interventions that reduce the fluctuation in the exchange rate reduce speculator's risk and thus foster the attack. However, raising the interest rate, widening the fluctuation band, and conducting random interventions in the currency can prevent an attack. The random intervention reduces the informativeness of the exchange rate and aggravates coordination. Ceteris paribus this policy allows a smaller stock of reserves than deterministic intervention. Angeletos et al. (2007) introduce dynamics through a repeated global game, where speculators learn about the underlying fundamentals. Then, they examine equilibrium properties of different exogenous changes. Information as well as fundamentals can be the trigger for a shift from tranquility to distress. They state, without explicitly modelling, that defense is possible through higher interest rates, where the required increase depends on the quality of information of speculators about the fundamentals. Hence, defense is more costly when information improves. Guimarães (2006) introduces a Poisson process that admits a random fraction of speculators to adapt their positions. This allows to model the evolution of a crisis, where the currency can be overvalued for a long time until an attack is triggered. Admitting less speculators to change their position, raising the interest rate, or reducing the overvaluation each lower the probability of a crisis.

Nearly all approaches focus on modelling information, neglecting—particularly in dynamic setups—the crucial influence of the central bank's choice of the interest rate on the underlying fundamentals. Therefore, we present an approach that models currency crises as an intertemporal optimization problem that accounts for the reflexive nature of policy decisions. Each decision has different consequences for the future path of the economy and the future position of the central bank.

3 Model

There are two actors: the central bank and speculators. The central bank maximizes utility

$$U_0\left(\theta_S, A_S\right) = \int_0^T e^{-\rho t} u\left(\theta\left(t\right)\right) dt + e^{-\rho T} \frac{1}{\rho} \upsilon\left(\theta\left(T\right) - c\right), \tag{1}$$

where instantaneous utility u is derived from the state of the fundamentals $\theta\left(t\right)$ and is discounted by factor ρ. The initial values of the fundamentals and the attack are $\theta_S = \theta\left(0\right)$ and $A_S = A\left(0\right)$. The overall utility U is the sum of the aggregated discounted instantaneous utility up to terminal time T plus the discounted terminal value.[1] The terminal time denotes the time when the central bank is forced to devalue and is endogenously determined by the state processes. The terminal value υ is a function of the fundamentals at terminal time less an amount c representing the costs of the regime change. For the remainder of the paper, we assume that the proceeding regime is in a steady state, so that the terminal value υ is constant.

The central bank maximizes the objective function (1) by setting the interest rate $r\left(t\right)$, which is always nonnegative $r\left(t\right) \geq 0$. The optimization problem is subject to the state of the system which is summarized by the state vector x that evolves according to

$$\dot{x} = \begin{pmatrix} \dot{\theta}\left(t\right) \\ \dot{A}\left(t\right) \end{pmatrix} = \begin{pmatrix} f\left(r\left(t\right), \theta\left(t\right)\right) \\ g\left(r\left(t\right), \theta\left(t\right), A\left(t\right)\right) \end{pmatrix}. \tag{2}$$

There are two state variables, the fundamentals $\theta\left(t\right)$ and the strength of the attack $A\left(t\right)$. The first state variable $\theta\left(t\right)$ enters utility directly, while the second $A\left(t\right)$ determines the terminal time $T = \inf\left\{t : A\left(t\right) > D\right\}$. This is, the first time when the strength of the attack exceeds the defensive measure D, e.g. the amount of reserves held by the central bank.[2] Hence, the central bank's control has two effects: firstly, it influences the fundamentals and thereby directly the utility. Secondly, it influences the terminal time in which utility can be accumulated and simultaneously the effect of the terminal value.[3]

The change of the fundamentals depends on their own current state and the interest rate. The central bank influences the fundamentals by setting the interest rate in relation to the

[1] For the given setup $\lim_{T\to\infty} e^{-\rho T} \upsilon\left(\theta\left(T\right) - c\right) = 0$, i.e. without devaluation the second term of equation 1 vanishes.

[2] Naturally, we restrict the initial state vector to be feasible, i.e. $A\left(0\right) \leq D$.

[3] As we describe later, utility might also decrease, independent of the policy chosen, so that an early opt-out is favorable.

natural rate $\bar{r}$. For interest rates below the natural rate, the cost of credit is below the possible return on investment. As a consequence investment increases and the economic fundamentals improve and vice versa (cf. Wicksell 1898). The motion of fundamentals is often represented by a Brownian motion (cf. Morris and Shin 1999 or Guimarães 2006), where deviations of the fundamentals from the natural rate $\bar{\theta}$ tend to be reversed over time. Therefore, we define the evolution of the fundamentals by

$$\dot{\theta} = f\left(r\left(t\right), \theta\left(t\right)\right) = -f_1\left(r\left(t\right)\right) - f_2\left(\theta\left(t\right)\right). \tag{3}$$

Where $\frac{\partial f_1(.)}{\partial r(t)} > 0$ is the interest rate elasticity of the fundamentals and $\frac{\partial f_2(.)}{\partial \theta(t)} \geq 0$ is the mean reversion elasticity of the fundamentals. The mean reversion works as a stabilizing mechanism that improves bad fundamentals (below the natural level) and reduces good fundamentals (higher than the natural level). Obviously, such a fundamentals process possesses a steady state $(\theta, r) = \left(\bar{\theta}, \bar{r}\right)$ if $f_1\left(\bar{r}\right) = f_2\left(\bar{\theta}\right) = 0$.

The motion of the attack depends on the costs $r\left(t\right)$, the fundamentals $\theta\left(t\right)$, and on strategic complementarities, i.e. a herding effect $A\left(t\right)$. When speculators expect a currency to devalue, they borrow the currency and sell it against foreign money. If the devaluation takes place, the position is closed. The profit equals the amount of the devaluation minus the costs for the loan. Increasing the interest rate raises the costs for speculators causing them to refrain from attacking (cf. e.g. Angeletos et al. 2007, Chamley 2003 and Daniëls et al. 2011). Here, the interest rate has only a defensive effect if it is higher than the natural rate $\bar{r}$. Below, the attack rises due to low costs of speculation. The success of an attack depends on the fundamentals of the economy: the expected payoff of the speculators decreases when fundamentals improve (cf. Obstfeld 1996 and Morris and Shin 1998). Hence, speculators refrain from attacking if the fundamentals are above their natural rate and vice versa. However, speculators also tend to imitate the behavior of other speculators without considering their own information (cf. Banerjee 1992 and Bikhchandani et al. 1992). Due to this herding effect an increase of the attack is ceteris paribus higher if more speculators already hold positions against the currency. We treat the attack strength as a reduced form equation of the aforementioned effects. Its evolution is given by

$$\dot{A} = g\left(r\left(t\right), \theta\left(t\right), A\left(t\right)\right) = -g_1\left(r\left(t\right)\right) - g_2\left(\theta\left(t\right)\right) + g_3\left(A\left(t\right)\right), \tag{4}$$

where $\frac{\partial g_1(.)}{\partial r(t)} > 0$ is the interest rate elasticity of the attack, $\frac{\partial g_2(.)}{\partial \theta(t)} \geq 0$ is the fundamentals elasticity, and $\frac{\partial g_3(.)}{\partial A(t)} \geq 0$ is the herding elasticity. If we assume, as above, that $g_1\left(\bar{r}\right) = g_2\left(\bar{\theta}\right) = 0$ and additionally that $g_3\left(0\right) = 0$ the attack is in steady state at $\left(\bar{\theta}, \bar{r}, 0\right)$. This equals the fundamental's steady state without speculative pressure and determines a steady state of the economy.[4]

Let $V\left(\theta, A\right)$ be the value function of this optimization problem, i.e. the total utility of the

[4] For every state $x^* = \left(r^*, \theta^*, A^*\right)$, with $\dot{\theta}\left(x^*\right) = 0$ and $\dot{A}\left(x^*\right) = 0$, the economy is in a steady state. We will show in section 3.2.2 that the economy possesses a steady state in addition to no pressure $A = 0$ at maximum pressure $A = D$. We call this steady states convergence or focal points.

central bank given it chooses an optimal control r^*

$$\begin{aligned} V(\theta_S, A_S) &= \sup_{r:[0;\infty[\to[0;\infty[} \{U_0(\theta_S, A_S)\} \\ &= U_0(\theta_S, A_S) \text{ with } \begin{pmatrix} \dot{\theta}(t) \\ \dot{A}(t) \end{pmatrix} = \begin{pmatrix} f(r^*(t), \theta(t)) \\ g(r^*(t), \theta(t), A(t)) \end{pmatrix} \\ &\qquad \text{and } \begin{pmatrix} \theta(0) \\ A(0) \end{pmatrix} = \begin{pmatrix} \theta_S \\ A_S \end{pmatrix}. \end{aligned}$$

From the value V we obtain the following Bellman equation (cf. Waelde 2008)

$$\rho V(\theta, A) = \sup_r \left\{ u(\theta) + \frac{dV(\theta, A)}{dt} \right\}. \tag{5}$$

Since V is not continuously differentiable at any feasible point,[5] a more general interpretation of this partial differential equation is necessary. As we will show, the concept of viscosity solutions applies.

3.1 Linear Version

For a first illustration of the model behavior, we set the mean reversion elasticity $\frac{\partial f_2(.)}{\partial \theta(t)}$, the fundamentals elasticity $\frac{\partial g_2(.)}{\partial \theta(t)}$, and the elasticity of herding $\frac{\partial g_3(.)}{\partial A(t)}$ equal to zero. The interest rate elasticities are assumed to be constant, where $\frac{\partial f_1(.)}{\partial r(t)} = \alpha$ and $\frac{\partial g_1(.)}{\partial r(t)} = \gamma$. With this modification, the motion of the state vector is

$$\begin{pmatrix} \dot{\theta} \\ \dot{A} \end{pmatrix} = \begin{pmatrix} -\alpha(r(t) - \bar{r}) \\ -\gamma(r(t) - \bar{r}) \end{pmatrix},$$

with $\alpha, \gamma > 0$. α is the interest rate elasticity of the fundamentals and γ the interest rate elasticity of the attack. In this simple model the central bank is confronted with a perfect correlation of fundamentals and attack. When it chooses a low interest rate to improve fundamentals, speculative pressure rises as well, and vice versa.

As a first step, we take an "educated guess" on the optimal control r^*, then show that the corresponding value function indeed satisfies the Bellman equation, and finally take a closer look at the Bellman equation at the border of the state space.

The optimal control r^* depends on the state, and two cases have to be analyzed separately: the interior $A < D$ and the border case $A = D$, where any further increase in the attack would lead to a breakdown of the regime.

1. The interior case $A < D$:

   The Bellman equation is given by (cf. Waelde (2008), ch. 6; Fleming and Soner (2006),

[5] If the reserves are exhausted and a regime switch is forced, the utility jumps.

ch. 1.7)

$$\begin{aligned}\rho V(\theta, A) &= \sup_r \left\{ u(\theta) + \frac{dV(\theta, A)}{dt} \right\} \\ &= \sup_r \left\{ u(\theta) + DV \cdot \begin{pmatrix} \dot{\theta} \\ \dot{A} \end{pmatrix} \right\} \\ &= \sup_r \left\{ u(\theta) - (V_\theta \alpha + V_A \gamma)(r - \bar{r}) \right\},\end{aligned} \tag{6}$$

with ρ as the discount factor and DV as the total derivative. The argument in the supremum is linear in r and the optimization problem (6) has a border solution $r = 0$, if and only if

$$V_\theta \alpha + V_A \gamma > 0. \tag{7}$$

As we show later and proof in appendix 5.2.1, this condition is valid.

2. The border case $A = D$:

   The value of abandoning the regime $\upsilon(\theta - c)$ is strictly lower than the value of defending the regime $V(\theta, A = D)$ for all possible values of θ (see appendix 5.2.2). Any further increase in A would lead to an infinitely negative slope of V and is therefore avoided. Thus, the optimization problem is to maximize θ subject to $\frac{dA}{dt} \leq 0$. Since $\frac{dA}{dr} < 0$ and $\frac{d\theta}{dr} < 0$, i.e. any control increasing θ also increases A, the optimal solution is to not let A decrease. Hence,

$$r^* = \bar{r}; \ \frac{dA}{dt} = 0; \ \frac{d\theta}{dt} = 0. \tag{8}$$

Summarizing, the optimal control is

$$r^*(\theta, A) = \begin{cases} 0 & \text{if } A < D \\ \bar{r} & \text{else} \end{cases}.$$

Starting at an arbitrary point (θ_S, A_S), where the strength of the attack is less than the reserves $A_S < D$, the central bank maximizes the fundamentals to improve utility (1). Therefore, the central bank conducts expansion policy, i.e. sets the interest rate to zero.[6] Hence, the fundamentals increase depending on their initial value θ_S, the interest rate elasticity α, the natural interest rate $\bar{r}$, and obviously the elapsed time t. Thus, we get as time path of the fundamentals:

$$\theta(t) = \theta_S + \int_0^t \alpha \bar{r} d\tau = \theta_S + \alpha \bar{r} t. \tag{9}$$

Expansion policy $(r(t) = 0)$ reduces the costs of attacking, implying that stress increases with improving fundamentals. The attack state is a function of the initial attack level A_S, the interest rate elasticity γ, the natural interest rate $\bar{r}$, and the elapsed time t. Hence, the time

[6] As noted earlier, we require the interest rate to be nonnegative. Obviously, without this condition the optimal interest rate would be minus infinity.

path of the attack is given by:

$$A(t) = A_S + \int_0^t \gamma \bar{r} d\tau = A_S + \gamma \bar{r} t. \tag{10}$$

The optimal policy of the central bank, to set the interest rate to zero, is accompanied by increasing stress, i.e. an increasing attack. To keep the exchange rate peg, the central bank has to intervene in the currency market, i.e. to sell foreign currency. Thereby, it reduces the reserves D. Since a devaluation involves costs c that decrease the central bank's utility, it starts to defend the peg additionally through raising the interest rate in the instant before the reserves are exhausted. The time when the central bank raises the interest rate to stop speculation, but does not yet devalue, is thus denoted by $T^{A=D}$ and is called defense time, with $T^{A=D} = \min\{t : A(t) = D\}$. $T^{A=D}$ is reached, when the strength of the attack equals the reserves $A\left(T^{A=D}\right) = D$. Inserting in (10) gives the defense time

$$T^{A=D} = \frac{D - A_S}{\gamma \bar{r}}. \tag{11}$$

The central bank has to defend earlier the lower the reserves D, the higher the initial attack level A_S, the interest rate elasticity of the attack γ, and the natural interest rate $\bar{r}$ are.

When the central bank applies a restrictive monetary policy, both, stress and fundamentals stop growing and the economy is in a steady state. Therefore, we get the following time paths given the optimal control r^*

$$A(t) = \begin{cases} A_S + \gamma \bar{r} t & \text{if } t < T^{A=D} \\ D & \text{else} \end{cases}$$
$$\theta(t) = \begin{cases} \theta_S + \alpha \bar{r} t & \text{if } t < T^{A=D} \\ \theta_S + \alpha \bar{r} T^{A=D} & \text{else} \end{cases}.$$

Assuming exponential utility $u(\theta) = -\exp(-\chi\theta)$, where χ is the risk aversion parameter, the value function is:[7]

$$\begin{aligned} V &= U_0(r^*) = \\ &= -\int_0^{T^{A=D}} \exp(-\rho t) \exp(-\chi(\theta_S + \alpha \bar{r} t))\, dt \\ &\quad - \int_{T^{A=D}}^{\infty} \exp(-\rho t) \exp\left(-\chi\left(\theta_S + \alpha \bar{r} T^{A=D}\right)\right) dt \\ &= -\frac{\exp(-\chi\theta)}{\rho + \chi\alpha\bar{r}} \left(\frac{\chi\alpha\bar{r}}{\rho} \exp\left(-(\rho + \chi\alpha\bar{r})\, T^{A=D}\right) + 1 \right). \end{aligned}$$

Now, we can show that this value function indeed solves the Bellman equation (6).[8] Rearranging and deriving with respect to the state variables θ and A delivers the costate variables

[7] A derivation of the value function and the costate variables is given in appendix 5.2.1.

[8] Inserting in the Bellman equation shows that the solution is feasible.

