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Ludwig von Auer
Tu Anh Pham

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Imperfect Collusion On Surveilled Markets With Free Entry

Ludwig von Auer (Universität Trier)
Tu Anh Pham (Universität Trier)[1]

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Abstract: Surveys of cartel proceedings reveal that illegal cartels usually (1) attempt to minimize the risk of detection, (2) achieve merely imperfect levels of collusion, (3) compete against some fringe firms, and (4) adjust to market entries and exits. By contrast, existing oligopoly models of collusive behavior consider only some of the four listed stylized facts and, thus, run the risk of missing important interdependencies between them. Therefore, the present paper develops a general quantity leadership model that simultaneously accommodates all four stylized facts. Within this model, an imperfectly colluding group of firms competes against independent fringe rivals. The market is surveilled by an antitrust authority that has three different policy instruments at its disposal: Ensuring free market access, obstructing collusion, and discouraging collusion through law enforcement. The results of the model indicate that the latter two instruments are rather ineffective.

JEL-Classification: L0, L1
Keywords: antitrust, fringe, oligopoly, stability, sustainability.

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1 Introduction

Surveys of cartel cases reveal many interesting facets about collusive behavior.[2] Four stylized facts stand out. Cartels (1) introduce elaborate arrangements to minimize the risk of being caught and punished by antitrust authorities, (2) struggle to enforce their agreement and, therefore, develop sophisticated means to improve compliance, (3) usually compete against some fringe firms, and (4) operate on markets characterized by occasional market entries and exits. The four listed empirical facts translate into four features that a comprehensive model of collusive behavior should accommodate.

The first empirical fact (exposure to antitrust surveillance) is the *antitrust* issue of the colluding firms (henceforth, "the cartel"). The most common instruments of antitrust policy are (a) ensuring free market access, (b) making collusion more difficult, and (c) discouraging collusion through law enforcement. To compare the efficacy of these instruments, a model of collusive behavior must accomodate them in an appropriate form.

The antitrust surveillance forces the cartel to renounce legally binding contracts and to operate in secrecy. This limits the cartel's ability to coordinate its actions and to prevent new competitors from entering the market. Therefore, the cartel develops other means of enforceable coordination. This is the second empirical fact (incomplete internal compliance). It highlights the issue of *cartel sustainability.* A cartel agreement is sustainable if no member of the cartel wants to deviate from this agreement. Therefore, a model's equilibrium solution requires that cartel members have no incentives to deviate from that solution.

The third empirical fact (competition from fringe firms) raises the issue of *cartel stability.* A cartel is considered as stable if no member has an incentive to become a fringe firm and, at the same time, no fringe firm has an incentive to become a member of the cartel. This requirement provides another condition for the derivation of a model's equilibrium solution.

The fourth empirical fact (market entries and exits) emphasizes a problem that is labelled here as *market stability.* A market is regarded as stable, if no firm wants to enter or exit it. This is a third condition for a model's equilibrium solution.

To the best of our knowledge, none of the existing oligopoly models that consider the issues of cartel sustainability or cartel stability (few consider both) also include the issues of antitrust surveillance and market stability. However, such a comprehensive approach is necessary to compare the efficacy of the three instruments of antitrust policy. Therefore, the overall contribution of the present paper is the joint analysis of all four issues within one comprehensive model. The results of the model indicate that ensuring free market access is a very effective instrument, while making collusion more difficult and discouraging collusion through law enforcement provides no significant additonal welfare gains.

The model is based upon the leadership approach, that is, the cartel acts as Stackelberg leader and the fringe firms as independent Stackelberg followers. The studies of d'Aspremont *et al.* (1983), Donsimoni (1985), Donsimoni *et al.* (1986), and Prokop (1999) utilize the price leadership model. The quantity leadership model is applied by Shaffer (1995), Lofaro (1999), Konishi and Lin (1999), Zu *et al.* (2012),

[2] See, for example, Harrington (2006) or Levenstein and Suslow (2006).

Auer and Pham (2021), and by the model of the present study.[3] Martin (1990) considers both variants. All listed studies neglect the problem of market stability. Instead, the primary topic of the leadership approach is cartel stability. The cartel's sustainability problem is usually evaded by simply assuming that the group of cartel members can act as if they were a merged firm.[4]

The issue of sustainability is extensively studied in supergames (repeated oligopoly games) with grim-trigger strategies or some other strategies that form a subgame perfect Nash equilibrium. This strand of literature has been pioneered by Friedman (1971). Sustainability is exclusively ensured by the threat of penalties imposed by the other cartel members. Other motivations for compliance (e.g., mutual trust and/or transparency) are not considered. Supergames that simultaneously tackle the issues of penalty enforced cartel sustainability and cartel stability include Escrihuela-Villar (2008, 2009), Bos (2009), and Bos and Harrington (2010). In Bos and Harrington (2015) the analysis of cartel stability and penalty enforced cartel sustainability is amended by an antitrust policy. Same as the existing models of the leadership approach, these supergames do not deal with the problem of market stability.

The supergame approach is a dynamic framework of imperfect competition. The leadership approach can be viewed as a "reduced-form" representation in the sense that the controversies and complexities associated with discounting a distant future are avoided, while the economic aspects of the cartel's fragility and the fringe-cartel relationship are fully preserved. The simplified time dimension of the leadership approach creates the possibility to design a comprehensive oligopoly model that simultaneously addresses all four features of collusive behavior and, therefore, their interdependencies.

The comprehensive oligopoly model is developed in three stages and solved by backward induction. The paper is organized accordingly. Section 2 introduces the last stage of the game. At this stage, the number of operating firms and also the status of each firm (fringe or cartel) are given. For each firm, the profit-maximizing output is derived. It satisfies cartel sustainability. The second stage is presented in Section 3. The firms can choose their preferred status and output, though the number of operating firms is still given. The solution satisfies cartel sustainability and cartel stability. In Section 4, the first stage of the game is presented. The firms decide on their market entry or exit, that is, the number of operating firms becomes endogenous, too. The unique equilibrium solution is derived. It satisfies cartel sustainability, cartel stability, and market stability. Furthermore, the implications for the design of an effective antitrust policy are discussed. Concluding remarks are contained in Section 5.

2 Sustainability of Cartels

This section is devoted to the final stage of the games' three stages. Thus, the number of operating firms and the status of each firm (fringe or cartel) has been

[3]Some important results derived in Martin (1990) and Shaffer (1995) coincide with findings presented in Selten's (1973) pioneering study.

[4]An interesting exception is the model of Lofaro (1999) described in Section 3.3.

determined on the previous stages.

2.1 Overview

The inverse demand function for a homogeneous product is $P = a - bQ$, where P is the market price and Q is the aggregate quantity produced. The industry consists of a given finite number of $n \geq 1$ identical firms. Only integer numbers of firms are considered. All n firms have a constant marginal cost equal to c and a positive fixed cost that can represent an entry cost or a cost of production.