V_θ and V_A

$$V_\theta = -\chi V, \quad V_A = \frac{1}{\gamma \bar{r}} \left(\left(\rho + \chi \alpha \bar{r} \right) V + \exp\left(-\chi\theta\right) \right).$$

The costate variables show how much a marginal increase in the respective state changes the overall value. Inserting into (7) and using the value function gives

$$\rho \left(-\frac{\exp\left(-\chi\theta\right)}{\rho + \chi\alpha\bar{r}} \left(\frac{\chi\alpha\bar{r}}{\rho} \exp\left(-\left(\rho + \chi\alpha\bar{r}\right) T^{A=D} \right) + 1 \right) \right) + \exp\left(-\chi\theta\right) > 0$$

which is true.[9] Hence, for the interior case $A < D$, the Bellman equation

$$\rho V\left(\theta, A\right) = \sup_{r} \left\{ u\left(\theta\right) - \left(V_\theta \alpha + V_A \gamma\right) \left(r - \bar{r}\right) \right\} \tag{12}$$

has an argument which is linear in $r\left(t\right)$ with a negative slope. Therefore, the solution to the optimization problem (6) is the minimum value of r, i.e. $r\left(t\right) = 0$.

For the border case $A = D$ we utilize the Hamiltonian notation of the problem as used in (Fleming and Soner 2006, ch. 2, lemma 8.1) and define the subsolutions $D^{-}V$ and supersolutions $D^{+}V$. A value function belonging to both $D^{-}V$ and $D^{+}V$ is called a viscosity solution. For infinite horizon time-homogeneous optimization problems with discounted utility the value function takes the form $V\left(t, x\right) = \exp\left(-\rho t\right) V\left(x\right)$, where ρ is the discount factor and x the state vector (cf. Fleming and Soner 2006, ch. 1.7).

Proposition 1 *For infinite horizon time-homogeneous optimization problems with discounted utility each feasible value function is continuously differentiable with respect to the time variable t. Thus, $\frac{\partial}{\partial t} V\left(t, x\right)$ enters each element in $D^{-}V$ and $D^{+}V$ and it is sufficient to define $D^{-}V$ and $D^{+}V$ without the time differential.*

We now define the subsolutions $D^{-}V$ and supersolutions $D^{+}V$

$$D^{+}V\left(\theta, A\right) = \left\{ \left(p, q\right) \in \mathbb{R}^2 : \limsup_{\substack{(y,a) \to (\theta, A) \\ a \leq D}} \frac{V\left(y, a\right) - V\left(\theta, A\right) - p\left(y - \theta\right) - q\left(a - A\right)}{\left\| \left(y, a\right) - \left(\theta, A\right) \right\|} \leq 0 \right\}, \tag{13}$$

$$D^{-}V\left(\theta, A\right) = \left\{ \left(p, q\right) \in \mathbb{R}^2 : \liminf_{\substack{(y,a) \to (\theta, A) \\ a \leq D}} \frac{V\left(y, a\right) - V\left(\theta, A\right) - p\left(y - \theta\right) - q\left(a - A\right)}{\left\| \left(y, a\right) - \left(\theta, A\right) \right\|} \geq 0 \right\}. \tag{14}$$

Since $V\left(\theta, A\right)$ is continuously differentiable in all feasible states, we have $D^{+}V\left(\theta, A\right) = D^{-}V\left(\theta, A\right) = \left(V_\theta\left(\theta, A\right), V_A\left(\theta, A\right)\right)$, which solve the Bellman equation. In addition to this standard definition we also define the sub- and supersolutions from beyond the feasible state, i.e. the region of states in which the regime ends. We will apply this to the Bellman equation

[9] See appendix 5.2.1.

to include controls which might end the regime:

$$D^+_{out}V(\theta,D) = \left\{(p,q)\in\mathbb{R}^2 : \limsup_{\substack{(y,a)\to(\theta,D)\\ a>D}} \frac{V(y,a)-V(\theta,D)-p(y-\theta)-q(a-D)}{\|(y,a)-(\theta,D)\|}\leq 0\right\}, \qquad (15)$$

$$D^-_{out}V(\theta,D) = \left\{(p,q)\in\mathbb{R}^2 : \liminf_{\substack{(y,a)\to(\theta,D)\\ a>D}} \frac{V(y,a)-V(\theta,D)-p(y-\theta)-q(a-D)}{\|(y,a)-(\theta,D)\|}\geq 0\right\}. \qquad (16)$$

Since the value after the regime change $\nu(\theta-c)=V(y,a)$ is strictly smaller than the value of remaining in the regime $V(\theta,D)$, we have $D^+_{out}V(\theta,D)=\left\{(p,q)\in\mathbb{R}^2 : \limsup\mathbb{R}^2\right\}=(\infty,\infty)$ and $D^-_{out}V(\theta,D)=\left\{(p,q)\in\mathbb{R}^2 : \liminf\emptyset\right\}=(-\infty,-\infty)$. We know that for all $(p,q)\in D^+_{out}V(\theta,D)$ we have $\rho V(\theta,A)\geq\sup_{r<\bar{r}}\{u(\theta)-(p\alpha+q\gamma)(r-\bar{r})\}$ and for all $(p,q)\in D^-_{out}V(\theta,D)$ we have $\rho V(\theta,A)\leq\sup_{r<\bar{r}}\{u(\theta)-(p\alpha+q\gamma)(r-\bar{r})\}$. The optimal control at the border, i.e. $A=D$, must satisfy the following viscosity formalization of the Bellman equation:

$$\begin{aligned}\rho V(\theta,A)\geq u(\theta) &- \sup_{r<\bar{r}}\{(p_{out}\alpha+q_{out}\gamma)(r-\bar{r})\}I(r<\bar{r})\\ &-\sup_{r\geq\bar{r}}\{(p\alpha+q\gamma)(r-\bar{r})\}I(r\geq\bar{r}),\\ &\text{for } (p_{out},q_{out})\in D^+_{out}V(\theta,D) \text{ and } (p,q)\in D^+V(\theta,D),\end{aligned}$$

$$\begin{aligned}\rho V(\theta,A)\leq u(\theta) &- \sup_{r<\bar{r}}\{(p_{out}\alpha+q_{out}\gamma)(r-\bar{r})\}I(r<\bar{r})\\ &-\sup_{r\geq\bar{r}}\{(p\alpha+q\gamma)(r-\bar{r})\}I(r\geq\bar{r}),\\ &\text{for } (p_{out},q_{out})\in D^-_{out}V(\theta,D) \text{ and } (p,q)\in D^-V(\theta,D),\end{aligned}$$

where $I(.)$ is the indicator function. The only control r that fulfills both conditions is $r(t)\equiv\bar{r}$. Any $r(t)<\bar{r}$ would violate at least one condition. Thus, the optimal behavior is to conduct expansion policy, $r(t)=0$, to maximize θ and immediately defend, $r(t)=\bar{r}$, when the regime is at stake.

This solution is a viscosity solution, i.e. the natural extension of the solution concept for the Bellman equation (6). It is well known that value functions in general are not continuously differentiable for some feasible states[10] and thus for these points the classical solutions do not apply. Viscosity solutions do apply also in many cases, where the value function is not continuously differentiable but necessarily coincides with the standard solution otherwise. Therefore, we could have restricted our analysis to the approach used for the border case $A=D$. However, for reasons of clarity and intuition, we first showed the classical approach and then the viscosity approach.

[10]In fact, the value function is differentiable only at regular points.

The viscosity solution implies that the optimal policy is to maximize the instantaneous utility and to not care about its fragility, i.e. rising stress. The fragility is recognized but not accounted for in the decision about the optimal interest rate until the immediate danger of a crisis emerges. Since an opt-out induces costs, the decision maker raises the interest rate to fend off the attack. Thereby, it is only necessary that costs exist no matter how big they are. Thus, it could also be private costs, which would arise with a breakdown of the regime, that prompt the decision maker to raise the interest rates. In the next section we discuss a linear model of the original differential equations (3) and (4).

3.2 Extended Linear Version

3.2.1 Differential Equations and Time Paths

Now, we consider the case where $f_2(.)$, $g_2(.)$, and $g_3(.)$ are also linear functions. We will continue to use the coefficients α and γ and define $\frac{\partial f_2(.)}{\partial \theta(t)} = \beta$,$\frac{\partial g_2(.)}{\partial \theta(t)} = \delta$, and $\frac{\partial g_3(.)}{\partial A(t)} = \varepsilon$. The difference to the simple model is that the attack not only increases due to a low interest rate but also due to herding (ε) and bad fundamentals (δ). This creates a far richer set of policy options, trade-offs, and realistic settings. E.g. expansion policy, $r(t) = 0$, not necessarily leads to an attack. It might be possible that the herding effect and the interest rate effect are outweighed through the effect of good fundamentals, implying that speculators refrain from attacking, $\dot{A} \leq 0$.

The state vector now evolves according to:

$$\begin{pmatrix} \dot{\theta} \\ \dot{A} \end{pmatrix} = \begin{pmatrix} -\alpha\left(r(t) - \bar{r}\right) - \beta\left(\theta(t) - \bar{\theta}\right) \\ -\gamma\left(r(t) - \bar{r}\right) - \delta\left(\theta(t) - \bar{\theta}\right) + \varepsilon A(t) \end{pmatrix}. \tag{17}$$

Analogously to (6), the Bellman equation is given by

$$\begin{aligned} \rho V(\theta, A) &= \sup_r \left\{ u(\theta) + \frac{dV(\theta, A)}{dt} \right\} \\ &= \sup_r \{ u(\theta) - \left(V_\theta \alpha + V_A \gamma\right)(r - \bar{r}) \\ &\quad - V_\theta \beta \theta(t) - V_A\left(\delta\theta(t) - \varepsilon A(t)\right)\}. \end{aligned} \tag{18}$$

Since the Bellman equation (18) is again linear in the control variable, a bang-bang solution is optimal. This solution describes immediate shifts in the policy, i.e. jumps from one endpoint to the other in the control interval. To assure that the control is bounded (cf. chapter 3, Feichtinger and Hartl 1986), we restrict the interest rate to:

$$0 \leq r(t) \leq R. \tag{19}$$

Hence, the central bank policy switches between expansion, $r(t) = 0$, and defense, $r(t) = R$.

In addition we restrict the attack to be nonnegative and less or equal the amount of reserves, i.e.

$$0 \leq A \leq D. \tag{20}$$

Solving the differential equations (17) gives the paths of the fundamentals for expansion policy $\theta(t)_{r=0}$ and for defense policy $\theta(t)_{r=R}$:[11]

$$\theta(t)_{r=0} = \theta_S \exp(-\beta t) + \theta_{\dot{\theta}=0,0}\left(1 - \exp(-\beta t)\right), \tag{21}$$

$$\theta(t)_{r=R} = \theta_S \exp(-\beta t) + \theta_{\dot{\theta}=0,R}\left(1 - \exp(-\beta t)\right). \tag{22}$$

[11] Derivations of all time paths are given in appendix 5.3.1.

The fundamentals process is driven by the policy decision and the mean reversion. Without policy intervention, i.e. the interest rate equals its natural rate, fundamentals converge to their natural level $\bar{\theta}$.

If the interest rate is set to $r(t) = 0$, an expansionary effect on the fundamentals is induced, that allows to boost the steady state of the fundamentals above their natural level to: $\theta_{\dot{\theta}=0,0} = \bar{\theta} + \frac{\alpha}{\beta}\bar{r}$. For defense policy, i.e. $r(t) = R$, the fundamentals steady state is lower than the natural level: $\theta_{\dot{\theta}=0,R} = \bar{\theta} + \frac{\alpha}{\beta}(\bar{r} - R)$, since $R > \bar{r}$.[12]

The paths (21) and (22) describe the evolution of θ in time t for an arbitrary starting point θ_S. If $t = 0$, then $\theta(0) = \theta_S$. When time passes the value of $\theta(t)$ moves from its starting point θ_S to its steady state $\theta_{\dot{\theta}=0,i}$ under the respective policy i.

Accordingly the paths of the attack for expansion policy $A(t)_{r=0}$ and for defense policy $A(t)_{r=R}$ are given by:[13]

$$\begin{aligned} A(t)_{r=0} &= A_S \exp(\varepsilon t) + \frac{1}{\varepsilon}\left(\frac{\alpha\delta}{\beta} - \gamma\right)\bar{r}\left(1 - \exp(\varepsilon t)\right) \\ &\quad + \frac{\delta}{\varepsilon + \beta}\left(\theta_S - \theta_{\dot{\theta}=0,0}\right)\left(\exp(-\beta t) - \exp(\varepsilon t)\right), \qquad (23) \\ A(t)_{r=R} &= A_S \exp(\varepsilon t) + \frac{1}{\varepsilon}\left(\frac{\alpha\delta}{\beta} - \gamma\right)(\bar{r} - R)\left(1 - \exp(\varepsilon t)\right) \\ &\quad + \frac{\delta}{\varepsilon + \beta}\left(\theta_S - \theta_{\dot{\theta}=0,R}\right)\left(\exp(-\beta t) - \exp(\varepsilon t)\right). \qquad (24) \end{aligned}$$

The first term shows the herding effect of the attack. For positive initial values A_S, this effect increases the attack over time. The second term describes the interest rate elasticity of the attack. Reducing the interest rate is equal to reducing the financing costs of speculators, which increases the attack. For defense policy $R > \bar{r}$ the costs of speculation are high and the second term decreases the attack level. The third term links the attack to the fundamental state. For good fundamentals, i.e. $\theta_S > \bar{\theta} + \frac{\alpha}{\beta}\bar{r}$, a successful attack is unlikely. The good fundamentals lead to a reduced expected payoff and the attack decreases over time. For states worse than the respective steady state the expected payoff is high and the attack increases.

An overview of the used variables describing the different fundamental states is given in appendix 5.1.

3.2.2 Model Dynamics

For a better understanding, we briefly summarize the policy options of the central bank that will be discussed in greater detail afterwards. The bang-bang solution only allows expansion

[12] To clarify the notation we introduce labels for several important fundamental states. The indices show whether the motion in a state variable is zero, the policy chosen, and, if necessary, the state of the attack. E.g. the label $\theta_{\dot{\theta}=0,R}$ gives the location, where the motion of the fundamentals stops ($\dot{\theta} = 0$) for defense policy ($r = R$). An overview of the labels is given in appendix 5.1 on page 31.

[13] Obviously, the paths are only valid as long as the state restriction $0 \leq A \leq D$ admits to maintain a respective interest rate policy.

or defense policy. Expansion policy boosts the fundamentals but also increases stress. At some time the attack will exhaust the reserves ($A = D$) and the central bank remains with three options: it can opt out, stop the attack ($\dot{A} = 0$), or conduct defense policy to completely fend off the attack ($\dot{A} < 0$). Option one requires no action, the regime would simply collapse. When choosing option two, the central bank has to choose the interest rate that stops the motion of the attack. Option three is to set the maximum interest rate, which allows to fend of the attack completely over time.[14] Necessarily, at some time the attack will cease ($A = 0$), giving the central bank the option to start again with expansion policy or to preserve the no stress state.

Figure 13 on page 42 in the appendix shows vector fields of expansion and defense policy with sample trajectories of the central bank's options.

These options of the central bank and the resulting dynamics of the system are now discussed in detail. Thereby, we follow the order just presented and start with expansion policy, i.e. $r(t) = 0$. For a first orientation we draw a phase diagram in the state space (θ, A) (cf. figure 1).

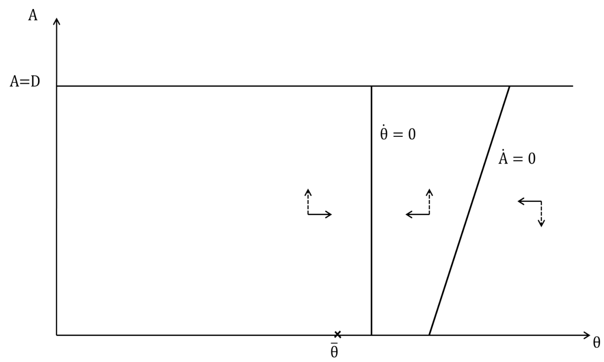


Figure 1: **Dynamics of expansion policy:** The arrows indicate the direction of the movement of the attack (dashed) and the fundamentals (solid). Fundamentals are drawn to their ZML ($\dot{\theta} = 0$), while the attack is pushed away of its ZML ($\dot{A} = 0$).

The θ, A space is crossed by zero motion lines (ZMLs), on which a differential equation equals zero, i.e. the motion in the respective state stops, i.e. $\dot{\theta} = \dot{A} = 0$.[15]

Proposition 2 *The expansion policy ZMLs do not intersect in the feasible attack state* $0 < A < D$*, whereas the attack ZML is to the right of the fundamental ZML* $\theta_{\dot{\theta}=0,0} \leq \theta_{\dot{A}=0,0}$.