Without loss of generality, we can change the currency and the units in which output is measured (e.g., Selten, 1973, p. 144). An original unit of output is equivalent to $b/(a-c)$ new units of output and a unit of the original currency is equivalent to $b/(a-c)^2$ units of the new currency. With this normalization, the new values of the parameters a, b, and c yield $a - c = 1$ and $b = 1$. Thus, the market volume (perfect competition output), $(a - c)/b$, is normalized to 1. The normalized fixed cost is denoted by $f \leq 1/9$. The upper bound is the gross profit (revenues minus variable cost) of Cournot duopolists.

Of the n operating firms, a given group of $k \in (2, \ldots, n)$ firms colludes. We denote this group as "the cartel". If all n firms join the cartel, it is denoted as *complete*. Otherwise, it is an *incomplete* cartel that competes against $(n-k)$ "fringe firms". Then, the cartel acts as a Stackelberg leader and the $(n - k)$ fringe firms as Stackelberg followers.[5] These fringe firms consider the cartel's output as given and compete on the residual demand.

Given their feasible level of collusion and the reaction function of the fringe, the cartel members collectively determine their profit maximizing joint *sustainable* output Q_K (details in Section 2.3). The output Q_K is sustainable in the sense that no member of the cartel has an incentive to deviate from its output decision. Inserting the output Q_K into the reaction function of the fringe firms yields the aggregated fringe output Q_F, the total output $(Q = Q_K + Q_F)$, the profit of the $(n - k)$ fringe firms, and the profit of the k cartel members.

For each industry size, n, and cartel size, k, such a sustainable equilibrium can be derived. Only in Section 3, the firms can choose whether they want to be a fringe firm or a member of the cartel, that is, we add the issue of cartel stability and the size of the cartel, k, becomes endogenous. The issue of market stability is introduced in Section 4 through the endogeneity of the number of operating firms, n.

2.2 Reaction Function of Fringe Firms

The fringe firms consider the output of the cartel as given and compete on the residual demand. Since the number of fringe firms is limited, they recognize the interdependency of their individual quantity decisions. Therefore, quantity leadership models assume that the fringe firms are in Cournot competition to each other.

[5]Brito and Catalão-Lopes (2011, p. 3) summarize the justifications for assuming that the cartel acts as a leader. Huck et al. (2007) provide some experimental evidence that firms that cooperate in a binding manner show leadership behavior, whereas the remaining firms exhibit follower behavior.

The profit of a fringe firm is

$$\pi_F = P(Q)q_F - cq_F - f = (1 - Q_K - Q_{-F})\, q_F - q_F^2 - f \ ,$$

where q_F is the fringe firm's output and Q_{-F} is the aggregate output of all other fringe firms. Each fringe firm considers Q_{-F} and the cartel output, Q_K, as given. Exploiting the symmetry of the fringe firms, their profit maximizing total output is

$$Q_F = (n-k)\frac{1 - Q_K}{n-k+1} \ . \tag{1}$$

This is the reaction function of the fringe, taking Q_K as given.

Inserting this result in the demand function yields

$$P(Q) - c = \frac{1 - Q_K}{n-k+1} \ . \tag{2}$$

This mark-up incorporates the profit maximizing reaction of the $(n-k)$ fringe firms to the output Q_K chosen by the cartel. The mark-up does not depend on the process by which the cartel output, Q_K, is determined. In our model, the cartel decides for an output that is sustainable. What is sustainable depends on the cartel's level of collusion.

2.3 Antitrust Authority and Imperfect Collusion

To address the antitrust issue, the model includes an antitrust authority that monitors the market and attempts to detect and punish collusive behavior. Let $\Pr \in [0\,,1]$ denote the probability of a successful conviction of the cartel. Such a conviction requires not only effective market surveillance but also a success in court. Only if the conviction is successful, the cartel must pay a fine. The fine is proportional to the convicted cartel member's gross profit. The factor of proportion is denoted by $\phi \in [0\,, 1/\Pr]$.

Thus, the gross profit of a member of a detected cartel is $(P-c)q_K\,(1-\phi)$ and a cartel member's expected profit is

$$\begin{aligned} E(\pi_K) &= (1 - \Pr)\,(P-c)q_K + \Pr(P-c)q_K\,(1-\phi) - f \\ &= (P-c)q_K\,(1-p) - f \ , \end{aligned} \tag{3}$$

with $p = \Pr \cdot \phi \in [0,1]$ denoting the antitrust policy's *rigour*. Analogously, the expected average profit of the other cartel members is

$$E\left(\bar{\pi}_{-K}\right) = (P-c)\bar{q}_{-K}\,(1-p) - f \ , \tag{4}$$

where $\bar{q}_{-K}$ is the average output of the other cartel members.

A single firm cannot proclaim itself as cartel, that is, as the Stackelberg leader. To be part of a cartel requires collusion and this involves at least two firms: $k \geq 2$.[6] In quantity leadership models, the cartel's collusion has two levels. The basic

[6] This condition requires that $n \geq 2$. In the monopoly case ($n = 1$) collusion is redundant. The monopolist's gross profit is $\pi = 1/4$ and total output is $Q = 1/2$.

level is the appropriation of the Stackelberg leadership and the upper level is the coordination of the cartel output and its allocation.

The standard quantity leadership model assumes that perfect collusion prevails, that is, not only the cooperation at the basic level of collusion (appropriation of Stackelberg leadership) is perfectly smooth but also at the upper level of collusion (coordination of quantities). The members of the cartel act as if they were the subsidiaries of a company that determines its joint profit maximizing quantity without worrying that individual subsidiaries may deviate from it (that is, produce a larger quantity).[7]

Our own quantity leadership model makes the same basic level assumption as the standard model (the cartel successfully appropriates the Stackelberg leadership), but generalizes the standard model with respect to the upper level of collusion. More specifically, the upper level of collusion allows for all possible degrees of *quantity coordination*, that is, from completely ineffective coordination of the cartel members' output decisions (the members act like perfect competitors) to perfect coordination of these decisions (the members act like a merged firm).

In other words, we start out by interpreting the notion of collusion in a very broad non-legal sense. Collusion is any joint effort of a group of firms to attain a leadership role and to push total output below the market volume, that is, below the output arising on a perfectly competitive market. This general approach is useful from a theoretical perspective. However, in the U.S. antitrust policy and the EU competition policy, such broadly defined collusion is not necessarily illegal. Only explicit (or formal) collusion is prosecuted, while tacit collusion is not (e.g., Martin, 2006, p. 1300). Thus, when our model considers the antitrust policy of prosecution and punishment (see Section 4.3), it is important to distinguish between explicit and tacit collusion. Our model will provide a natural dividing line (see p. 8).

Perfect quantity coordination is rarely feasible. However, antitrust proceedings reveal that cartels are impressively innovative in establishing mechanisms that ensure at least imperfect coordination. The achieved degree of coordination varies widely. It depends not only on aspects such as the antitrust authority's diligence and resources, the nature of the judicial system, the type of product, the number of firms, the market size, the cost function, the degree of product differentiation, or the potential for regional separation, but also on the design of the cartel agreement (market strategy, trust building, surveillance machanism, and system of internal sanctions) as well as behavioral fundamentals of the members of the cartel.[8]

To accomodate in our model the broadest possible spectrum of institutional arrangements and behavioral assumptions, we modify a game-theoretic concept that Cyert and DeGroot (1973, p. 25) coined as the "coefficient of cooperation"-

[7] The case of a perfectly colluding cartel competing against a Cournot fringe is explored in the quantity leadership models of Martin (1990, pp. 9-12, 1993, pp. 98-100), Shaffer (1995, pp. 744-749) as well as Auer and Pham (2021). Note that a perfect quantity coordination of a cartel is not sufficient to establish a monopoly. The latter also requires that the cartel is complete, that is, no fringe firms exist.

[8] For example, Matsumura and Matsushima (2012) and Matsumura *et al.* (2013) point out that managers often do not only care about their own firm's profit (the Cournot-Nash case), but also about their firm's profit relative to the profit of the competing firms. The perfect competition outcome arises when each firm cares only about its relative profit.

approach.[9] In our modified approach, we define the *coefficient of coordination* $\lambda \in [0,1]$. Instead of its own expected profit, $E(\pi_K)$, each cartel member maximizes the following *compound profit*:[10]

$$\hat{\pi}_K = E(\pi_K) + (k\lambda - 1)E(\bar{\pi}_{-K}) \ . \tag{5}$$

In this objective function, the expected average profit's weight, $(k\lambda - 1)$, is a strictly positive monotonic transformation of the coefficient of coordination, λ. In a completely dysfunctional cartel, the coefficient of coordination is $\lambda = 0$ and the weight becomes (-1): $\hat{\pi}_K = E(\pi_K) - E(\bar{\pi}_{-K})$. Each cartel member maximizes the difference between its own expected profit and the expected average profit of the other cartel members, that is, only relative performance matters. For $\lambda = 1/k$, the weight is 0: $\hat{\pi}_K = E(\pi_K)$. Each cartel member maximizes only its own expected profit. The performance of the other cartel members is irrelevant. For $\lambda = 1$, the weight is $(k-1)$: $\hat{\pi}_K = E(\pi_K) + (k-1)E(\bar{\pi}_{-K})$. Each cartel member maximizes the expected total cartel profit, that is, only the cartel's joint performance matters.

Inserting Eqs. (2), (3), and (4) into the cartel member's compound profit defined by Eq. (5) yields

$$\hat{\pi}_K = (1-p)\,\frac{1 - q_K - (k-1)\bar{q}_{-K}}{n-k+1}\,[q_K + (k\lambda - 1)\bar{q}_{-K}] - k\lambda f \ . \tag{6}$$

2.4 Profit Maximizing Sustainable Quantities

Each cartel member maximizes its compound profit, $\hat{\pi}_K$, and takes the average output of the other cartel members, $\bar{q}_{-K}$, as given.[11] Differentiating Eq. (6) with respect to q_K leads to the first-order condition

$$[k(1+\lambda) - 2]\,\bar{q}_{-K} + 2q_K = 1 \ .$$

Since all cartel members are identical, a symmetric solution arises. Substituting $\bar{q}_{-K}$ by q_K yields the sustainable total cartel output

$$Q_K = kq_K = \frac{1}{1+\lambda} \ . \tag{7}$$

It is independent from the number of cartel members, k, and the number of firms on the market, n. With this cartel output, no member of the cartel has an incentive to deviate from its own output decision. Eq. (7) reveals that $\partial Q_K / \partial \lambda < 0$.

Since n and k are given, each member of the cartel considers the risk of being prosecuted by the antitrust authority as an unavoidable cost. Thus, q_K is independent from the antitrust authority's rigour, $p = \Pr \cdot \phi$.

[9] Martin (1993, pp. 30-31) and Escrihuela-Villar (2015, pp. 476-479) demonstrate the close correspondence between the coefficient of cooperation and Bowley's (1924) concept of conjectural variation.

[10] Edgeworth (1881, p. 53, fn. 1) proposes a similar specification of the objective function.

[11] From a formal game-theoretic perspective, this behavior of the individual members of the cartel could be considered as "non-cooperative". Note, however, that for $\lambda > 1/k$ this "non-cooperative" behavior is quite "cooperative" because each member maximizes the compound profit, $\hat{\pi}_K$, instead of its own expected profit, $E(\pi_K)$. For $\lambda < 1/k$, the behavior takes on a "destructive" component.

Inserting Eq. (7) in Eq. (1) gives the sustainable total output of the fringe:

$$Q_F = (n-k)q_F = \frac{n-k}{n-k+1}\,\frac{\lambda}{1+\lambda}\ . \tag{8}$$

Therefore, $\partial Q_F/\partial\lambda > 0$ and $Q_F < Q_K$. When $\lambda = 0$, the output of the cartel is equal to the market volume ($Q = 1$) and the fringe produces no output.

Eqs. (7) and (8) define the sustainable equilibrium for given values of n and k. Thus, total (sustainable) output is

$$Q = Q_K + Q_F = \frac{n-k+1/(1+\lambda)}{n-k+1}\ . \tag{9}$$

Q increases with the number of fringe firms, $\partial Q/\partial(n-k) > 0$, and decreases when the level of coordination increases, $\partial Q/\partial\lambda < 0$. When n and k simultaneously change by the same number (e.g., a member of the cartel leaves the market), Q is not affected. The antitrust policy's rigour, p, has no direct effect on Q.

Eq. (9) is a very general expression of total output, because it covers the equilibrium output of all standard oligopoly models that use quantities as the strategic variable. For $k = n$ and $\lambda = 1/n$, the total output of the standard Cournot model arises: $Q = n/(n+1)$. For $k = 1$ (which we have ruled out) and $\lambda = 1$, Eq. (9) gives $Q = (2n-1)/(2n)$ which is the total output of the standard Stackelberg model. The monopoly output, $Q = 1/2$, arises for $k = n$ and $\lambda = 1$, while the market volume arising on perfectly competitive markets ($Q = 1$) is obtained for $\lambda = 0$.

In practice, it is easier to observe the market price P than the total quantity Q. If the total output arising on the cartelized market were equal to the Cournot output, $n/(n+1)$, the observable market price would be indistinguishable from the market price arising in Cournot competition. As a consequence, the courts are likely to consider the market as unsuspicious. Only if the market price were above the Cournot price, the courts would suspect collusion. Thus, in our theoretical model, we consider cartels as legal as long as the n firms' total output, Q, is not less than the total output arising in Cournot competition with n firms. For legal cartels, the probability of a successful conviction, Pr, would become 0, and so would the antitrust authority's rigour, $p = \Pr\cdot\phi$. This aspect is taken up in Section 4.3.