Solving the differential equations (17) for $r(t) = 0$ according to some initial value $\theta(0) = \theta_S$ respectively and equating, gives a negative attack level.[16]

The vertical line in figure (1) is the ZML of the fundamentals, $0 = \alpha\bar{r} - \beta\left(\theta(t) - \bar{\theta}\right)$, which is independent of the attack state. The diagonal line is the ZML of the attack, $0 = \gamma\bar{r} - \delta\left(\theta(t) - \bar{\theta}\right) + \varepsilon A(t)$. It states, that to offset a change in the attack at a higher attack level the fundamentals have to increase. Through the herding effect more speculators attack when the

[14] Due to the restrictions on the control not every option is possible in every state (cf. proposition 3).
[15] Derivations of the ZMLs are given in section 5.3.2 of the appendix.
[16] Detailed proofs of propositions (2) - (5) are given in section 5.3.2 of the appendix.

overall attack level increases. For a given interest rate, only a reduction in the expected payoff, i.e. higher fundamentals, can offset the motion in the attack, causing the positive slope of the attack ZML.

If the central bank sets the natural interest rate, $r(t) = \bar{r}$, the mean reversion pushes the fundamentals to their natural level $\bar{\theta}$. When choosing expansion policy, $r(t) = 0$, an economic growth effect of $\frac{\alpha}{\beta}\bar{r}$ is realized in addition to the natural level. Thus, the steady state for expansion policy equals $\bar{\theta} + \frac{\alpha}{\beta}\bar{r}$, which is also the location of the fundamental ZML. Fundamental states worse than this steady state exhibit a positive mean reversion, where better states exhibit a negative mean reversion. To the right of the attack ZML, the good fundamentals reduce the expected payoff of attacking so much, that, even if an attack is free of cost ($r(t) = 0$), stress declines. Thus, the fundamentals converge to their ZML whereas the attack diverges from its ZML. Depending on the starting point it is possible that a state trajectory crosses the attack ZML from right to left (cf. figure 13).

Left to the attack ZML, expansion policy leads to increasing stress. Hence, after some time the attack will exhaust the reserves of the central bank. This attack state, where $A(t) = D$, we term high stress. In this case, the central bank's options—expansion and defense policy—widen by the possibility to stop the attack without fending it off completely. Therefore, the central bank chooses the smallest interest rate that offsets the motion of the attack, i.e. $r = \min\left\{r : \dot{A} = 0\right\}$. This stops the attack immediately, but causes an adaption of the fundamentals. To obtain a time path of the fundamentals during high stress, we solve the differential equations (17) for $\theta(t)$ with the additional restriction $\dot{A} = 0$ and get:[17]

$$\theta(t)_{A=D} = \theta_S \exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right) + \widehat{\theta}_{A=D}\left(1 - \exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right)\right). \tag{25}$$

Where θ_S is the fundamental value, in which the attack exhausts the reserves. If $\frac{\alpha\delta}{\gamma} - \beta > 0$, the fundamentals would converge to $\pm\infty$, depending on the initial position of θ_S. For fundamental states better than $\widehat{\theta}_{A=D}$, fundamentals would infinitely grow. For states worse, fundamentals would infinitely decline. Therefore, we imply the following condition on the parameters:

$$\frac{\alpha\delta}{\gamma} - \beta < 0. \tag{26}$$

Proposition 3 *Defense in high stress leads to a convergence point* $\left(\widehat{\theta}_{A=D}, D\right)$, *with* $\widehat{\theta}_{A=D} = \bar{\theta} + \frac{\frac{\alpha\varepsilon}{\gamma}D}{\frac{\alpha\delta}{\gamma} - \beta}$. *For* $R > \bar{r} - \frac{\varepsilon D}{\left(\frac{\alpha\delta}{\beta} - \gamma\right)}$ *(cf. 45) the point is accessible. Its fundamental state* $\widehat{\theta}_{A=D} \in \left[\theta^{A=D}_{\dot{A}=0,R}; \bar{\theta}\right[$, *i.e. its location, is between the attack ZML of defense policy and the no stress steady state.*

Defense in high stress stops the attack and leads to a convergence of the fundamentals. This is achieved by setting the interest rate to $r(t) = \bar{r} - \frac{\delta}{\gamma}\left(\theta_S - \bar{\theta}\right) + \frac{\varepsilon}{\gamma}D$, which is given

[17] The derivation is given in appendix 5.3.1.

by the differential equation of the attack (17). Note that the interest rate has to rise when fundamentals deteriorate. Bad fundamentals increase the chance of a successful attack and hence raise the expected payoff, which induces more speculators to attack. This can only be offset, if the central bank raises the costs of speculation through raising the interest rate. The worse the fundamentals, the higher the interest rate has to be to fend off additional speculators. The control restriction (19) on the interest rate implies a fundamental state θ, where $R = \max\left\{r : \dot{A} = 0\right\}$. This fundamental state coincides with the attack ZML of defense policy at $A = D$, which precisely defines the point where the growth of the attack stops. In every state worse, defense would require interest rates higher than R. Thus, defense is not possible and the central bank is forced to abandon the regime. On the other hand, better fundamental states reduce the expected payoff of an attack and thus induce more speculators to refrain from attacking. Hence, the control restriction implies another fundamental state θ, where $0 = \min\left\{r : \dot{A} = 0\right\}$. This fundamental state coincides with the attack ZML of expansion policy. For better states the attack decreases without an intervention of the central bank. Therefore, the path of the fundamentals in high stress is only valid in the interval $[\theta^{A=D}_{\dot{A}=0,R}, \theta^{A=D}_{\dot{A}=0,0}]$, i.e. between the attack ZMLs. Figure (2) shows the evolution of the interest rate (red line) depending on the underlying fundamental state.

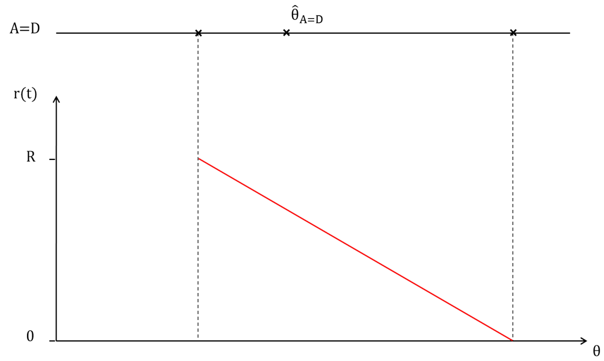


Figure 2: **Convergence in high stress:** The figure shows the evolution of the interest rate (red line) depending on the fundamental state. Deteriorating fundamentals induce more speculators to attack and require higher interest rates to stop the attack. Convergence in high stress $A = D$ is only possible between the attack ZMLs $[\theta^{A=D}_{\dot{A}=0,R}, \theta^{A=D}_{\dot{A}=0,0}]$. See also figure 3.

Proposition 4 *The defense policy ZMLs do not intersect in the feasible attack state* $0 < A < D$*, whereas the attack ZML is to the left of the fundamental ZML* $\theta_{\dot{A}=0,R} \leq \theta_{\dot{\theta}=0,R}$.

Solving the differential equations (17) for $r(t) = R$ according to $\theta(0) = \theta_S$, leads to an attack level higher than the stock of reserves.

When the central bank decides to defend, i.e. $r(t) = R$, the ZMLs shift and the dynamics change. The high interest rate increases the cost of credit and dampens the fundamentals by the amount $\frac{\alpha}{\beta}R$, compared to expansion policy. Hence, the steady state for defense policy is $\bar{\theta} + \frac{\alpha}{\beta}(\bar{r} - R)$, which equals the location of the ZML of the fundamentals. The high interest rate increases the costs for speculators and thereby reduces stress by $\frac{\gamma}{\delta}R$. Figure 3 shows the phase diagram from above extended by defense policy.

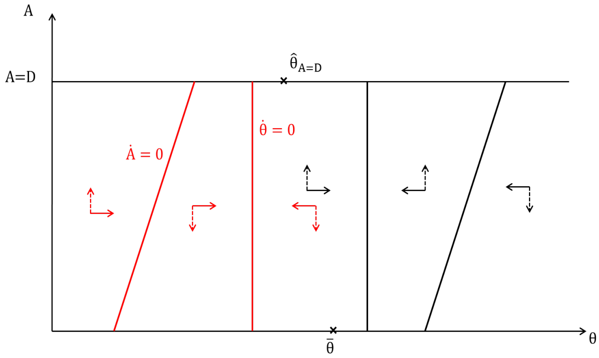


Figure 3: **Dynamics of expansion and defense policy:** The red arrows indicate the direction of the movement of the attack (dashed) and the fundamentals (solid) under defense policy. Again fundamentals are drawn to their ZML ($\dot{\theta} = 0$), while the attack is pushed away of its ZML ($\dot{A} = 0$).

Again fundamentals converge to their ZML, whereas the attack diverges from its ZML. Depending on the starting point it is possible that a state trajectory crosses the attack ZML from left to right.[18]

Defense policy leads to decreasing stress, so that the attack ceases after some time and the no stress region is reached: $A(t) = 0$. At this lower boundary the central bank has the choice to start again with expansion policy or to preserve the no stress state ($\dot{A} = 0$).

With deteriorating fundamentals the expected payoff of attacking rises, inducing more speculators to attack. To preserve the no stress state, the central bank has to raise the interest rate appropriately (cf. figure 4). This increases the costs of speculation and induces more speculators to refrain from the attack. For fundamentals worse than the attack ZML of defense policy interest rates higher than the upper limit, R, would be required to successfully keep the no stress state. In this region the attack increases independent of the central bank policy.[19] For fundamentals better than the attack ZML of expansion policy, expected payoffs decrease so much, that even for an interest rate of zero the attack declines. Since the attack is restricted to nonnegative values it is assumed to equal zero in this region. Therefore, the time path of the fundamentals in no stress is

$$\theta(t)_{A=0} = \theta_S \exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right) + \widehat{\theta}_{A=0}\left(1 - \exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right)\right). \tag{27}$$

The path is valid for $\theta_S \geq \bar{\theta} + \frac{\gamma}{\delta}(\bar{r} - R)$. Note that this implies that the path is valid beyond the attack ZML of expansion policy in no stress. The red line in figure 4 shows the evolution of the interest rate in no stress.

Proposition 5 *The convergence point in no stress* $\left(\widehat{\theta}_{A=0}, 0\right)$ *equals the natural rate of the fundamentals* $\bar{\theta} \in \left]\widehat{\theta}_{A=D}; \theta_{\dot{\theta}=0,0}\right[$, *i.e. its location is between the convergence point in high stress and the fundamental ZML of expansion policy.*

[18] A vector field with sample trajectories is given in appendix 5.3.2.

[19] Obviously, the speed of change is still influenced through the policy decision.

Due to the herding effect more speculators attack for a given fundamental state with increasing stress. To stop the attack the interest rate has to increase according to the level of stress. Consequently, the fundamentals in high stress are affected more than in no stress and converge to a lower fundamental state. Thus, the convergence point in no stress is in a better fundamental state than the convergence point in high stress $\widehat{\theta}_{A=0} > \widehat{\theta}_{A=D}$. Therefore, we term $\widehat{\theta}_{A=0}$ good focal point and $\widehat{\theta}_{A=D}$ bad focal point.

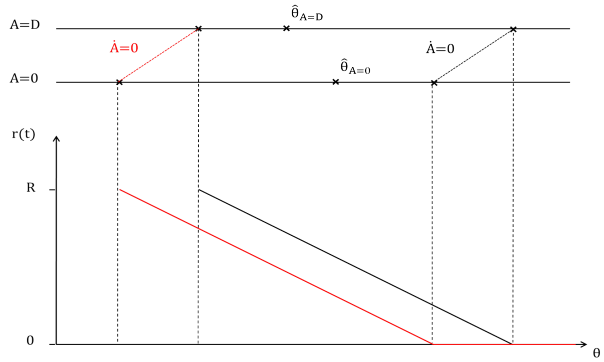


Figure 4: **Convergence in no stress:** Deteriorating fundamentals require higher interest rates to stop speculators from attacking. Convergence in no stress $A = 0$ is possible to the right of the attack ZML of defense policy, $[\theta^{A=0}_{\dot{A}=0,R}, \infty[$. The red line shows the interest rate that is necessary to stop the attack in no stress.

3.2.3 Optimal Behavior

Numerical Example Due to the imposed control restriction, state restriction, and terminal condition we could not obtain a closed solution of the Bellman and the Hamiltonian approach. Therefore, we present numerical solutions of optimal policies in specified areas of the state space. The following parameters resemble a heuristic calibration of a developed country:

$$\alpha = 0.1, \beta = 0.2, \gamma = 0.2, \delta = 0.3, \varepsilon = 0.05, R = 14, \bar{r} = 3, \bar{\theta} = 2, D = 8.$$

Starting in the no stress steady state $(\theta_S, A_S) = (\bar{\theta}, 0)$ the periodic natural growth rate $\bar{\theta}$ is equal to 2%, the according natural interest rate $\bar{r}$ is 3%. If the central bank conducts expansion policy, $r(t) = 0$, this improves fundamentals' growth by $-\alpha(0 - \bar{r}) = 0.3\%$ inducing a mean reversion effect of $-\beta(2.3 - \bar{\theta}) = -0.015\%$. Note that this example should only give an intuition about the impact of the effects and does not represent the continuous effects exactly. In the long run, expansion policy can improve the growth rate from 2% to $\bar{\theta} + \frac{\alpha}{\beta}\bar{r} = 3.5\%$. However, expansion policy allows to speculate at zero costs. Hence, the motion of the attack, initially at zero, increases by $-\gamma(R - \bar{r}) = 0.6\%$. The increase is amplified through the herding effect: $\varepsilon \cdot 0.6 = 0.03\%$. However, both effects are offset through the improving fundamentals and the accompanied decrease in the expected payoff to attacking. This reduces the growth of the attack by: $-\delta(2.3 - \bar{\theta}) = -0.69\%$. In this example the initial point was in the no stress steady state. Expansion policy increased the fundamentals at the cost of increased stress. Here, it would take 16.7 periods until the ongoing attack exhausts the reserves. In initial states worse, the terminal time is significantly smaller, e.g. for $\theta_S = -8$ it takes only 2.6 periods from an environment with initially no stress to reach high stress, with the regime being at stake.

Identity Line When comparing the value of expansion policy and defense policy a crucial question determining the overall outcome is: where is the locus[20] of the fundamental path and how long does it take, till one of the state constraints of the attack is reached? Since the terminal time T has no closed solution, we can only argue that e.g. the state at termination for expansion policy is smaller than for defense policy, $\theta_{T_{r=0}} < \theta_{T_{r=R}}$. This is the case if the slope of the trajectory in the state space $\left(\frac{\partial A}{\partial \theta}\right)$ is always higher under expansion policy than under defense policy. Therefore, we look for a curve on which the slopes of the trajectories of expansion and defense are equal. We call this curve identity line.

Proposition 6 *On an identity line the slope of the state space trajectory under expansion policy equals the slope of the state space trajectory under defense policy, i.e.* $\frac{\partial A_{r=0}}{\partial \theta_{r=0}} = \frac{\partial A_{r=R}}{\partial \theta_{r=R}}$. *Since the direction of the motion changes depending on the location, it is necessary to also compute:* $\frac{\partial A_{r=0}}{\partial \theta_{r=0}} = -\frac{\partial A_{r=R}}{\partial \theta_{r=R}}$. *Consequently, we get two equations that define the identity lines.*

$$A_S = \left(\theta_S - \bar{\theta}\right) \frac{\frac{\alpha\delta}{\gamma} - \beta}{\frac{\alpha\varepsilon}{\gamma}} \text{ and } A_S = -\frac{1}{\varepsilon}\left(\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \frac{\gamma R\left(\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)\right)}{\alpha R - 2\left(\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)\right)}\right) \tag{28}$$

The focal points $\widehat{\theta}_{A=0}$ and $\widehat{\theta}_{A=D}$ lie on the identity line.[21]

The identity lines separate the θ, A space into four areas (cf. figure 5). Since the motion of the fundamentals stops at the fundamental ZML, the slope of the trajectory rises infinitely. Hence, around the fundamental ZML of expansion policy, the slope of the trajectory under expansion is higher than under defense. Since the motion in the attack stops at the attack ZML, the slope of the trajectory converges to zero. Consequently, around the attack ZML of expansion policy, defense policy leads to a higher slope of the trajectory. Figure 5 shows the identity lines as well as the four areas and marks which policy alternative leads to a higher slope of a state space trajectory.[22]

Value of Convergence Points

Proposition 7 *For sufficiently high time preference the bad focal point $\widehat{\theta}_{A=D}$ is stable, while the good focal point $\widehat{\theta}_{A=0}$ is unstable. Choosing defense in the bad focal point leads to an immediate loss in the fundamentals, while choosing expansion in the good focal point increases the fundamentals further.*

[20] Locus refers to the location of a path in the state space.
[21] The proof and the derivation of the identity lines are in appendix 5.3.3.
[22] A formal proof is given in appendix 5.3.3.