2.5 Sustainable Profits

Inserting Eq. (7) in Eq. (2), the mark-up simplifies to

$$P(Q) - c = \frac{\lambda}{(n-k+1)\,(1+\lambda)} \qquad (= 1-Q)\ . \tag{10}$$

It increases with the level of coordination, λ. The general design of the model allows for different means of coordination. These include mutual surveillance and penalties (as in the supergame literature) or regular communication, transparancy, and other trust building measures (as documented in many real world cartel cases).

Inserting $q_K = 1/[(1+\lambda)k]$ and Eq. (10) in Eq. (3) yields the sustainable expected profit of each member of the cartel:

$$E[\pi_K(k)] = \frac{\lambda}{(1+\lambda)^2}\frac{(1-p)}{(n-k+1)k} - f\ . \tag{11}$$

The notation $E[\pi_K(k)]$ emphasizes that the expected profit depends on the number of cartel members, k. Similarly, inserting $q_F = \lambda/[(1+\lambda)(n-k+1)]$ and Eq. (10) in $\pi_F(k) = (P-c)q_F - f$, gives the sustainable profit of each fringe firm:

$$\pi_F(k) = \left(\frac{\lambda}{1+\lambda}\right)^2 \frac{1}{(n-k+1)^2} - f \ . \qquad (12)$$

Both, $E[\pi_K(k)]$ and $\pi_F(k)$ are increasing with the coefficient of coordination, λ.

When $p = 1$ or $\lambda = 0$, the expected profits of the members of the cartel are negative. For given $p < 1$, an increase in the level of coordination, λ, reduces the cartel output and, therefore, increases the residual demand available for the fringe. As a consequence, not only the members of the cartel, but also the $(n-k)$ fringe firms benefit from an improved coordination of the cartel. The optimal situation for the cartel and the fringe firms is a perfectly colluding cartel ($\lambda = 1$).

The profit functions (11) and (12) directly yield the following findings:

Lemma 1 *The sustainable profit of a fringe firm exceeds the expected sustainable profit of a cartel member, if and only if* $k > (n+1)(1-p)/(1+\lambda-p)$.

Furthermore, we can derive the following result:

Lemma 2 *The profit function of a cartel member,* $E[\pi_K(k)]$*, is convex with the minimum profit at* $k = (n+1)/2$*. The profit function of a fringe firm,* $\pi_F(k)$*, is convex, too, but with the minimum profit at* $k = 0$.

*P*roof: See Appendix A.

The graphical implications of Lemmas 1 and 2 are depicted in Figure 1. It shows the $E[\pi_K(k)]$-curve, the $\pi_F(k)$-curve and their intersection. All other elements of Figure 1 will be explained shortly.

In the derivation of the sustainable total output defined in Eq. (9), the status of the firms was fixed during the previous stages of the game, that is, n and k were given. As a consequence, the solution was independent from the antitrust policy's rigour, p. Even for $p = 1$, a cartel member would keep its status and would make a loss of f; see the profit function (11). In the next section, each firm decides on its own status, that is, k becomes endogenous. This is the penultimate stage of the game.

3 Stability of Sustainable Cartels

Sustainability merely ensures that no cheating occurs within the cartel. It does not preclude situations in which a cartel member wants to exit the cartel (violation of internal stability; k decreases by one) or a fringe firm wants to enter the cartel (violation of external stability; k increases by one). In other words, the sustainable solution defined by Eq. (9) is not necessarily a stable solution. For given n, we derive the unique integer value k which ensures that the solution defined by Eq. (9) is not only sustainable but also stable.

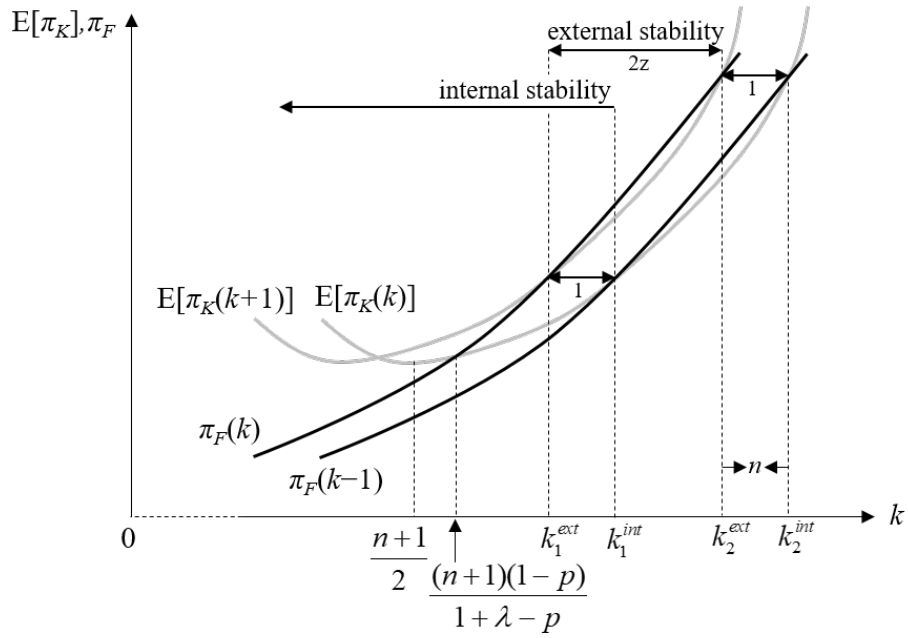

Figure 1: Profits As a Function of the Size of the Cartel.

3.1 Definition of Stability and Important Thresholds

Following Selten (1973) and d'Aspremont *et al.* (1983), stability of a cartel with k members requires that both the condition for internal stability,

$$E[\pi_K(k)] > \pi_F(k-1)\ , \tag{13}$$

and the condition for external stability,

$$E[\pi_K(k+1)] \leq \pi_F(k)\ , \tag{14}$$

are satisfied.[12] The concept of stability implicitly assumes that only one firm at a time can change its status. The only exception is the case of a cartel with two members. Since a cartel with a single member does not exist, internal stability of a cartel with two members requires that its expected profit is larger than the expected profit that arises when the cartel is "empty", that is, larger than the profit in the standard Cournot model: $E[\pi_K(2)] > 1/(n+1)^2 - f$.[13]

Let k_1^{ext} denote the smallest k-value at which $E[\pi_K(k+1)] \leq \pi_F(k)$. It defines the minimum k-value for external stability (see Figure 1). Similarly, let k_1^{int} denote the smallest k-value at which $E[\pi_K(k)] \leq \pi_F(k-1)$. All k-values smaller than k_1^{int} ensure that internal stability prevails. The formulas for the compilation of k_1^{ext} and k_1^{int} are derived in Section 3.2. Figure 1 reveals that $k_1^{\text{int}} = k_1^{\text{ext}} + 1$. Therefore, in

[12] In his formulation of stability, Selten (1973, pp. 179-181) denotes the members of the cartel as "participators" while the fringe firms are called the "non-participators". Note that in d'Aspremont *et al.* (1983) the internal stability condition (13) has a weak inequality sign, while the external stability condition (14) has a strict inequality sign. We prefer our own definition, because it implies that firms prefer the status of cartel membership only if the profit is strictly larger than that associated with the legal status of a fringe firm.

[13] Conversely, external stability of an "empty cartel" requires that $E[\pi_K(2)] \leq 1/(n+1)^2 - f$.

the interval $[k_1^{\mathrm{ext}}, k_1^{\mathrm{int}})$ only a single integer exists. We denote this integer by k^*. It is the *unique* integer value of k that satisfies both, the internal stability condition (13) and the external stability condition (14). In other words, k^* is the unique number of cartel members that ensures a sustainable and stable equilibrium solution.

3.2 Equilibrium Analysis

For $\lambda = 0$, all firms make a loss that is equal to their fixed cost f. In the following, we consider the case $\lambda > 0$. It is helpful to distinguish between the case of a complete cartel ($k = n$) and the case of an incomplete cartel ($n - k \geq 1$). When a complete cartel arises, the expected profit of each cartel member is $E[\pi_K(n)] = \lambda(1-p)/[n(1+\lambda)^2] - f$. For such a cartel, the external stability condition is redundant because there is no fringe firm that could enter the cartel. The internal stability condition of a complete cartel is $E[\pi_K(n)] > \pi_F(n-1)$. If one member leaves the complete cartel, this firm becomes the first fringe firm. The associated profit is $\pi_F(n-1) = \lambda^2/[4(1+\lambda)^2] - f$. Therefore, defining

$$\Omega = \frac{1-p}{\lambda} ,$$

we directly obtain the following result:

Theorem 1 *For*

$$n < 4\Omega =: n' , \tag{15}$$

a sustainable complete cartel is internally stable, that is, the equilibrium is a cartel with $k^* = n$ *members.*

When n is at least as large as n', an incomplete cartel arises. It is characterized by the following finding:[14]

Theorem 2 *For* $n \geq n'$, *the sustainable and stable equilibrium number of cartel members,* k^*, *is either* 0 *(empty cartel) or it is the unique integer satisfying the condition* $1 < k_1^{ext} \leq k^* < k_1{}^{int} \leq n$, *where* $k_1^{int} = k_1^{ext} + 1$ *and*

$$k_1^{ext} = \frac{n - 1 + 2\Omega\,(n+1)}{2\,(\Omega+1)} - z \tag{16}$$

with

$$z = \frac{\sqrt{(n+1)^2 - 4\Omega\,(n+2)}}{2\,(\Omega+1)} . \tag{17}$$

Proof: See Appendix B

[14] In Appendix B (proof of Theorem 2) it is shown that the largest k-value consistent with external stability is $k_2^{\mathrm{ext}} = k_1^{\mathrm{ext}} + 2z$ and that all k-values larger than $k_2^{\mathrm{int}} = k_1^{\mathrm{int}} + 2z$ satisfy the internal stability condition. However, it is also shown that, for $n \geq n'$, the integer in the interval $[k_2^{\mathrm{ext}}, k_2^{\mathrm{int}})$ is n (this case is depicted in Figure 1). Then, the threshold $k_2^{\mathrm{int}}(> n)$ is irrelevant for the formal analysis.

Corollary 1 *For $n \geq n'$, the first-order derivatives of k_1^{ext} with respect to λ, n, and p yield*

$$\frac{\partial k_1^{ext}}{\partial \lambda} < 0 \ ,q \quad \frac{\partial k_1^{ext}}{\partial n} > 0 \ , \quad \textit{and} \qquad \frac{\partial k_1^{ext}}{\partial p} < 0 \ .$$

Proof: See Appendix C.

The numerical implications of Theorems 1 and 2, and Corollary 1 are illustrated in Figure 2. It comprises two diagrams. Both correspond to $p = 0$, while the f-values differ. Both diagrams look like a big flight of stairs. The height of each step (measured from the bottom of the diagram) shows, for the given p-f-combination, the equilibrium number of fringe firms, $(n - k^*)$, corresponding to the respective values of λ and n. The dark area in front of the steps represents the λ-n-combinations leading to a complete cartel.[15]

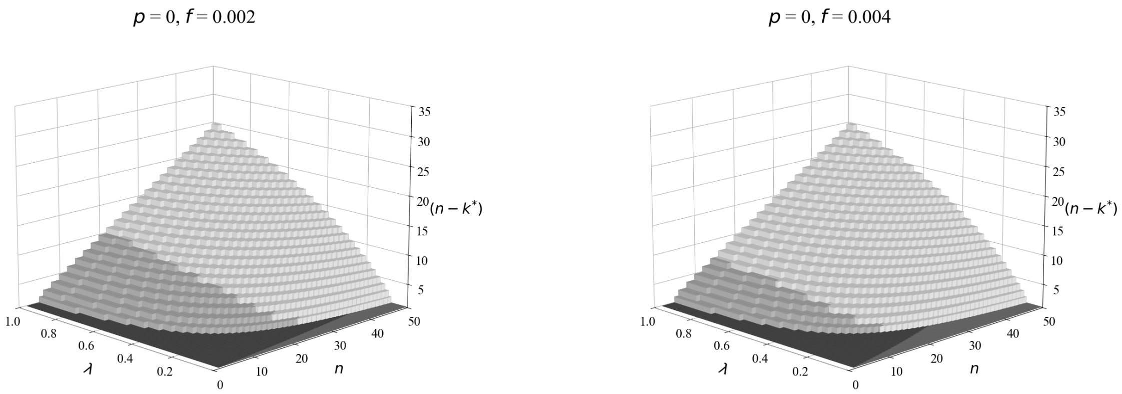

Figure 2: The Equilibrium Values $(n - k^*)$ As a Function of the Total Number of Firms, n, and the Degree of Quantity Coordination, λ, for $p = 0$ and $f = 0.002$ as well as $f = 0.004$.

For given n, the equilibrium size of the fringe *increases* with the cartel's degree of quantity coordination, λ. It is worthwhile to explain this somewhat counterintuitive relationship. A strengthened coordination, λ, increases the output of the fringe firms, q_F, but reduces the cartel's output, q_K, and total output, Q; see Eq. (9). As a consequence, the mark-up, $(P-c)$, increases. The increase of $(P-c)$ in conjunction with the increase of q_F and the reduction of q_K implies that, for given cartel size k, the increase of λ raises the profit of a fringe firm by more than the expected profit of a cartel member. If this difference is sufficiently large, the internal stability condition may no longer hold and one member of the cartel may want to leave the strengthened cartel, that is, k^* falls and $(n - k^*)$ increases. This reasoning confirms our earlier claim that important interdependencies between the sustainability issue and the stability issue exist.