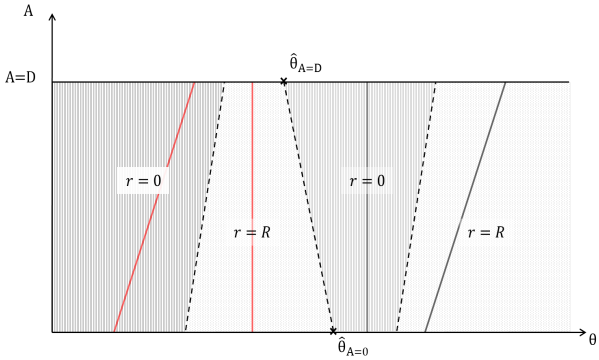


Figure 5: **Identity lines:** On the identity lines (dashed) the slope of the state space trajectories are equal for expansion and defense. The shaded areas show, whether the trajectory under expansion policy ($r = 0$) has a higher slope or the trajectory under defense policy ($r = R$).

Starting from the bad focal point $\widehat{\theta}_{A=D}$, the central bank can produce a closed loop (cf. figure 6). Therefore, it defends for some time, then expands till the reserves are again exhausted[23] and finally stops the attack to converge back to the bad focal point. During defense the slope of the trajectory exceeds the slope during expansion. After passing the identity line, the slope under expansion exceeds the slope under defense. This track leads to a fundamental state at the time the reserves are exhausted again that is better than the bad focal point. For a sufficiently high time preference, the central bank will avoid current losses and will not deviate from the bad focal point. Panel (a) of figure 7 shows the evolution of the instantaneous utility (red line) for a one period deviation from the bad focal point (dashed red line).

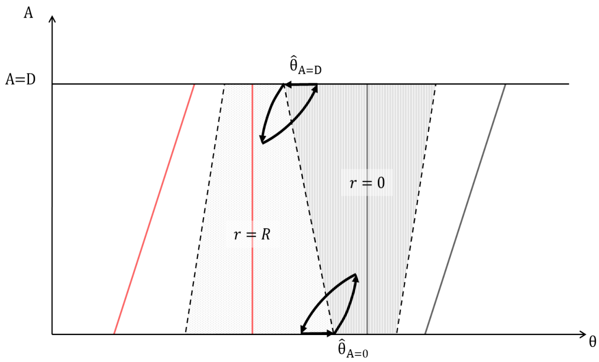
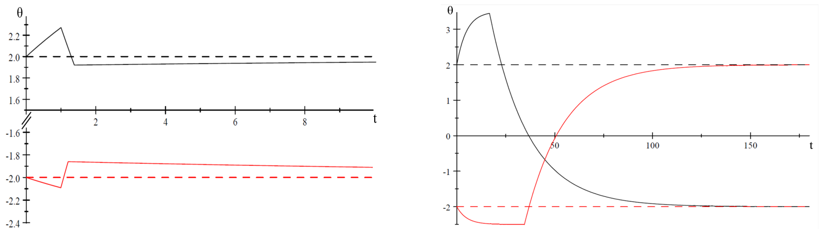


Figure 6: **Closed loops:** The figure shows the paths of short-term deviations from the focal points.

The central bank can also produce a closed loop starting from the good focal point $\widehat{\theta}_{A=0}$ (cf. figure 6). Therefore, it expands for some time, then defends till the attack ceases and converges back to the good focal point. During expansion the slope of the trajectory exceeds the slope during defense. After passing the identity line, the slope under defense exceeds the slope under expansion. Thus, the fundamental state in which no stress is reached is lower than the good focal point. Hence, a current profit in instantaneous utility can be exchanged with a future loss in instantaneous utility. A sufficiently high time preference induces the central bank to deviate

[23]When speculators refrain from attacking, they buy back the currency to settle their accounts, thereby restoring the reserves of the central bank.

from the good focal point. But, having a high time preference, the central bank has no reason to defend after some time, since defense would cause lower fundamentals than expansion. When the central bank sticks to the expansion policy, it will reach high stress after some time and consequently end up in the bad focal point, which is stable if time preference is high. However, having a low time preference, the central bank will bear the current loss, induced by a deviation from the bad focal point, and defend to reach the good focal point, which is stable for a low time preference. Panel (b) of figure 7 shows the evolution of the instantaneous utilities for a transition from the good to the bad focal point, in case of high time preference, as well as the transition from the bad to the good focal point, in case of low time preference.



(a) short-term (b) long-term

Figure 7: **Focal points:** The panels show the evolution of the instantaneous utility for short-term deviations and long-term convergence from the good ($\widehat{\theta}_{A=0} = 2$) and bad ($\widehat{\theta}_{A=D} = -2$) focal points. The plots are based on the aforementioned parameter values.

Comparison of Values In section 3.2.2 we showed that stopping the attack ($\dot{A} = 0$) in no stress is only possible to the right of the attack ZML of defense policy. In high stress the attack can only be stopped between the attack ZMLs. The options of the central bank depend on the fundamental state at the time when the attack meets the state restriction (20). Therefore, we define sets of starting points that lead to the same state restrictions. Thereby, we identify three areas (cf. figures 8 and 13). From area one all paths lead to a forced opt-out. From area three all paths lead to a temporary convergence in no stress.[24,25] Note that an evolution into area one and three is not possible if the starting point is outside these areas. In area two, defense leads to convergence in no stress, while expansion leads to high stress and the choice to converge or to opt out.[26]

In the remainder of the section, we compare the values of expansion and defense policy for starting points from the three areas.

[24] With expansion policy leading to the higher slope in area one, the state trajectory of defense policy with starting point $\left(\theta^{A=D}_{\dot{A}=0,R}, D\right)$ and time running in reverse gives the area of starting points that finally lead to an opt-out.
In area three we look for all paths that lead to a no stress state. Here, defense policy leads to the higher slope, implying that the state trajectory of expansion policy with starting point $\left(\theta^{A=0}_{\dot{A}=0,r}, 0\right)$ and time running in reverse gives the area of starting points leading to no stress.

[25] Obviously, convergence in no stress leads to area two, but for reasons of clarity, area three is analyzed separately.

[26] In section 5.3.2 of the appendix we plot vector fields of the differential system for expansion and defense policy (figure 13 on page 42). As illustration, we highlighted some sample trajectories that show the evolution from starting points of the different areas. The trajectories are marked with the according area.

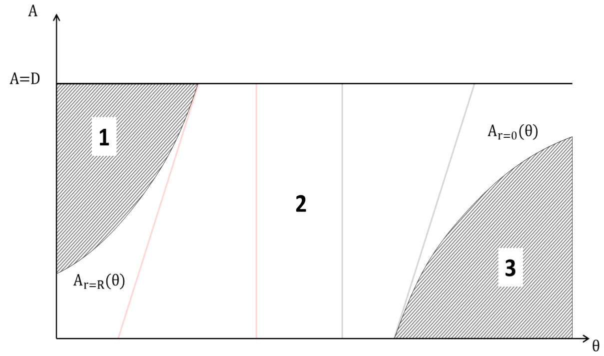


Figure 8: **Separation of paths:** The shaded areas contain the sets of initial values (θ_S, A_S) from which both policies lead to the same state restriction. For sample trajectories see also figure 13.

Proposition 8 *Area 1: for a sufficiently high time preference, the value of expansion policy and opt-out is higher than the value of defense policy and opt-out:*

$$\begin{aligned}&\int_0^{T_{r=0}} \exp(-\rho t)\left(\theta(t)_{r=0}\right) dt + \exp(-\rho T_r)\frac{1}{\rho}\left(\theta_{T_{r=0}} - c\right) \\ &> \int_0^{T_{r=R}} \exp(-\rho t)\left(\theta(t)_{r=R}\right) dt + \exp(-\rho T_R)\frac{1}{\rho}\left(\theta_{T_{r=R}} - c\right). \end{aligned} \tag{29}$$

Where $\theta(t)_{r=0}$ is the path of the fundamentals for expansion policy and $\theta(t)_{r=R}$ the path for defense policy. The starting point (θ_S, A_S) is in area one. The moment the level of the attack reaches the defensive measure is indicated by the terminal time T_i. Where $T_{r=0} = \inf\{t : A_{r=0}(t) > D\}$ and $T_{r=R} = \inf\{t : A_{r=R}(t) > D\}$. The corresponding value of the fundamentals is θ_{T_i}. Expansion policy increases the fundamentals $\dot{\theta}_{r=0} > \dot{\theta}_{r=R} > 0$ but does not restrain the attack $\dot{A}_{r=0} > \dot{A}_{r=R} > 0$. However, the regime under expansion is terminated earlier: $T_{r=0} < T_{r=R}$. In area one the slope of the state trajectory is higher under expansion than under defense. Hence, the fundamental state in which reserves are exhausted is lower for expansion policy: $\theta_{T_{r=0}} < \theta_{T_{r=R}}$.

From the perspective of expansion policy, the trade-off is: there is a faster increase in fundamentals with early costs of opt-out, opposite to a slower, but higher, increase in fundamentals with postponed costs of opt-out. Figure 9 shows a numerical sample plot of the instantaneous utilities of both policies with starting points in area one (defense policy is red, expansion policy is black).[27]

The lower the initial attack level, the longer expansion policy can accumulate a higher utility compared to defense policy. However, the postponed regime switch allows defense policy to achieve a higher instantaneous utility in the long run. What policy is better, is calculated by discounting the instantaneous utilities and adding them up to the current value of the respective path. Thus, the discount factor ρ is crucial for the overall outcome. Even if defense policy leads to a higher instantaneous utility in the long run, a high discount factor, i.e. a high preference for the present, can lead to a higher current value of expansion and thus make it optimal.

[27]Since utility is the identity function it is necessary that $\theta = u(\theta)$.

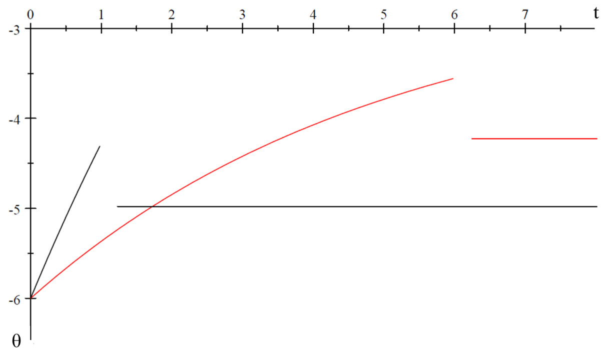


Figure 9: Instantaneous utility of expansion (black) and defense (red) with inevitable opt-out.

Proposition 9 *Area 2: for a sufficiently high time preference, the value of expansion policy and converging in high stress is higher than the value of defense policy and converging in no stress:*

$$\int_0^{T_{r=0}^{A=D}} \exp(-\rho t)\left(\theta(t)_{r=0}\right) dt + \int_{T_{r=0}^{A=D}}^{\infty} \exp(-\rho t)\left(\theta(t)_{A=D}\right) dt$$
$$> \int_0^{T_{r=R}^{A=0}} \exp(-\rho t)\left(\theta(t)_{r=R}\right) dt + \int_{T_{r=R}^{A=0}}^{\infty} \exp(-\rho t)\left(\theta(t)_{A=0}\right) dt \tag{30}$$

Where $\theta(t)_{A=D}$ is the path of the fundamentals for convergence in high stress $A = D$ and $\theta(t)_{A=0}$ is the path of fundamentals for convergence in no stress $A = 0$. The starting point (θ_S, A_S) is in area two. The moment the level of the attack reaches no stress is indicated by time $T_{r=R}^{A=0} = \min\{t : A_R(t) = 0\}$.

For an arbitrary starting point (θ_S, A_S) the instantaneous utility through expansion policy grows faster in bad fundamental states and falls slower in good fundamental states, i.e. $\dot{\theta}_{r=0} > \dot{\theta}_{r=R}$. Defense policy reduces the level of stress in good states and slows down the increase in stress in bad states: $\dot{A}_{r=0} > \dot{A}_{r=R}$. Hence, expansion policy generates a higher instantaneous utility, but also admits higher stress that has to be dealt with. The bigger the attack in the initial starting point, the earlier expansion policy has to opt out or defend and there is less time to accumulate gains in instantaneous utility over defense policy.

The trade-off from the perspective of expansion policy is: initially higher fundamentals are followed by a long-term convergence to the bad focal point $\widehat{\theta}_{A=D}$ opposite to initially lower fundamentals with a long-term convergence to the good focal point $\widehat{\theta}_{A=0}$. However, for a sufficiently high time preference, the central bank values current more than future profits and losses, giving expansion policy a higher value than defense policy. Again a numerical example is plotted in figure 10.

Proposition 10 *Area 2: for sufficiently high time preference, the value of expansion policy and*

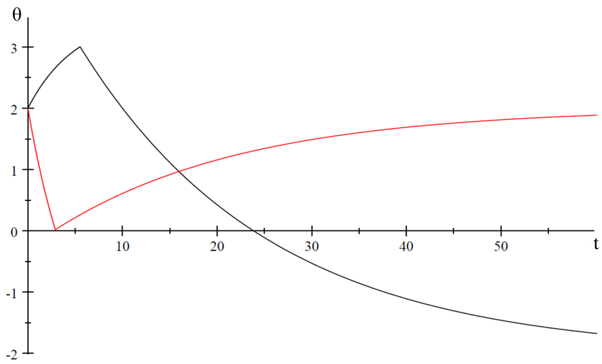


Figure 10: Instantaneous utility of expansion policy and convergence in high stress (black) versus defense policy and convergence in no stress (red).

opt-out is higher than the value of defense policy and converging in no stress:

$$\int_0^{T_{r=0}} \exp(-\rho t)\left(\theta(t)_{r=0}\right) dt + \exp(-\rho T_{r=0}) \frac{1}{\rho}\left(\theta_{T_{r=0}} - c\right)$$
$$> \int_0^{T_{r=R}^{A=0}} \exp(-\rho t)\left(\theta(t)_{r=R}\right) dt + \int_{T_{r=R}^{A=0}}^{\infty} \exp(-\rho t)\left(\theta(t)_{A=0}\right) dt \tag{31}$$

The dynamics are unchanged compared to the previous example except for the immediate opt-out of the central bank, when the reserves are exhausted. The trade-off is: the more favorable evolution of the fundamentals through expansion is now followed by an instant drop in value imposed through the costs of the regime change opposite to a less favorable evolution of fundamentals and convergence to the good focal point. Despite the costs of opt-out, a regime switch can be optimal since it allows the proceeding of a regime being in a no stress steady state at $\theta(T_{r=0}) - c$. If $\theta(T_{r=0}) - c$ is smaller than $\widehat{\theta}_{A=0}$, defense policy reaches a higher instantaneous utility over time and again the outcome of the value comparison depends on the time preference. If, however, $\theta(T_{r=0}) - c$ is greater or equal $\widehat{\theta}_{A=0}$, expansion and opt-out is optimal independent of the time preference (cf. figure 11).

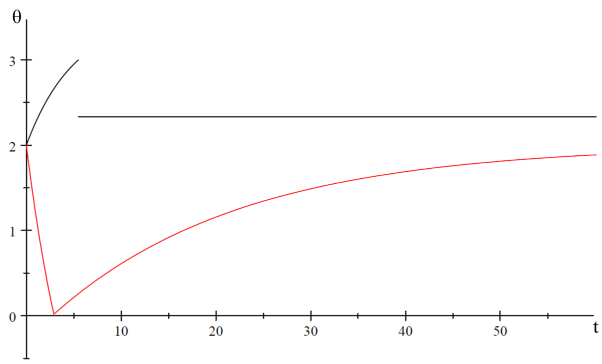


Figure 11: Instantaneous utility of expansion and opt-out (black) versus defense and convergence in no stress (red).

But, why should the central bank bear the costs of a change immediately, when it could also converge in high stress?