3.3 An Alternative Interpretation

In some respects, our model resembles the model introduced by Lofaro (1999). However, his model includes neither an antitrust authority nor a fixed cost (or entry cost)

[15] The meaning of the different shades of gray will be described in Section 4.1.

and the number of operating firms, n, is exogenously given. In our model, n will be endogenized; see Section 4. A seemingly less relevant difference between the models is the weighting in Eq. (5). The weight in our model is $(k\lambda - 1)$, while the Lofaro model uses $\lambda(k-1)$. However, it turns out that our weighting yields a much smoother algebra. As a consequence, we could derive an explicit characterization of the sustainable and stable equilibrium.

With the difference in the weighting comes a difference in content. In our model, collusion is interpreted as any successful attempt to lower total output below the market volume. This notion covers tacit collusion as well as explicit collusion. In the Lofaro model, collusion requires that the degree of quantity coordination within the cartel exceeds $\lambda = 1/k$.

The key question of Lofaro's study is the following: taking into account that a larger degree of coordination, λ, increases the incentive to defect to the fringe, which level of coordination would maximize the profits of the cartel members? In our own model, the combination of Theorem 1, Lemma 2, and Corollary 1 provides the answer. The cartel would choose the maximum degree of coordination, λ, that still ensures a complete cartel. This degree of coordination is well below 1 and negatively related to the number of firms, n. The Lofaro model arrives at exactly the same answer.

However, our model even yields the precise relationship between the number of firms, n, and the degree of coordination chosen by these firms, λ. It is given by Eq. (15), with $p = 0$. In the diagrams of Figure 2, this relationship is depicted by the arcuate rim separating the bottom from the lowest stairs. Clearly, λ falls when n increases. Furthermore, for a given n, our model yields not only the profit maximizing value of λ, but also the corresponding output, Q. For example, when $n = 9$, the cartel's profit maximizing value of λ is 0.44 and the corresponding output is $Q = 0.694$.

In contrast to all existing quantity leadership models, we were able to derive a closed solution for the sustainable and stable equilibrium. As a consequence, we can proceed to study the consequences of market entries and exits. That is, n becomes endogenous. We consider this as an important step forward, because ensuring free market access is a viable antitrust policy of its own and previous models were unable to compare its effectiveness to that of surveillance, prosecution, and punishment of cartels.

4 Market Stability With Sustainable and Stable Cartels and Antitrust Policy

Section 2 was concerned with *cartel sustainability* and Section 3 enhanced the analysis by the concept of *cartel stability*. In both sections, the *antitrust policy* issue was captured by the parameters p and λ. The parameter p indicates the rigour of the implemented antitrust policy. The parameter λ reflects the cartel's degree of quantity coordination. The antitrust authority can influence this level. For example, it can introduce a stricter surveillance that complicates the cartel's coordination. However, the policy portfolio of antitrust authorities is not limited to a change of the parameters p and λ. Another important policy option is to ensure free market access (e.g., effective measures against predatory pricing and against firms that penalize clients that order from new entrants). If this policy succeeds, positive profits trigger market entries and negative profits cause market exits. In other words, the total number of firms, n, becomes endogenous raising the issue of *market stability.* A market is stable if no firm wants to enter or exit the market.

The entry decision is the first stage of our comprehensive oligopoly model. In its entry decision, each firm anticipates its status decision (the second stage, discussed in Section 3) and its profit maximizing output decision (the final stage, discussed in Section 2).

With the endogeneity of n comes a minor modification in notation. Henceforth, the expected profit of a cartel member is denoted by $E\left[\pi_K(n,k)\right]$ instead of $E\left[\pi_K(k)\right]$. The profits of a fringe firm are represented by $\pi_F(n,k)$ instead of $\pi_F(k)$.

As a starting point, we analyze a situation in which the antitrust authority successfully ensures free market access. We derive the long-run equilibrium (n^{**}, k^{**}) as a function of the three remaining parameters λ, p, and f (Section 4.1). Our analysis reveals that ensuring free market access can have a strong positive welfare effect. Furthermore, we investigate the welfare consequences of the other two antitrust policy options: the reduction of the degree of quantity coordination λ (Section 4.2) and the increase of the antitrust policy's rigour p (Section 4.3). Our analysis examines whether these additional measures further increase the level of welfare. Even though policy measures exist that possibly affect the barriers to entry as well as the parameters p and λ (e.g., a more generous leniency program or a more effective protection of employees that become whistle-blowers), we keep the analysis of these three policy options separate from each other. This allows us to identify the individual contributions of each option.

4.1 Antitrust Policy I: Free Market Access

When firms are free to enter or exit the market, we can derive the long-run equilibria (n^{**}, k^{**}) instead of the short-run equilibria (n, k^*) considered in Section 3. For a short-run equilibrium (n, k^*) to represent the long-run equilibrium (n^{**}, k^{**}), it must satisfy the following additional conditions:

$$E\left[\pi_K(n,k^*)\right] \geq 0 \qquad \text{and} \qquad \pi_F(n,k^*) \geq 0 \tag{18}$$

but

$$E\left[\pi_K(n+1,k^{*\prime})\right] < 0 \qquad \text{and/or} \qquad \pi_F(n^*+1,k^{*\prime}) < 0\ , \tag{19}$$

where $(n+1, k^{*\prime})$ is the short-run equilibrium corresponding to $(n+1)$ operating firms. Conditions (18) and (19) say that the short-run equilibrium (n, k^*) with the largest value of n ensuring non-negative profits for all n firms, represents the unique long-run equilibrium (n^{**}, k^{**}).

The implications are illustrated in Figure 2 (p. 12). One can see that for each given λ-value the height of the stairs is non-decreasing in n. For example, we know from Theorem 1 that for the parameter values $\lambda = 0.4$, $f = 0.002$, and $p = 0$ (left diagram of Figure 2), the maximum size of a complete cartel is $n = 9$. When n increases to 10, the entering firm does not want to join the cartel, but prefers to become the first fringe firm. The expected profit of the resulting incomplete cartel is still positive. Therefore, further firms enter until $n = 22$ firms operate on the market. From Eq. (16) of Theorem 2 we know that the corresponding equilibrium size of the cartel is $k^* = 17$, while the remaining five firms form the fringe. From the profit function (3) it can be seen that the cartel's expected profit is still positive. If an additional firm entered the market, the corresponding equilibrium size of the cartel would increase to $k^* = 18$. Inserting $n = 23$ and $k^* = 18$ in the profit function (3) reveals that the cartel's expected profit would become negative. In the left diagram of Figure 2, this transition from profitability to loss making is highlighted by the changeover to the lighter shade of gray.