Proposition 11 *In high stress to abandon the regime instantly is the optimal decision for fundamental states* $\theta > \widehat{\theta}_{A=D} + c\frac{\frac{\alpha\delta}{\gamma}-\beta-\rho}{\frac{\alpha\delta}{\gamma}-\beta}$.[28]

When the reserves are exhausted in a very good fundamental state, it is better to abandon the regime immediately, bearing the costs, and prevent the fundamentals from further deterioration till the convergence point. The better the fundamentals the higher is the mean reversion effect and hence the loss through convergence. With higher costs of opt-out the immediate loss grows and therefore the fundamental state θ_c, above which an opt-out is optimal, increases too. Note that for high costs, i.e. $c > \frac{\bar{r}(\alpha\delta-\beta\gamma)-\beta\varepsilon D}{\delta\left(\frac{\alpha\delta}{\gamma}-\beta-\rho\right)}$, θ_c is to the right of the attack ZML of expansion policy and cannot be reached. If the time preference increases, the costs have a higher influence on the decision of the central bank. Hence, an opt-out must permit a higher fundamental state if the time preference increases.

Proposition 12 *Area 3: for good fundamentals, independent of the time preference, expansion policy and converging in no stress is better than defense policy and converging in no stress:*

$$\begin{aligned} &\int_0^{T_{r=0}^{A=0}} \exp(-\rho t)\left(\theta(t)_{r=0}\right) dt + \int_{T_{r=0}^{A=0}}^{\infty} \exp(-\rho t)\left(\theta(t)_{A=0}\right) dt \\ &> \int_0^{T_{r=R}^{A=0}} \exp(-\rho t)\left(\theta(t)_{r=R}\right) dt + \int_{T_{r=R}^{A=0}}^{\infty} \exp(-\rho t)\left(\theta(t)_{A=0}\right) dt \end{aligned} \tag{32}$$

For starting points in area three of figure 8 we have $\dot{\theta}_{r=R} < \dot{\theta}_{r=0} < 0$. Again, expansion policy leads to more favorable fundamentals than defense, but defense has a major effect on the attack $\dot{A}_{r=R} < \dot{A}_{r=0} < 0$. Therefore, the attack ceases earlier under defense policy $T_{r=0}^{A=0} > T_{r=R}^{A=0}$. To keep the no stress state, it is sufficient to set the interest rate to zero (cf. proposition 5). Hence, both policy alternatives converge to the good focal point. Since expansion started with a zero interest rate the fundamental state during expansion is greater or equal than the fundamental state during defense, so that the realized value during expansion is greater. Consequently, expansion policy is optimal independent of the time preference. Figure 12 shows a numerical example of both policy alternatives.

[28]The proof is given in appendix 5.3.3.

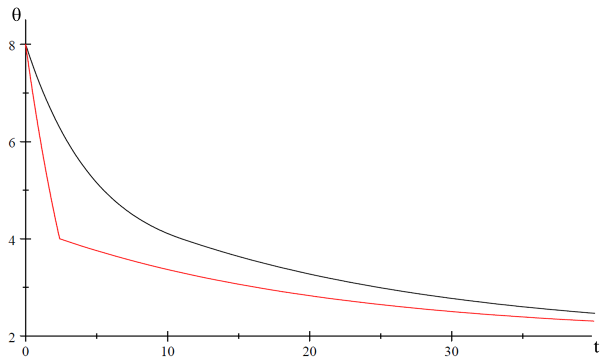


Figure 12: Instantaneous utility of defense and convergence in no stress (red) versus expansion and convergence in no stress (black).

4 Discussion

We applied an infinite horizon intertemporal optimization model with endogenous exit to a simple speculative attack framework. The central bank sets the interest rate which influences both fundamentals and attack strength. Hence, with one variable the central bank improves fundamentals but also increases stress. The central bank's role is beyond solely responding to speculative pressure or signaling. Depending on the current state the decision of the central bank has different implications for its own future position. Consequently, the central bank has to weigh the different outcomes against each other. If there is no dominant path, the optimal policy depends on the time preference of the central bank.

In the reduced linear model the interest rate is the only variable that influences and controls the state variables. Since there are no feedback effects, the motion in the attack can be stopped in no stress as well as in high stress without causing an adaption in the fundamentals. The central bank expands till the reserves are exhausted and the decision is between: stop the attack or allow a costly regime switch. Then the central bank raises the interest rate and remains in high stress.

We extend the model through allowing for fundamental feedback effects and a herding effect. In this extended linear model two focal points emerge. For a given policy and the necessary adaption at the state restriction the state trajectory ends up in one of the focal points.

The good focal point is characterized by a better fundamental state and lower interest rate. In high stress, through the herding effect, more speculators have to be fend off, which is done by raising the interest rate. However, the higher interest rate reduces the fundamentals which also induces higher speculative pressure that has to be fend off through an even higher interest rate.

Through a temporary deviation, expansion from the good focal point improves the fundamental state. With a subsequent defense and convergence back to the good focal point the central bank can produce a closed loop. In our numeric example the accumulated utility through the loop, after an expansion for 1 period, is for the first 3.7 periods higher than remaining at the good focal point. Thus, there is a huge incentive problem, if favorable short-term results give an advantage to the policy maker, who is subject to short mandates or reelections. Independency and low time preference, through long mandates and intermediate to long-term policy goals, are necessary to not put too much weight on the present. With a high time preference and the resulting high value of the present the central bank deviates from the good focal point. But, there is no reason to stop the expansion after a short time because further expansion further increases the fundamentals. With this reasoning going on, the reserves will be exhausted through an increasing attack. Then, the high time preference will only allow one decision: to fend off additional speculators, since this avoids both, the immediate costs an of an opt-out as well as the high costs of reducing the attack. Hence, the high time preference central bank will end up in the bad focal point.

From the bad focal point a deviation is only possible through defense. A closed loop is then produced if the central bank expands and finally converges back to the bad focal point. Hence, current costs have to be weighed against future profits. In our numerical example it takes only 1.5 periods till the cumulated utility turns positive compared to remaining in the bad focal point. For a sufficiently high time preference, the bad focal point is stable, since current losses are weighed more. However, a low time preference central bank will deviate to achieve a higher future utility and higher overall profits. But, this argument is also valid after a short defense, hence for sufficiently low time preference the central bank will defend till the attack ceases and converge to the good focal point to achieve the maximum utility in the long run.

For some initial values the state and control restrictions prevent the convergence to the focal points. Therefore, we categorized the state space into three areas that lead to different policy options at the state restrictions (cf. figure 8).

All areas have in common that expansion policy leads to more favorable fundamental states compared to defense policy. This holds as long as the policy is not changed or a state restriction is met that requires an adaption. On the contrary, defense policy leads to lower attack levels compared to expansion policy.

In area one, which is characterized through bad fundamentals and medium to high stress, expansion as wells as defense, both, lead to further increasing stress and improving fundamentals. The trajectories reach high stress at fundamental states where defense is not possible and a regime switch is forced. Expansion gives the advantage of a currently superior fundamental state at the cost of a lower steady state after the opt-out. Consequently, defense policy will have the higher cumulated utility over time, but time preference determines which policy is valued more. A high time preference central bank values current profits more than future losses and thus chooses expansion. Since an opt-out is inevitable, the outcome is analogous to the first generation models and the 'hell' of the second generation models.[29] The main difference is, that, with the option to defend, the central bank can influence the terminal time as well as the terminal value.

In area two, better fundamentals allow defense policy to succeed in reducing stress.[30] This gives the central bank the opportunity to reach no stress and converge to the good focal point. Alternatively the central bank can conduct expansion policy and decide in high stress to either converge to the bad focal point or to opt out by choice. This regime switch induces a proceeding regime that—through conducting sound policy—persists in a steady state at a fundamental value of the terminal value minus the costs for the regime switch. Such an opt-out can only be optimal, if the costs of a regime switch are negligible and the time preference is not too high. Hence, an immediate opt-out is better than convergence to a lower fundamental state, if the costs and the time preference are low. For very good initial fundamental states the terminal value of the regime after costs might be better than the good focal point. In this case, expansion and opt-out is the dominant strategy independent of the time preference. This area

[29] For a categorization of the states in the second generation see Jeanne (1999).

[30] Starting with bad fundamentals it takes some time till stress decreases under defense, since the trajectory must first cross the attack ZML of defense policy.

can be compared to the 'purgatory' from the second generation and also to global games. While in second generation models the outcome—crisis or no crisis—depends on the expectations of investors, here, the time preference of the central bank sets the direction and yields a unique equilibrium. Moreover, loops as described in the global game literature are possible (cf. e.g. Angeletos et al. 2007). However, for a constant time preference the outcome is unique.

For higher costs, a high time preference central bank will avoid the detriments in fundamentals through defense. In fact, it will conduct expansion policy and converge in high stress. Conversely, a low time preference central bank will accept the preliminary lower fundamentals, through defense, to achieve a higher long run growth in the good focal point.

In area three, which is characterized through good fundamentals and medium to no stress, both policies lead to decreasing stress and deteriorating fundamentals. The trajectories reach no stress in an area where the attack can be stopped through a zero interest rate. Expansion profits from the good fundamental state that allows to continue expansion policy without having to bear increasing stress as in other states. When converging to the good focal point, the trajectories move into area two. Henceforward, the decision which policy is optimal is ambiguous and again depends on the time preference and the costs of an opt-out. Area three can be compared to the 'heaven' in the second generation models. Though, here area three is only a temporary stage before area two is reached and the central bank policy determines whether the economy evolves into the high stress region or stays in no stress.

This temporary nature of 'heaven' and 'hell' explains why there is no empirical confirmation for these states. They only exist temporarily after big shocks.

In contrast to the second generation, where—ex post—the realization of the expectations is used to justify whether the regime is abandoned or not, here, the state of the economy justifies the outcome and the economy evolves according to the initial position and the policy chosen.

In summary, a high time preference central bank will end up in the bad focal point or, if bad fundamentals do not allow high stress defense and the opt-out is forced, a proceeding regime with lower steady state. Conversely, a low time preference central bank will end up in the good focal point or a proceeding regime with comparatively high steady state. For low costs of a regime change, not too high time preference, and good fundamentals an immediate opt-out is optimal compared to convergence in high stress as well as convergence in low stress. Therefore, the proceeding regime must have a steady state, i.e. terminal value minus costs, that is higher than the good focal point.

If one follows the principle that more is better, it is suggested that measures should be taken that reduce the time preference of the central bank. This assures that the central bank is willing to bear currently lower fundamentals to profit from a higher long-term growth, although naturally present consumption is preferred to future consumption. Hence, a central bank should be independent, have a long-term mandate, and pursue long-term goals like price stability.

Also this model is too rudimentary to provide a realistic setting of the emergence of currency crisis it offers useful insights in the various complications associated with policy choices and the paths that emerge.

Therefore, I apply our model to the Swedish currency crisis 1992. For a detailed description of the events during the Swedish crisis see Hörngren and Lindberg (1993). In 1991 the Riksbank decided to switch the peg from a trade weighted currency basket to a unilateral ECU peg. In September 1992 Sweden experienced a recession while its currency remained stable with overnight interest rates of 12%. However, on September 8th Finland abandoned its fixed exchange rate and simultaneously the confidence in the Swedish peg vanished with capital outflows increasing. To prevent this outflow the Riksbank raised the overnight interest rate to 75% which successfully fend off speculators. On September 15th speculation rose again, so that the central bank raised the overnight interest rate to 75% and to 500% the next day. After four days the situation relaxed and interest rates were lowered. However, in November pressure and—as a reply—interest rates rose again. Due to high costs the Riksbank decided to abandon the fixed exchange rate. In the context of our model Sweden was in area two to the right of the bad focal point. There, expansion policy improves fundamentals but also allows increasing stress. With the speculative attack increasing, defense becomes necessary, which results in decreasing fundamentals that become an increasing burden for the economy. Thereby, every expansion that is followed by a defense, in sum, reduces the fundamentals (area above the identity line). With the fundamentals decreasing, the required interest rate for a successful defense increases. The implied damaging effects for the economy—facing a convergence to the bad focal point—induced the Riksbank to finally opt-out.

For future research it would be useful to study different terminal regimes, not only a pressure free state. This would allow to analyze recurring attacks so that an initial avoidance of an attack is rewarded more. Including stress in the utility function might also give useful insights in that only the relative gain of fundamentals over stress is then pursued. Moreover, the linearity assumptions could be relaxed to allow for nonlinearities in the herding effect, the fundamental feedback effects, and the interest rate effects.

5 Appendix

5.1 List of Fundamental States

Label	Description	Expression
θ	fundamental state	
$\bar{\theta}$	natural growth rate of fundamentals, convergence point in no stress, no stress steady state	
$\theta_{\dot{\theta}=0,0}$	steady state for expansion policy, locus of the fundamental ZML	$\bar{\theta} + \frac{\alpha}{\beta}\bar{r}$
$\theta_{\dot{\theta}=0,R}$	steady state for defense policy, locus of the fundamental ZML	$\bar{\theta} + \frac{\alpha}{\beta}\left(\bar{r} - R\right)$
$\theta_{\dot{A}=0,0}$	locus of the attack ZML for expansion policy in no stress	$\bar{\theta} + \frac{\gamma}{\delta}\bar{r}$
$\theta^{A=D}_{\dot{A}=0,0}$	locus of the attack ZML for expansion policy in high stress	$\bar{\theta} + \frac{\gamma}{\delta}\bar{r} + \frac{\varepsilon}{\delta}D$
$\theta_{\dot{A}=0,R}$	locus of the attack ZML for defense policy in no stress	$\bar{\theta} + \frac{\gamma}{\delta}\left(\bar{r} - R\right)$
$\theta^{A=D}_{\dot{A}=0,R}$	locus of the attack ZML for defense policy in high stress	$\bar{\theta}+\frac{\gamma}{\delta}\left(\bar{r} - R\right)+\frac{\varepsilon}{\delta}D$
$\widehat{\theta}_{A=0}$	convergence point in no stress, natural growth rate	$\bar{\theta}$
$\widehat{\theta}_{A=D}$	convergence point in high stress	$\bar{\theta} + \frac{\frac{\alpha\varepsilon}{\gamma}D}{\frac{\alpha\delta}{\gamma}-\beta}$
θ_c	point of indifference between opt-out and convergence in high stress	$\widehat{\theta}_{A=D} + c\frac{\frac{\alpha\delta}{\gamma}-\beta-\rho}{\frac{\alpha\delta}{\gamma}-\beta}$

5.2 Linear Version

5.2.1 Value Function

Assuming exponential utility $u(\theta) = -exp(-\chi\theta)$, where χ is the risk aversion parameter we derive the value function for expansion policy and defense at $A = D$:

$$
\begin{aligned}
V \quad &= \sup_r (U_0) \\
&= -\int_0^{T^{A=D}} \exp(-\rho t)\exp(-\chi(\theta+\alpha\bar{r}t))\,dt - \int_{T^{A=D}}^{\infty} \exp(-\rho t)\exp\left(-\chi\left(\theta+\alpha\bar{r}T^{A=D}\right)\right)dt \\
&= -\int_0^{T^{A=D}} \exp(-\rho t-\chi(\theta+a\bar{r}t))\,dt - \exp\left(-\chi\left(\theta+\alpha\bar{r}T^{A=D}\right)\right)\int_{T^{A=D}}^{\infty} \exp(-\rho t)\,dt \\
&= \left[\frac{\exp(-\rho t-\chi(\theta+\alpha\bar{r}t))}{\rho+\chi\alpha\bar{r}}\right]_0^{T^{A=D}} - \exp\left(-\chi\left(\theta+\alpha\bar{r}T^{A=D}\right)\right)\left[-\frac{\exp(-\rho t)}{\rho}\right]_{T^{A=D}}^{\infty} \\
&= \frac{\exp\left(-\rho T^{A=D}-\chi\left(\theta+\alpha\bar{r}T^{A=D}\right)\right)}{\rho+\chi\alpha\bar{r}} - \frac{\exp(-\chi\theta)}{\rho+\chi\alpha\bar{r}} - \frac{1}{\rho}\exp\left(-\rho T^{A=D}-\chi\left(\theta+\alpha\bar{r}T^{A=D}\right)\right) \\
&= \frac{\exp\left(-\rho T^{A=D}-\chi\left(\theta+\alpha\bar{r}T^{A=D}\right)\right)}{\rho+\chi\alpha\bar{r}} - \frac{\exp(-\chi\theta)}{\rho+\chi\alpha\bar{r}} \\
&\qquad - \frac{\exp\left(-\rho T^{A=D}-\chi\left(\theta+\alpha\bar{r}T^{A=D}\right)\right)\left(1+\frac{1}{\rho}\chi\alpha\bar{r}\right)}{\rho\left(1+\frac{1}{\rho}\chi\alpha\bar{r}\right)} \\
&= \frac{-\exp\left(-\rho T^{A=D}-\chi\left(\theta+\alpha\bar{r}T^{A=D}\right)\right)\frac{1}{\rho}\chi\alpha\bar{r} - \exp(-\chi\theta)}{\rho+\chi a\bar{r}} \\
&= -\frac{\exp(-\chi\theta)}{\rho+\chi\alpha\bar{r}}\left(\frac{\chi\alpha\bar{r}}{\rho}\exp\left(-(\rho+\chi\alpha\bar{r})T^{A=D}\right)+1\right) < 0,
\end{aligned}
\tag{33}
$$

which is always negative.

The partial derivative of V with respect to θ is

$$
\begin{aligned}
&\frac{d}{d\theta}\left(-\frac{\exp(-\chi\theta)}{\rho+\chi\alpha\bar{r}}\left(\frac{\chi\alpha\bar{r}}{\rho}\exp\left(-(\rho+\chi\alpha\bar{r})T^{A=D}\right)+1\right)\right) \\
&= -\chi V,
\end{aligned}
$$

which is always positive.

The partial derivative of V with respect to A is[31]

$$
\begin{aligned}
&\frac{d}{dA}\left(-\frac{\exp(-\chi\theta)}{\rho+\chi\alpha\bar{r}}\left(\frac{\chi\alpha\bar{r}}{\rho}\exp\left(-(\rho+\chi\alpha\bar{r})T^{A=D}\right)+1\right)\right) \\
&= -\frac{\exp(-\chi\theta)}{\rho+\chi\alpha\bar{r}}\left(\frac{\chi\alpha\bar{r}}{\rho}\exp\left(-T^{A=D}(\rho+\chi\alpha\bar{r})\right)\right)\frac{(\rho+\chi\alpha\bar{r})}{\gamma\bar{r}},
\end{aligned}
$$

[31] From (11) we know that $T^{A=D} = \frac{D-A_S}{\gamma\bar{r}}$ and therefore $\frac{dT^{A=D}}{dA} = -\frac{1}{\gamma\bar{r}}$.

which is always negative. Using the expression for the value function (33), the partial derivative can also be written as

$$
\begin{aligned}
&= \frac{\rho+\chi\alpha\bar{r}}{\gamma\bar{r}}\left(V+\frac{\exp\left(-\chi\theta\right)}{\rho+\chi\alpha\bar{r}}\right)\\
&= \frac{1}{\gamma\bar{r}}\Bigg(\left(\rho+\chi\alpha\bar{r}\right)V+\exp\left(-\chi\theta\right)\Bigg).
\end{aligned}
$$

Using (7) we proof that an exterior solution exists. Therefore, it is necessary that:

$$
\begin{aligned}
&V_\theta\alpha+V_A\gamma>0\\
&V_\theta\alpha+V_A\gamma = -\chi V\alpha+\frac{1}{\gamma\bar{r}}\Bigg(\left(\rho+\chi\alpha\bar{r}\right)V+\exp\left(-\chi\theta\right)\Bigg)\gamma\\
&\qquad\qquad\quad = -\chi V\alpha+\frac{1}{\bar{r}}\rho V+\chi\alpha V+\frac{1}{\bar{r}}\exp\left(-\chi\theta\right)\\
&\qquad\qquad\quad = \frac{1}{\bar{r}}\left(\rho V+\exp\left(-\chi\theta\right)\right),
\end{aligned}
$$

for the proof, it is necessary that:

$$
\rho V+\exp\left(-\chi\theta\right)>0.
$$

Inserting for V gives

$$
\begin{aligned}
\rho\left(-\frac{\exp\left(-\chi\theta\right)}{\rho+\chi\alpha\bar{r}}\left(\frac{\chi\alpha\bar{r}}{\rho}\exp\left(-\left(\rho+\chi\alpha\bar{r}\right)T^{A=D}\right)+1\right)\right)+\exp\left(-\chi\theta\right)&>0\\
-\frac{1}{\rho+\chi\alpha\bar{r}}\left(\chi\alpha\bar{r}\exp\left(-\left(\rho+\chi\alpha\bar{r}\right)T^{A=D}\right)+\rho\right)+1&>0\\
\frac{1}{\rho+\chi\alpha\bar{r}}\left(\chi\alpha\bar{r}\exp\left(-\left(\rho+\chi\alpha\bar{r}\right)T^{A=D}\right)+\rho\right)&<1\\
\chi\alpha\bar{r}\exp\left(-\left(\rho+\chi\alpha\bar{r}\right)T^{A=D}\right)+\rho&<\rho+\chi\alpha\bar{r}\\
\exp\left(-\left(\rho+\chi\alpha\bar{r}\right)T^{A=D}\right)&<1\\
-\left(\rho+\chi\alpha\bar{r}\right)T^{A=D}&<0,
\end{aligned}
$$

which is true, since $T^{A=D}\geq 0$.

5.2.2 Comparison of Values

For $t > T^{A=D}$ the central bank sets $r = \bar{r}$, which implies that $\theta(t)$ is constant, i.e. $\dot{\theta} = 0$ and thus $\theta(t) = \theta_T$. For $\upsilon(\theta_T - c)$ we write:

$$\begin{aligned} V(\theta_T - c, A = D) &= \int_0^\infty \exp(-\rho\tau)\,\upsilon(\theta_T - c)\,dt \\ &= \upsilon(\theta_T - c)\frac{1}{\rho} \\ &= -\frac{1}{\rho}\exp(-\chi(\theta_T - c)). \end{aligned} \tag{34}$$

In comparison, V at point $A = D$ equals:

$$\begin{aligned} V(\theta_T, A = D) &= \int_0^\infty \exp(-\rho t)\,u(\theta)\,dt \\ &= u(\theta_T)\frac{1}{\rho} \\ &= -\frac{1}{\rho}\exp(-\chi\theta_T). \end{aligned} \tag{35}$$

If we compare equations (34) and (35), we see that indeed $V(\theta_T, A = D)$ is higher and thus, defending the regime at the corner is optimal.

5.3 Extended Linear Version

5.3.1 Differential Equations and Time Paths

Time Path of the Fundamentals The central bank sets the control $r(t)$ and chooses between expansion policy $r(t) = 0$ and defense policy $r(t) = R$ (cf. condition (19)). The differential equation of the fundamentals (17) for defense policy thus is:

$$\dot{\theta} = -\alpha (R - \bar{r}) - \beta \left(\theta (t) - \bar{\theta}\right). \tag{17}$$

Solving the differential equation for θ gives:

$$\begin{aligned}
\beta\theta (t) + \frac{d\theta}{dt} &= \alpha (\bar{r} - R) + \beta\bar{\theta} \\
\left(\beta\theta (t) + \frac{d\theta}{dt}\right) \exp (\beta t) &= \left(\alpha (\bar{r} - R) + \beta\bar{\theta}\right) \exp (\beta t) \\
\frac{d}{dt} (\theta (t) \exp (\beta t)) &= \left(\alpha (\bar{r} - R) + \beta\bar{\theta}\right) \exp (\beta t) \\
\int \frac{d}{dt} (\theta (t) \exp (\beta t))\, dt &= \int \left(\alpha (\bar{r} - R) + \beta\bar{\theta}\right) \exp (\beta t)\, dt \\
\theta (t) \exp (\beta t) &= C_\theta + \frac{1}{\beta} \left(\alpha (\bar{r} - R) + \beta\bar{\theta}\right) \exp (\beta t) \\
\theta (t) &= C_\theta \exp (-\beta t) + \bar{\theta} + \frac{\alpha}{\beta} (\bar{r} - R),
\end{aligned}$$

for $t = 0$

$$\theta_S - \bar{\theta} - \frac{\alpha}{\beta} (\bar{r} - R) = C_\theta$$

and thus

$$\theta (t) = \left(\theta_S - \bar{\theta} - \frac{\alpha}{\beta} (\bar{r} - R)\right) \exp (-\beta t) + \bar{\theta} + \frac{\alpha}{\beta} (\bar{r} - R)$$

which we write as

$$\boxed{\theta (t) = \theta_S \exp (-\beta t) + \left(\bar{\theta} + \tfrac{\alpha}{\beta} (\bar{r} - R)\right) (1 - \exp (-\beta t)).} \tag{22}$$

If the central bank conducts expansion policy, we get

$$\boxed{\theta (t) = \theta_S \exp (-\beta t) + \left(\bar{\theta} + \tfrac{\alpha}{\beta} \bar{r}\right) (1 - \exp (-\beta t)).} \tag{21}$$

Time Path of the Attack The differential equation of the attack is

$$\dot{A} = -\gamma (R - \bar{r}) - \delta \left(\theta (t) - \bar{\theta}\right) + \varepsilon A (t) \tag{17}$$

$$\dot{A} - \varepsilon A (t) = \gamma (\bar{r} - R) - \delta\theta (t) + \delta\bar{\theta}.$$

Inserting for $\theta(t)$ gives

$$\dot{A} - \varepsilon A(t) = \gamma(\bar{r} - R) - \delta\left(\theta_S - \bar{\theta} - \frac{\alpha}{\beta}(\bar{r} - R)\right)\exp(-\beta t) - \delta\bar{\theta} - \frac{\alpha\delta}{\beta}(\bar{r} - R) + \delta\bar{\theta}$$

$$\dot{A} - \varepsilon A(t) = -\delta\left(\theta_S - \bar{\theta} - \frac{\alpha}{\beta}(\bar{r} - R)\right)\exp(-\beta t) - \left(\frac{\alpha\delta}{\beta} - \gamma\right)(\bar{r} - R).$$

Solving for A gives:

$$\begin{aligned}
\int \frac{d}{dt}A(t)\exp(-\varepsilon t)\,dt = &-\int \delta\left(\theta_S - \bar{\theta} - \frac{\alpha}{\beta}(\bar{r} - R)\right)\exp((-\varepsilon - \beta)t)\,dt \\
&-\int\left(\frac{\alpha\delta}{\beta} - \gamma\right)(\bar{r} - R)\exp(-\varepsilon t)\,dt \\
A(t)\exp(-\varepsilon t) = &\frac{\delta}{\varepsilon + \beta}\left(\theta_S - \bar{\theta} - \frac{\alpha}{\beta}(\bar{r} - R)\right)\exp((-\varepsilon - \beta)t) \\
&+\frac{1}{\varepsilon}\left(\frac{\alpha\delta}{\beta} - \gamma\right)(\bar{r} - R)\exp(-\varepsilon t) + C_A \\
A(t) = &\frac{\delta}{\varepsilon + \beta}\left(\theta_S - \bar{\theta} - \frac{\alpha}{\beta}(\bar{r} - R)\right)\exp(-\beta t) + \frac{1}{\varepsilon}\left(\frac{\alpha\delta}{\beta} - \gamma\right)(\bar{r} - R) + C_A\exp(\varepsilon t).
\end{aligned}$$

For $t = 0$

$$A_S - \frac{1}{\varepsilon}\left(\frac{\alpha\delta}{\beta} - \gamma\right)(\bar{r} - R) - \frac{\delta}{\varepsilon + \beta}\left(\theta_S - \bar{\theta} - \frac{\alpha}{\beta}(\bar{r} - R)\right) = C_A$$

and thus

$$\begin{aligned}
A(t) = &\frac{1}{\varepsilon}\left(\frac{\alpha\delta}{\beta} - \gamma\right)(\bar{r} - R) + \frac{\delta}{\varepsilon + \beta}\left(\theta_S - \bar{\theta} - \frac{\alpha}{\beta}(\bar{r} - R)\right)\exp(-\beta t) \\
&+\left(A_S - \frac{1}{\varepsilon}\left(\frac{\alpha\delta}{\beta} - \gamma\right)(\bar{r} - R) - \frac{\delta}{\varepsilon + \beta}\left(\theta_S - \bar{\theta} - \frac{\alpha}{\beta}(\bar{r} - R)\right)\right)\exp(\varepsilon t).
\end{aligned}$$

The time path of the attack for defense policy is

$$\boxed{\begin{aligned} A(t) = &A_S\exp(\varepsilon t) + \tfrac{1}{\varepsilon}\left(\tfrac{\alpha\delta}{\beta} - \gamma\right)(\bar{r} - R)(1 - \exp(\varepsilon t)) \\ &+\tfrac{\delta}{\varepsilon+\beta}\left(\theta_S - \bar{\theta} - \tfrac{\alpha}{\beta}(\bar{r} - R)\right)(\exp(-\beta t) - \exp(\varepsilon t)), \end{aligned}} \tag{24}$$

for expansion policy we get

$$\boxed{\begin{aligned} A(t) = &A_S\exp(\varepsilon t) + \tfrac{1}{\varepsilon}\left(\tfrac{\alpha\delta}{\beta} - \gamma\right)\bar{r}(1 - \exp(\varepsilon t)) \\ &+\tfrac{\delta}{\varepsilon+\beta}\left(\theta_S - \bar{\theta} - \tfrac{\alpha}{\beta}\bar{r}\right)(\exp(-\beta t) - \exp(\varepsilon t)). \end{aligned}} \tag{23}$$

Time Path of the Fundamentals During High Stress Convergence At the upper boundary of the attack, $A = D$, the interest rate is chosen from the control interval $[0, R]$, to offset the motion of the attack, i.e. $\dot{A} = 0$. We apply these conditions to the differential equation of the attack (17) and get:

$$0 = -\gamma\left(r\left(t\right) - \bar{r}\right) - \delta\left(\theta\left(t\right) - \bar{\theta}\right) + \varepsilon D$$

Solving for the interest rate gives

$$r\left(t\right) = \bar{r} - \frac{\delta}{\gamma}\left(\theta\left(t\right) - \bar{\theta}\right) + \frac{\varepsilon}{\gamma}D. \tag{36}$$

Equation (36) shows that the interest rate $r\left(t\right)$ required to offset the attack increases when the fundamentals deteriorate and decreases when the fundamentals rise. Therefore, there are fundamental states where the attack cannot be offset since the required interest rate would exceed R. In this case, the central bank is forced to opt out. But there are also states in which the attack decreases, due to good fundamentals, even though the interest rate equals zero. The fundamental states corresponding to the control boundary are

$$\begin{aligned} r\left(t\right) = R &\leftrightarrow \theta = \bar{\theta} + \frac{\varepsilon}{\delta}D + \frac{\gamma}{\delta}\left(\bar{r} - R\right) \\ r\left(t\right) = 0 &\leftrightarrow \theta = \bar{\theta} + \frac{\varepsilon}{\delta}D + \frac{\gamma}{\delta}\bar{r}. \end{aligned} \tag{37}$$

Inserting $r\left(t\right)$ (eq. 36) in the differential equation of the fundamentals (17) gives

$$\begin{aligned} \dot{\theta} &= -\alpha\left(\left(\bar{r} - \frac{\delta}{\gamma}\left(\theta\left(t\right) - \bar{\theta}\right) + \frac{\varepsilon}{\gamma}D\right) - \bar{r}\right) - \beta\left(\theta\left(t\right) - \bar{\theta}\right) \\ &= \left(\frac{\alpha\delta}{\gamma} - \beta\right)\left(\theta\left(t\right) - \bar{\theta}\right) - \frac{\alpha\varepsilon}{\gamma}D. \end{aligned}$$

The path of the fundamentals at $A = D$ is thus

$$\frac{d\theta}{dt} - \left(\frac{\alpha\delta}{\gamma} - \beta\right)\theta\left(t\right) = -\left(\frac{\alpha\delta}{\gamma} - \beta\right)\bar{\theta} - \frac{\alpha\varepsilon}{\gamma}D$$

$$\int \frac{d}{dt}\left(\theta\left(t\right)\exp\left(-\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right)\right)dt = \int\left(-\left(\frac{\alpha\delta}{\gamma} - \beta\right)\bar{\theta} - \frac{\alpha\varepsilon}{\gamma}D\right)\exp\left(-\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right)dt$$

$$\theta\left(t\right)\exp\left(-\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right) = C + \frac{1}{-\left(\frac{\alpha\delta}{\gamma} - \beta\right)}\left(-\left(\frac{\alpha\delta}{\gamma} - \beta\right)\bar{\theta} - \frac{\alpha\varepsilon}{\gamma}D\right)\exp\left(-\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right)$$

$$\theta\left(t\right) = C\exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right) + \bar{\theta} + \frac{\frac{\alpha\varepsilon}{\gamma}D}{\frac{\alpha\delta}{\gamma} - \beta}.$$

For $t = 0$

$$\theta_S - \bar{\theta} - \frac{\frac{\alpha\varepsilon}{\gamma}D}{\frac{\alpha\delta}{\gamma} - \beta} = C$$

and thus

$$\theta\left(t\right) = \left(\theta_S - \bar{\theta} - \frac{\frac{\alpha\varepsilon}{\gamma}D}{\frac{\alpha\delta}{\gamma} - \beta}\right)\exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right) + \bar{\theta} + \frac{\frac{\alpha\varepsilon}{\gamma}D}{\frac{\alpha\delta}{\gamma} - \beta}.$$

Hence, the path of the fundamentals in high stress for $\theta \in \left[\bar{\theta} + \frac{\varepsilon}{\delta}D + \frac{\gamma}{\delta}(\bar{r} - R), \bar{\theta} + \frac{\varepsilon}{\delta}D + \frac{\gamma}{\delta}\bar{r}\right]$[32] is

$$\boxed{\theta(t) = \theta_S \exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right) + \left(\widehat{\theta}_{A=D}\right)\left(1 - \exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right)\right).} \tag{25}$$

where $\widehat{\theta}_{A=D} = \bar{\theta} + \frac{\frac{\alpha\varepsilon}{\gamma}D}{\frac{\alpha\delta}{\gamma} - \beta}$ is the upper convergence point.