In sum, a market with the parameter values $\lambda = 0.4$, $f = 0.002$, and $p = 0$ can support up to 22 firms, 17 of which form the cartel. Therefore, the short-run equilibrium $(22, 17)$ is the one that also represents the long-run equilibrium (n^{**}, k^{**}). It satisfies not only cartel sustainability and cartel stability, but also market stability. For each λ-p-f-combination a unique long-run equilibrium (n^{**}, k^{**}) exists. It is defined by the market stability conditions (18) and (19) in conjunction with Theorems 1 and 2.

What are the welfare consequences of the antitrust authority's free market access policy? We define welfare as the sum of consumer and producer rent. This sum is equal to $(Q-0.5Q^2)$, where the value of total output, Q, is defined by Eq. (9). Recall that the perfect competition output is 1. For Q-values smaller than 1, welfare and total output, Q, are positively correlated. Therefore, we can confine the welfare analysis to an analysis of total output, Q. From Eq. (9) we know that total output depends only on the degree of coordination, λ, and the long-run equilibrium number of fringe firms, $(n^{**}-k^{**})$. Therefore, for given λ, in Figure 2 a higher step represents a higher welfare.

Positive profits induce new firms to enter the market. However, the cartel output is $Q_K = \lambda/(1 + \lambda)$, regardless of the number of cartel members. Thus, only those new firms that join the fringe increase total output and, therefore, welfare.

Figure 2 illustrates the welfare effect of the free market access policy. Returning to the example with $\lambda = 0.4$, $f = 0.002$, and $p = 0$ (left diagram in Figure 2), the short-run equilibrium relating to $n = 9$ is a complete cartel: $(n, k^*) = (9, 9)$. Eq. (9) implies that the associated total output is $Q = 0.714$. Free market access induces thirteen firms to enter the market, five of which become fringe firms: $(n^{**}, k^{**}) = (22, 17)$. Only the fringe firms affect total output. More specifically, total output increases to $Q = 0.952$. This increase represents a considerable welfare gain.

In many models, the fixed cost f is interpreted as a cost of market entry. The larger f, the larger the barriers to entry. Figure 2 illustrates the welfare consequences of a change of f. Comparing the two diagrams reveals that decreasing the entry cost, f, from 0.004 (right diagram) to 0.002 (left diagram) does not affect the flight of stairs, but shifts the borderline between profitable and unprofitable short-run equilibria upwards, that is, away from the origin ($\lambda = 0$ and $n = 0$). Thus, for each given λ-p-combination, the long-run equilibrium number of fringe firms and, therefore, output and welfare increase. This merely reinforces our previous result: a reduction of the barriers of entry (here the entry cost f) increases welfare.

4.2 Antitrust Policy II: Obstructing Collusion

The antitrust authority could complement its free market access policy by a more rigorous obstruction of collusive behavior. For example, it could improve its surveillance of the industry federation and its protection of whistle-blowers. Such measures are likely to reduce the cartel's degree of quantity coordination, λ.

We know from Eq. (9) that, for given n and k, a reduction of λ generates an increase in total output, Q. This is the direct effect of λ on Q. However, with free market access, there is also a less obvious indirect effect. A sufficiently strong reduction of λ also changes the long-run equilibrium (n^{**}, k^{**}). Somewhat paradoxically, the cartel's deteriorating collusion tends to *raise* cartel membership, k^{**}, (see Corollary 1) and to *increase* the number of operating firms, n^{**}. Since the k^{**}-increasing effect dominates the n^{**}-increasing effect, the number of fringe firms, $(n^{**} - k^{**})$, decreases and so does total output, Q.

Before discussing the logic behind this detrimental indirect effect, it is worthwhile to re-consider Figure 2. It illustrates the negative relationship between λ and n^{**}. For each given λ-value, the equilibrium value n^{**} is indicated by the (last) changeover between the darker and lighter shade of gray. This boundary point separates the short-run equilibria with positive cartel profits from those with negative cartel profits. The combination of all boundary points shows the relationship between λ and n^{**}. A decrease in the degree of quantity coordination, λ, leads to a moderate increase of n^{**} until a complete cartel is reached.

Why does a deteriorating coordination of an incomplete cartel raise k^{**} and, to a lesser extent, also n^{**}? The decline in λ reduces the profits of the fringe even more than the profits of the cartel; see the profit functions (11) and (12). Four different cases can arise.

Case 1 is the standard case. The reduction of λ causes a violation of the external stability condition and triggers a changeover of a fringe firm to the cartel. This reduces total output, Q, and, therefore, welfare. Figure 3 illustrates this effect. As in Figure 2, the parameter values are $f = 0.002$ and $p = 0$. The graph illustrates the impact of λ on the long-run equilibrium (n^{**}, k^{**}) and the associated output Q.

For $\lambda = 1$, the long-run equilibrium is $(n_1^{**}, k_1^{**}) = (21, 12)$ and total output is $Q = 0.9500$; see the right-hand side of Figure 3. When λ is gradually reduced and reaches 0.8547, external stability is violated and one of the nine fringe firms becomes a member of the cartel. Case 1 arises. The new long-run equilibrium is $(n_1^{**}, k_1^{**} + 1) = (21, 13)$. From Eq. (9) it follows that the changeover of the fringe firm reduces total output. In Figure 3, this fall in total output is shown by the kink at $\lambda = 0.8547$. When λ falls to 0.7213, case 1 arises again and the new long-run equilibrium is $(21, 14)$. Output is *lower* than with perfect quantity coordination $(\lambda = 1)$.

When λ reaches the value 0.6735, external stability is again violated and a fringe firm joins the cartel. However, the resulting increase in the fringe profits is so large that a new firm enters the market and joins the fringe. This is case 2. The new long-run equilibrium is $(22, 15)$. Since the number of fringe firms is the same as in the long-run equilibrium $(21, 14)$, output is not affected. Therefore, in Figure 3 no kink arises at $\lambda = 0.6735$.