Time Path of the Fundamentals During No Stress Convergence At the lower boundary of the attack, $A = 0$, the interest rate is chosen from the control interval $[0, R]$, to offset the motion of the attack, i.e. $\dot{A} = 0$. Using these conditions with the differential equation of the attack (17) gives:

$$0 = -\gamma(r(t) - \bar{r}) - \delta\left(\theta(t) - \bar{\theta}\right)$$

and solving for the interest rate gives

$$r(t) = \bar{r} - \frac{\delta}{\gamma}\left(\theta(t) - \bar{\theta}\right). \tag{38}$$

As in high stress, the control restriction imposes a restriction on the fundamental state space in which the motion of the attack can be offset. For bad fundamentals, the attack grows even though the interest rate is set to R. The fundamental states corresponding to the control boundary is

$$r(t) = R \leftrightarrow \theta = \bar{\theta} + \frac{\gamma}{\delta}(\bar{r} - R). \tag{39}$$

Inserting $r(t)$ (eq. 38) in the differential equation of the fundamentals (17) gives

$$\begin{aligned} \dot{\theta} &= -\alpha\left(\left(\bar{r} - \frac{\delta}{\gamma}\left(\theta(t) - \bar{\theta}\right)\right) - \bar{r}\right) - \beta\left(\theta(t) - \bar{\theta}\right) \\ &= \left(\frac{\alpha\delta}{\gamma} - \beta\right)\left(\theta(t) - \bar{\theta}\right). \end{aligned}$$

The path of the fundamentals at $A = 0$, is

$$\frac{d\theta}{dt} - \left(\frac{\alpha\delta}{\gamma} - \beta\right)\theta(t) = -\left(\frac{\alpha\delta}{\gamma} - \beta\right)\bar{\theta}$$

$$\int \frac{d}{dt}\left(\theta(t)\exp\left(-\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right)\right)dt = -\int\left(\frac{\alpha\delta}{\gamma} - \beta\right)\bar{\theta}\exp\left(-\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right)dt$$

$$\theta(t) = C\exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right) + \bar{\theta}.$$

For $t = 0$

$$\theta_S - \bar{\theta} = C$$

[32]Defense in high stress is only possible between the attack ZMLs.

and thus

$$\theta\left(t\right)=\left(\theta_S-\bar{\theta}\right)\exp\left(\left(\frac{\alpha\delta}{\gamma}-\beta\right)t\right)+\bar{\theta}.$$

or

$$\theta\left(t\right)=\theta_S\exp\left(\left(\frac{\alpha\delta}{\gamma}-\beta\right)t\right)+\bar{\theta}\left(1-\exp\left(\left(\frac{\alpha\delta}{\gamma}-\beta\right)t\right)\right).$$

For good states, there is an expected loss on attacking that actually allows to fend off additional speculators, even for negative interest rates. Since we restricted the interest rate to be nonnegative (19), we set $A=0$ for states better than the attack ZML of expansion policy and neglect the theoretical negativity of A. The path of the fundamentals in no stress for $\theta\in\left[\bar{\theta}+\frac{\gamma}{\delta}\left(\bar{r}-R\right),\infty\right[$ is:

$$\boxed{\theta\left(t\right)=\ \theta_S\exp\left(\left(\tfrac{\alpha\delta}{\gamma}-\beta\right)t\right)+\widehat{\theta}_{A=0}\left(1-\exp\left(\left(\tfrac{\alpha\delta}{\gamma}-\beta\right)t\right)\right),}\tag{27}$$

where $\widehat{\theta}_{A=0}=\bar{\theta}$.

5.3.2 Model Dynamics

Zero-Motion-Lines To describe the direction of the movement of the state variable we calculate zero-motion-lines (ZMLs). The differential equation of the fundamentals (17) for $r(t) = R$ is: $\dot{\theta}_{r=R} = -\alpha(R-\bar{r}) - \beta(\theta_{\dot{\theta}=0,R} - \bar{\theta})$. Equating with zero and solving for θ gives:

$$\theta_{\dot{\theta}=0,R} = \bar{\theta} - \frac{\alpha}{\beta}(R-\bar{r}) \tag{40}$$

accordingly for $r(t) = 0$

$$\theta_{\dot{\theta}=0,0} = \bar{\theta} + \frac{\alpha}{\beta}\bar{r} \tag{41}$$

Since the ZMLs of the fundamentals are independent of A they are vertical lines in the θ,A space.

Equating the differential equation of the attack (17) for $r(t) = R$ with zero $\dot{A} = -\gamma(R-\bar{r}) - \delta(\theta_{\dot{A}=0,R} - \bar{\theta}) + \varepsilon A = 0$ and solving for θ gives:

$$\theta_{\dot{A}=0,R} = \bar{\theta} - \frac{\gamma}{\delta}(R-\bar{r}) + \frac{\varepsilon}{\delta}A \tag{42}$$

accordingly for $r(t) = 0$

$$\theta_{\dot{A}=0,0} = \bar{\theta} + \frac{\gamma}{\delta}\bar{r} + \frac{\varepsilon}{\delta}A \tag{43}$$

The ZMLs of the attack have a positive slope of $\frac{\varepsilon}{\delta}$ in the θ,A space.

Proof of proposition 2: we proof by contradiction and show that the intersection of the expansion policy ZMLs (41) and (43) violates the state restriction (20). That is $\bar{\theta} + \frac{\alpha}{\beta}\bar{r} \neq \bar{\theta} + \frac{\gamma}{\delta}\bar{r} + \frac{\varepsilon}{\delta}A$, for $A \in [0, D]$. Reorganizing gives: $\frac{1}{\varepsilon}\left(\frac{\alpha\delta}{\beta} - \gamma\right)\bar{r} \neq A$. With restriction (26), $\frac{\alpha\delta}{\beta} - \gamma < 0$, we get: $A < 0$ for an intersection of the expansion policy ZMLs.

Since expansion policy ZMLs only intersect for a negative attack value and the attack ZML has a positive slope, the fundamental ZML lies on the left side of the attack ZML in the feasible state.

Proof of proposition 3: when the central bank stops the attack ($\dot{A} = 0$) in high stress ($A = D$) the evolution of fundamentals is given by

$$\theta(t)_{A=D} = \theta_S \exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right) + \left(\bar{\theta} + \frac{\frac{\alpha\varepsilon}{\gamma}D}{\frac{\alpha\delta}{\gamma} - \beta}\right)\left(1 - \exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right)\right). \tag{25}$$

It is obvious that for

$$\boxed{\frac{\alpha\delta}{\gamma} - \beta < 0} \tag{26}$$

the path converges to

$$\widehat{\theta}_{A=D} = \bar{\theta} + \frac{\frac{\alpha\varepsilon}{\gamma}D}{\frac{\alpha\delta}{\gamma} - \beta}. \tag{44}$$

If condition (26) would not apply, fundamentals would grow or decrease infinitely depending on the initial fundamental state at $A = D$. Since an infinite expansion is economically not reasonable we apply the parameter restriction.

The convergence point is only accessible, if it's in the interval defined by the control restriction (cf. 37), i.e. $\left[\bar{\theta} + \frac{\varepsilon}{\delta}D + \frac{\gamma}{\delta}(\bar{r} - R), \bar{\theta} + \frac{\varepsilon}{\delta}D + \frac{\gamma}{\delta}\bar{r}\right]$. In other words, the convergence point is between the attack ZMLs $\left[\theta^{A=D}_{\dot{A}=0,R}, \theta^{A=D}_{\dot{A}=0,0}\right]$. Therefore, it is necessary that $\bar{\theta} + \frac{\frac{\alpha\varepsilon}{\gamma}D}{\frac{\alpha\delta}{\gamma} - \beta} > \bar{\theta} + \frac{\varepsilon}{\delta}D + \frac{\gamma}{\delta}(\bar{r} - R)$. Hence, the convergence point is feasible, if

$$R > \bar{r} - \frac{\varepsilon D}{\left(\frac{\alpha\delta}{\beta} - \gamma\right)}. \tag{45}$$

On the other side of the interval it is necessary that $\bar{\theta} + \frac{\frac{\alpha\varepsilon}{\gamma}D}{\frac{\alpha\delta}{\gamma} - \beta} < \bar{\theta} + \frac{\varepsilon}{\delta}D + \frac{\gamma}{\delta}\bar{r} \leftrightarrow \frac{\beta\delta\varepsilon D}{\gamma\left(\frac{\alpha\delta}{\gamma} - \beta\right)} < \bar{r}$, which is true (cf. condition 26). From (44) it is obvious that the maximum of $\widehat{\theta}_{A=D}$ is $\bar{\theta}$ for $D = 0$. Therefore $\widehat{\theta}_{A=D}$ is between the attack ZML of defense policy $\theta^{A=D}_{\dot{A}=0,R}$ and the no stress steady state $\bar{\theta}$.

Proof of proposition 4: we proof by contradiction and show that the intersection of the defense policy ZMLs (40) and (42) violates the state restriction (20). That is $\bar{\theta} + \frac{\alpha}{\beta}(\bar{r} - R) \neq \bar{\theta} + \frac{\gamma}{\delta}(\bar{r} - R) + \frac{\varepsilon}{\delta}A$, for $A \in [0, D]$. Reorganizing gives: $\frac{1}{\varepsilon}\left(\frac{\alpha\delta}{\beta} - \gamma\right)(\bar{r} - R) \neq A$. The left hand side is positive ($R > \bar{r}$). With the restriction to feasible convergence points (cf. proposition 3), i.e. $R > \bar{r} - \frac{\varepsilon D}{\frac{\alpha\delta}{\beta} - \gamma}$ it follows that $A > D$. Therefore, the defense policy ZMLs intersect above the defensive measure.

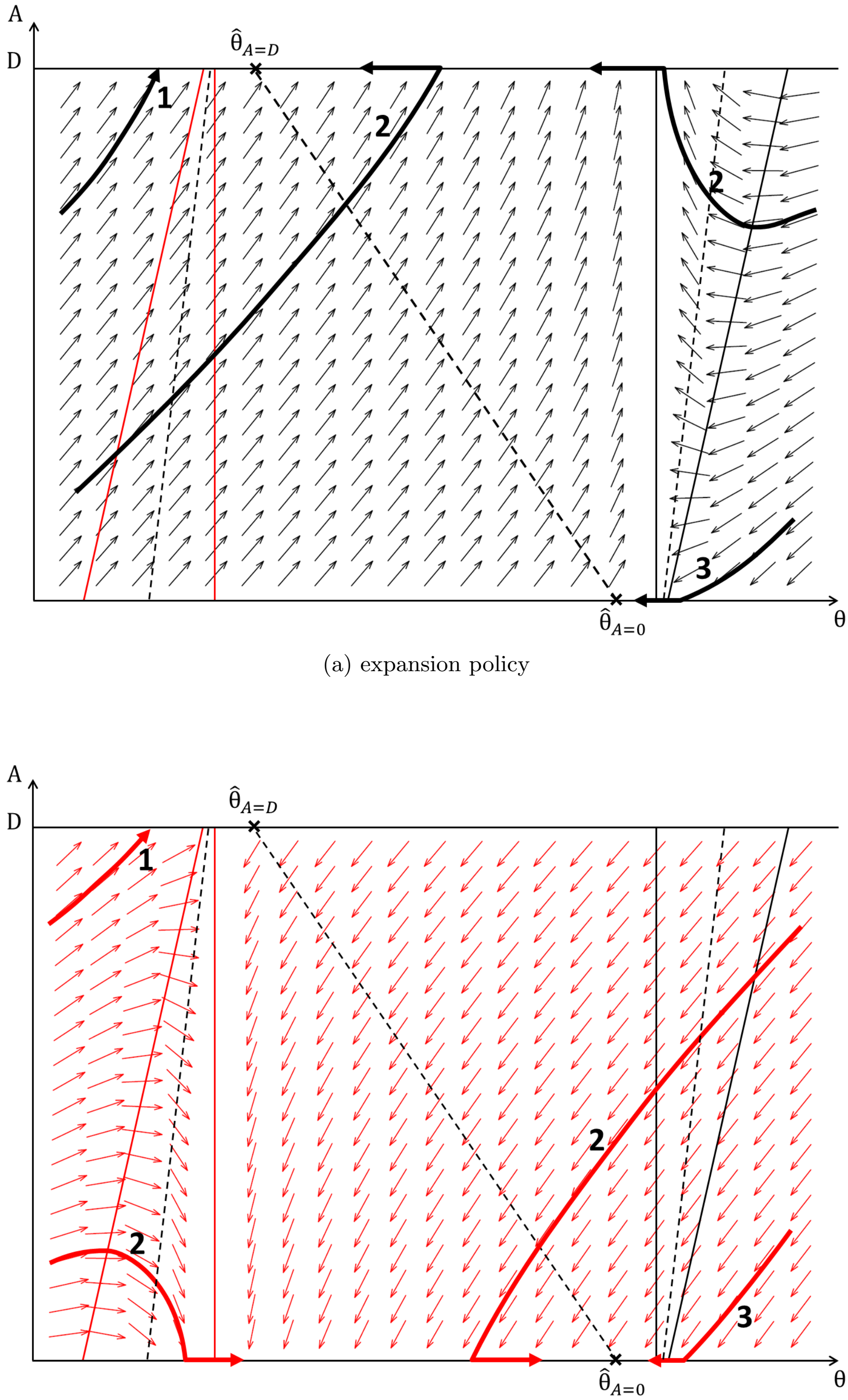
Proof of proposition 5: when the central bank chooses to preserve the attack in no stress ($\dot{A} = 0$ and $A = 0$), the fundamentals evolve according to

$$\theta(t) = \theta_S \exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right) + \bar{\theta}\left(1 - \exp\left(\left(\frac{\alpha\delta}{\gamma} - \beta\right)t\right)\right). \tag{27}$$

For $\frac{\alpha\delta}{\gamma} - \beta$ (condition 26) it is obvious that the path converges to

$$\widehat{\theta}_{A=0} = \bar{\theta}.$$

The position of the no stress convergence point $\widehat{\theta}_{A=0}$ is between the fundamental ZMLs of expansion policy and defense policy. It is obvious that $\theta_{\dot{\theta}=0,R} < \widehat{\theta}_{A=0} < \theta_{\dot{\theta}=0,0}$, i.e. $\bar{\theta} - \frac{\alpha}{\beta}(R - \bar{r}) < \bar{\theta} < \bar{\theta} + \frac{\alpha}{\beta}\bar{r}$.



(a) expansion policy

(b) defense policy

Figure 13: **Vector fields:** The figures show vectors that represent the directional motion of the system under expansion (black, panel a) and defense (red, panel b). The black lines are sample trajectories, with initial values corresponding to areas 1 to 3 of figure 8. Also shown are the ZMLs (expansion: solid black; defense: solid red), focal points, and the identity lines (dahed black).

5.3.3 Optimal Behavior

Proof of proposition 6: the identity line is derived from equating the slopes of the paths of expansion and defense policy. For positive slopes of the trajectories we have $\frac{dA_{r=0}}{d\theta_{r=0}} = \frac{dA_{r=R}}{d\theta_{r=R}}$, while for slopes of differing sign we have $\frac{dA_{r=0}}{d\theta_{r=0}} = -\frac{dA_{r=R}}{d\theta_{r=R}}$. These equalities are equal to $\frac{\dot{A}_{r=0}}{\dot{\theta}_{r=0}} = \frac{\dot{A}_{r=R}}{\dot{\theta}_{r=R}}$ and $\frac{\dot{A}_{r=0}}{\dot{\theta}_{r=0}} = -\frac{\dot{A}_{r=R}}{\dot{\theta}_{r=R}}$. Inserting the differential equations (17) in the positive slopes, we get:

$$\frac{\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \varepsilon A_S}{\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)} = \frac{-\gamma\left(R - \bar{r}\right) - \delta\left(\theta_S - \bar{\theta}\right) + \varepsilon A_S}{-\alpha\left(R - \bar{r}\right) - \beta\left(\theta_S - \bar{\theta}\right)}.$$

Reorganizing

$$\frac{\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right) - \alpha R}{\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)} = \frac{\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \varepsilon A_S - \gamma R}{\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \varepsilon A_S}$$

cancelling down

$$\frac{\alpha}{\gamma} = \frac{\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)}{\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \varepsilon A_S}$$

and solving for A_S gives the identity line:

$$\boxed{A_S = \left(\theta_S - \bar{\theta}\right)\frac{\alpha\delta - \beta\gamma}{\alpha\varepsilon} = \left(\theta_S - \bar{\theta}\right)\frac{\frac{\alpha\delta}{\gamma} - \beta}{\frac{\alpha\varepsilon}{\gamma}}.} \tag{28}$$

The function describes a straight line with negative slope in the θ, A space. The line crosses certain prominent points that are relevant for the characterization of the dynamics. The identity lines are plotted in figure 5 on page 20.

If $A_S = 0$, then $\theta_S = \bar{\theta} = \widehat{\theta}_{A=0}$; if $A_S = D$ then $\theta_S = \bar{\theta} + \frac{\frac{\alpha\varepsilon}{\gamma}D}{\frac{\alpha\delta}{\gamma} - \beta} = \widehat{\theta}_{A=D}$. Inserting in the attack ZML (43) for expansion policy gives: $\theta_S = \bar{\theta} + \frac{\gamma}{\delta}\bar{r} + \frac{\varepsilon}{\delta}\left(\theta_S - \bar{\theta}\right)\frac{\frac{\alpha\delta}{\gamma} - \beta}{\frac{\alpha\varepsilon}{\gamma}}$ and solving for θ_S gives: $\bar{\theta} + \frac{\alpha}{\beta}\bar{r}$. Using (42) for defense policy we get: $\theta_S = \bar{\theta} - \frac{\gamma}{\delta}\left(R - \bar{r}\right) + \frac{\varepsilon}{\delta}\left(\theta_S - \bar{\theta}\right)\frac{\frac{\alpha\delta}{\gamma} - \beta}{\frac{\alpha\varepsilon}{\gamma}}$ and solving for θ_S gives: $\bar{\theta} + \frac{\alpha}{\beta}\left(\bar{r} - R\right)$. Thus, the identity line crosses the intersection of the defense ZMLs, the upper and lower convergence points and the intersection of the expansion ZMLs.

To proof, which policy has a higher slope above the identity line we add a small $\xi > 0$ to the attack value, $A_S + \xi = \left(\theta_S - \bar{\theta}\right)\frac{\frac{\alpha\delta}{\gamma} - \beta}{\frac{\alpha\varepsilon}{\gamma}} + \xi$. We then proof that expansion policy has a higher slope than defense policy above the identity line. Therefore, it is necessary that:

$$\frac{\dot{A}_{r=0}}{\dot{\theta}_{r=0}}\left(\theta_S, A_S + \xi\right) > \frac{\dot{A}_{r=R}}{\dot{\theta}_{r=R}}\left(\theta_S, A_S + \xi\right)$$

$$\frac{\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \varepsilon\left(\left(\theta_S - \bar{\theta}\right)\frac{\alpha\delta - \beta\gamma}{\alpha\varepsilon} + \xi\right)}{\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)} > \frac{-\gamma\left(R - \bar{r}\right) - \delta\left(\theta_S - \bar{\theta}\right) + \varepsilon\left(\left(\theta_S - \bar{\theta}\right)\frac{\alpha\delta - \beta\gamma}{\alpha\varepsilon} + \xi\right)}{-\alpha\left(R - \bar{r}\right) - \beta\left(\theta_S - \bar{\theta}\right)}.$$

Depending on the location in the state space the directional movement of the fundamentals and the attack differs. The identity line derived is valid in areas where the slopes of the trajectories have positive signs. That is to the left of the attack ZML of defense policy, between the fundamental ZMLs and to the right of the attack ZML of expansion policy. Since the directions of the differential equations differ across these areas, we proof the inequality by cases.

1. To the left of the attack ZML of defense policy both trajectories have the same direction, with $\dot{\theta}_{r=0},\ \dot{A}_{r=0},\ \dot{\theta}_{r=R},\ \dot{A}_{r=R} > 0$. Therefore, we rearrange the inequality as follows:

$$\begin{aligned}
\frac{-\alpha\left(R-\bar{r}\right)-\beta\left(\theta_S-\bar{\theta}\right)}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)} &> \frac{-\gamma\left(R-\bar{r}\right)-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}+\xi\right)}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}+\xi\right)}\\
1-\frac{\alpha R}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)} &> 1-\frac{\gamma R}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}+\xi\right)}\\
\frac{\alpha}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)} &< \frac{\gamma}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}+\xi\right)}\\
\frac{\alpha}{\gamma} &< \frac{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}+\xi\right)},
\end{aligned}$$

which is false, since $\frac{\alpha}{\gamma}=\frac{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon A_S}$ (cf. equation 28). Consequently, the expansion policy trajectory has the higher slope to the left of the attack ZML for $0\leq A\leq D$.

2. Between the fundamental ZMLs both trajectories move in opposite directions, with $\dot{\theta}_{r=0}$, $\dot{A}_{r=0}>0$ and $\dot{\theta}_{r=R},\ \dot{A}_{r=R}<0$. Hence,

$$\begin{aligned}
\frac{-\alpha\left(R-\bar{r}\right)-\beta\left(\theta_S-\bar{\theta}\right)}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)} &< \frac{-\gamma\left(R-\bar{r}\right)-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}+\xi\right)}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}+\xi\right)}\\
1-\frac{\alpha R}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)} &< 1-\frac{\gamma R}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}+\xi\right)}\\
\frac{\alpha}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)} &> \frac{\gamma}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}+\xi\right)}\\
\frac{\alpha}{\gamma} &> \frac{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}+\xi\right)},
\end{aligned}$$

which is true, confirming that above the identity line and between the fundamental ZMLs expansion policy leads to a higher slope, while below defense policy leads to a higher slope.

3. To the right of the attack ZML of expansion policy both trajectories again have the same direction, with $\dot{\theta}_{r=0},\ \dot{A}_{r=0},\ \dot{\theta}_{r=R},\ \dot{A}_{r=R} < 0$. Hence,

$$\begin{aligned}
\frac{-\alpha\left(R-\bar{r}\right)-\beta\left(\theta_S-\bar{\theta}\right)}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)} &> \frac{-\gamma\left(R-\bar{r}\right)-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}+\xi\right)}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}+\xi\right)}\\
1-\frac{\alpha R}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)} &> 1-\frac{\gamma R}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}+\xi\right)}\\
\frac{\alpha}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)} &< \frac{\gamma}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}+\xi\right)}\\
\frac{\alpha}{\gamma} &> \frac{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}+\xi\right)},
\end{aligned}$$

which is false, since $\dot{\theta}_{r=0}=\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)<0$ and $\dot{A}_{r=0}=\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(\left(\theta_S-\bar{\theta}\right)\frac{\alpha\delta-\beta\gamma}{\alpha\varepsilon}\right)$ <0. This confirms that the trajectory of defense policy has the higher slope to the right of the attack ZML of expansion policy for $0\leq A\leq D$.

To find the identity line, we compared the slopes of the directional movement in the θ, A space. The identity line considered resulted from a positive slope of the trajectories of expansion and defense policy. In the state space between the expansion policy ZMLs and the defense policy ZMLs the slopes of the trajectories have a different sign (cf. figure 3). Therefore, there are two more identity lines given by

$$\frac{\dot{A}_{r=0}}{\dot{\theta}_{r=0}}=-\frac{\dot{A}_{r=R}}{\dot{\theta}_{r=R}}.$$

Inserting the differential equations (17)

$$\frac{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon A_S}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)}=-\frac{-\gamma\left(R-\bar{r}\right)-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon A_S}{-\alpha\left(R-\bar{r}\right)-\beta\left(\theta_S-\bar{\theta}\right)}$$

and rearranging

$$\frac{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)-\alpha R}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)}=\frac{-\gamma\bar{r}+\delta\left(\theta_S-\bar{\theta}\right)-\varepsilon A_S+\gamma R}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon A_S}$$

gives

$$1-\frac{\alpha R}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)}=-1+\frac{\gamma R}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon A_S}$$

or

$$2=\frac{\gamma R}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon A_S}+\frac{\alpha R}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)}.$$

Which can be solved for A_S

$$\boxed{A_S=-\frac{1}{\varepsilon}\left(\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\frac{\gamma R\left(\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)\right)}{\alpha R-2\left(\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)\right)}\right).} \tag{28}$$

Which is a function that defines two curves with positive slopes in the feasible attack space. The curves cross the intersection of the defense policy ZMLs and the expansion policy ZMLs. Again, we are interested, whether expansion policy has a higher slope above the identity line $(A_S + \xi)$. To compare the absolute value of the slope we consider only positive slopes. Since the direction of the motion changes, depending on the state, we have to proof by cases.

1. Between the defense policy ZMLs the differential equations have the following signs: $\dot{\theta}_{r=0}$, $\dot{A}_{r=0} > 0$ and $\dot{\theta}_{r=R} > 0$, $\dot{A}_{r=R} < 0$. Therefore, we get:

$$\frac{\dot{A}_{r=0}}{\dot{\theta}_{r=0}} (A_S + \xi) > -\frac{\dot{A}_{r=R}}{\dot{\theta}_{r=R}} (A_S + \xi)$$

after inserting, we get

$$\frac{\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \varepsilon\left(-\frac{1}{\varepsilon}\left(\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \frac{\gamma R\left(\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)\right)}{\alpha R - 2\left(\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)\right)}\right) + \xi\right)}{\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)} >$$
$$-\frac{-\gamma\left(R - \bar{r}\right) - \delta\left(\theta_S - \bar{\theta}\right) + \varepsilon\left(-\frac{1}{\varepsilon}\left(\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \frac{\gamma R\left(\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)\right)}{\alpha R - 2\left(\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)\right)}\right) + \xi\right)}{-\alpha\left(R - \bar{r}\right) - \beta\left(\theta_S - \bar{\theta}\right)}.$$

$$\frac{-\alpha\left(R - \bar{r}\right) - \beta\left(\theta_S - \bar{\theta}\right)}{\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)} >$$
$$-\frac{-\gamma\left(R - \bar{r}\right) - \delta\left(\theta_S - \bar{\theta}\right) + \varepsilon\left(-\frac{1}{\varepsilon}\left(\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \frac{\gamma R\left(\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)\right)}{\alpha R - 2\left(\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)\right)}\right) + \xi\right)}{\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \varepsilon\left(-\frac{1}{\varepsilon}\left(\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \frac{\gamma R\left(\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)\right)}{\alpha R - 2\left(\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)\right)}\right) + \xi\right)}$$

$$1 - \frac{\alpha R}{\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)} >$$
$$-1 + \frac{\gamma R}{\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \varepsilon\left(-\frac{1}{\varepsilon}\left(\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \frac{\gamma R\left(\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)\right)}{\alpha R - 2\left(\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)\right)}\right) + \xi\right)}$$

$$2 > \frac{\gamma R}{\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \varepsilon\left(-\frac{1}{\varepsilon}\left(\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \frac{\gamma R\left(\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)\right)}{\alpha R - 2\left(\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)\right)}\right) + \xi\right)} + \frac{\alpha R}{\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)},$$

which is true, since $2 = \frac{\gamma R}{\gamma\bar{r} - \delta\left(\theta_S - \bar{\theta}\right) + \varepsilon A_S} + \frac{\alpha R}{\alpha\bar{r} - \beta\left(\theta_S - \bar{\theta}\right)}$ (cf. equation 28). Hence, above the identity line expansion policy leads to the higher slope of the state trajectory.

2. Between the expansion policy ZMLs the differential equations have the following signs: $\dot{\theta}_{r=0} < 0$, $\dot{A}_{r=0} > 0$ and $\dot{\theta}_{r=R}$, $\dot{A}_{r=R} < 0$. To compare positive slopes of the trajectories

in this area we write

$$-\frac{\dot{A}_{r=0}}{\dot{\theta}_{r=0}}\left(A_S+\xi\right) > \frac{\dot{A}_{r=R}}{\dot{\theta}_{r=R}}\left(A_S+\xi\right).$$

after inserting, we get

$$-\frac{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(-\frac{1}{\varepsilon}\left(\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\frac{\gamma R\left(\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)\right)}{\alpha R-2\left(\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)\right)}\right)+\xi\right)}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)} >$$

$$\frac{-\gamma\left(R-\bar{r}\right)-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(-\frac{1}{\varepsilon}\left(\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\frac{\gamma R\left(\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)\right)}{\alpha R-2\left(\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)\right)}\right)+\xi\right)}{-\alpha\left(R-\bar{r}\right)-\beta\left(\theta_S-\bar{\theta}\right)}.$$

$$-\frac{-\alpha\left(R-\bar{r}\right)-\beta\left(\theta_S-\bar{\theta}\right)}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)} <$$

$$\frac{-\gamma\left(R-\bar{r}\right)-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(-\frac{1}{\varepsilon}\left(\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\frac{\gamma R\left(\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)\right)}{\alpha R-2\left(\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)\right)}\right)+\xi\right)}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(-\frac{1}{\varepsilon}\left(\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\frac{\gamma R\left(\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)\right)}{\alpha R-2\left(\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)\right)}\right)+\xi\right)}$$

$$-1+\frac{\alpha R}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)} <$$

$$1-\frac{\gamma R}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(-\frac{1}{\varepsilon}\left(\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\frac{\gamma R\left(\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)\right)}{\alpha R-2\left(\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)\right)}\right)+\xi\right)}$$

$$\frac{\gamma R}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon\left(-\frac{1}{\varepsilon}\left(\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\frac{\gamma R\left(\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)\right)}{\alpha R-2\left(\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)\right)}\right)+\xi\right)}+\frac{\alpha R}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)} < 2,$$

which is true, since $2=\frac{\gamma R}{\gamma\bar{r}-\delta\left(\theta_S-\bar{\theta}\right)+\varepsilon A_S}+\frac{\alpha R}{\alpha\bar{r}-\beta\left(\theta_S-\bar{\theta}\right)}$ (cf. equation 28). Therefore, above the identity line expansion policy leads to the higher slope, whereas below defense policy has the higher slope.

The identity lines and the areas with the respective higher slope are plotted in figure 5 on page 20.

Proof of proposition 11: we give a condition for which the value of an immediate opt-out is better than the value of converging in high stress:

$$\frac{\theta_S-c}{\rho} > \int_0^\infty \exp\left(-\rho t\right)\left(\left(\theta_S-\widehat{\theta}_{A=D}\right)\exp\left(\left(\frac{\alpha\delta}{\gamma}-\beta\right)t\right)+\widehat{\theta}_{A=D}\right)dt$$

$$\frac{\theta_S-c}{\rho} > -\frac{\theta_S-\widehat{\theta}_{A=D}}{\frac{\alpha\delta}{\gamma}-\beta-\rho}+\frac{\widehat{\theta}_{A=D}}{\rho}$$

Rearranging and solving for θ_S gives:

$$\theta_S > \widehat{\theta}_{A=D} + c\frac{\frac{\alpha\delta}{\gamma} - \beta - \rho}{\frac{\alpha\delta}{\gamma} - \beta}$$

$$\boxed{\theta_c \equiv \widehat{\theta}_{A=D} + c\tfrac{\frac{\alpha\delta}{\gamma} - \beta - \rho}{\frac{\alpha\delta}{\gamma} - \beta}.}$$

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