The different welfare consequences of cases 1 and 2 reinforce our claim that cartel

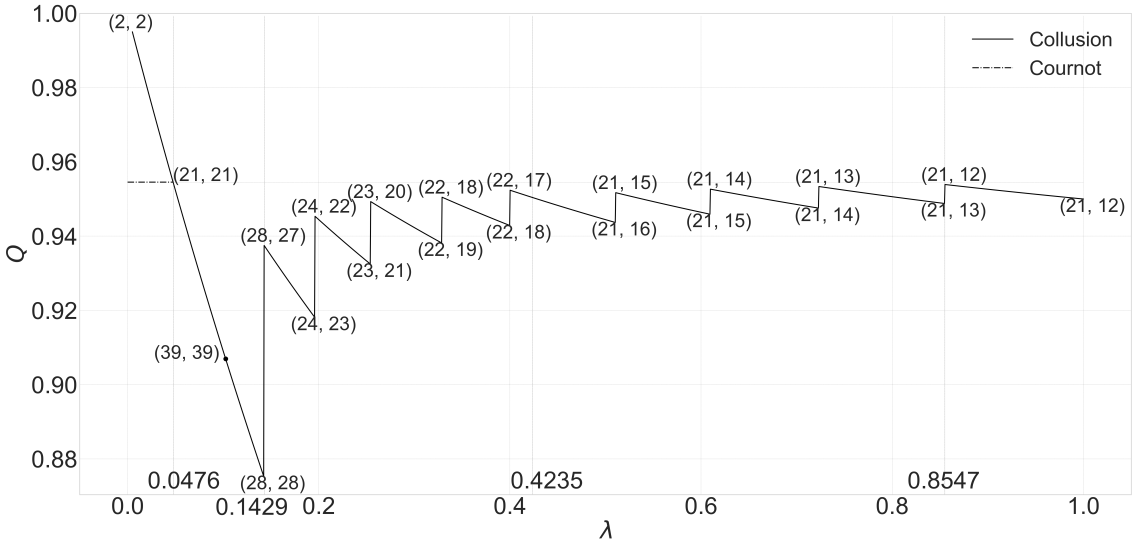

Figure 3: The Total Output, Q, and the Long-Run Equilibria, (n^{**}, k^{**}), Corresponding to Different Values of λ.

stability and market stability should be studied together. Additional support for this claim arises when λ falls to 0.6655. At this λ-level, external stability is not violated, but cartel profits become negative and one member of the cartel leaves the market. Thus, the profits of the remaining members of the cartel increase, while the profits of the fringe firms remain unchanged. Therefore, external stability must be examined. In the present situation ($\lambda = 0.6655$) it is still satisfied. Therefore, welfare remains unchanged. This is case 3. It reverses the effects of case 2 on n and k.[16]

The lowest output and, therefore, welfare arises when λ falls to 0.1429. This is the value at which the last fringe firm joins the cartel. At that moment, the cartel becomes a complete cartel with 28 members and the external stability condition becomes redundant. Since the profits are positive, a new firm may consider to enter the market and the cartel. However, it will abstain from an entry because it anticipates that this entry would lead to a violation of internal stability; see Eq. (15) of Theorem 1. One member of the cartel would become the first fringe firm and, as a consequence, the cartel profits would become negative.

Therefore, when $\lambda = 0.1429$, the internal stability condition (15) is the binding restriction. As λ falls further, this restriction is relaxed. New firms enter the market and join the complete cartel until it has 39 members. Note that the output of a complete cartel increases as λ falls and that this output is independent of the size of the cartel. Therefore, no kink arises to the left of $\lambda = 0.1429$ and reductions of λ reduce the aggregate cartel profit. New cartel members aggravate this effect.

[16]If the exit induced increase in cartel profits triggered a violation of external stability, the size of the cartel would return to its original level, while the number of fringe firms and, therefore, welfare, would fall. This would be case 4.

When λ reaches the value 0.0961, the binding restriction is no longer internal stability but profitability. Further reductions of λ make the existing cartel members unprofitable. Therefore, members are forced to leave the market.

When λ falls to $1/21 = 0.04762$, the size of the complete cartel falls to 21 members, that is, $\lambda = 1/k$. Therefore, the cartel's degree of quantity coordination is equivalent to the behavior of 21 firms that are in Cournot competition with each other (see Section 2.4). The associated output is 0.955.

Theorem 3 *If* (n^{**}, k^{**}) *and* Q' *is the long-run equilibrium corresponding to* $\lambda = 1$, *then* (n^{**}, n^{**}) *and* Q'' *is the long-run equilibrium corresponding to* $\lambda = 1/n^{**}$, *where*

$$1 < \frac{Q''}{Q'} < 1 + \frac{2}{(n+1)(n-2)} \,. \tag{20}$$

Proof: See Appendix D.

The implications of Theorem 3 are also illustrated in Figure 3. It reveals that the collusive long-run equilibrium corresponding to $\lambda = 1$, $f = 0.002$, and $p = 0$ is $(n^{**}, k^{**}) = (21, 12)$. Therefore, $\lambda = 1/21$ would result in 21 firms operating on the market. The associated output is only slightly larger than the output related to the market configuration with the perfectly colluding cartel ($\lambda = 1$) and free market access. Inequality (20) of Theorem 3 implies that this is a general result valid for all p-f-combinations. Figure 3 illustrates that intermediate values of λ generate smaller levels of total output than the case $\lambda = 1$. A comprehensive numerical analysis reveals that this is a general finding. Figure 3 shows the typical relationship between λ and Q. A reduction of λ leads to a zigzag pattern of declining output levels, Q, where the minimum output is reached when the cartel becomes a complete cartel. This output is considerably smaller than the output associated with a perfectly colluding cartel ($\lambda = 1$) competing against some fringe.

The welfare and policy implications of the preceding discussions (Sections 4.1 and 4.2) are rather obvious. The policy instrument of ensuring free market access is effective. Attempts to further increase welfare by obstructing collusion are largely futile.

4.3 Antitrust Policy III: Prosecution and Punishment

The third antitrust policy instrument is the prosecution and punishment of cartels. This can be captured by an increase of the antitrust policy's rigour, $p = \Pr \cdot \phi$. Formally, such an increase can be accomplished by a more severe punishment, ϕ, or by a higher probability of a successful conviction, Pr. Possible means of increasing Pr include more efficient auditing, an expanded leniency program, or changes in the judicial system.

However, the scope for increasing p is limited. One limitation (not considered in the present paper) is the budget of the antitrust authority. The other limitation is the definition of illegal collusion. When the n^{**} operating firms produce a total output that is not smaller than the Cournot output, $n^{**}/(n^{**}+1)$, courts are unlikely to consider the collusion as illegal. Thus, the probability of a successful conviction, Pr, approximates 0, and so does rigour, p.

To identify the long-run welfare consequences of an increase of p, it again suffices to study the effect of p on total output, Q. A formal analysis is rather tedious because various cases and subcases can arise. However, a numerical analysis is straightforward. It reveals a clear pattern.

When the fixed cost, f, is small, many firms operate on the market. A continuous increase in p may induce members of the cartel to exit the market but this does not lead to a continuous increase in the number of fringe firms ($n^{**} - k^{**}$). When an increase of p "generates" a new fringe firm, an additional increase of p usually reverts this increment because a fringe firm exits the market. In other words, the number of fringe firms fluctuates within a small range and so does total output, Q.

This is illustrated in Figure 4 which connects to our previous example. The level of coordination is $\lambda = 0.4$ and the fixed cost is $f = 0.002$. The rigour, p, ranges from 0 to 1. The number of fringe firms fluctuates between four and five. Accordingly, total output, Q, fluctuates between 0.94 and 0.96.

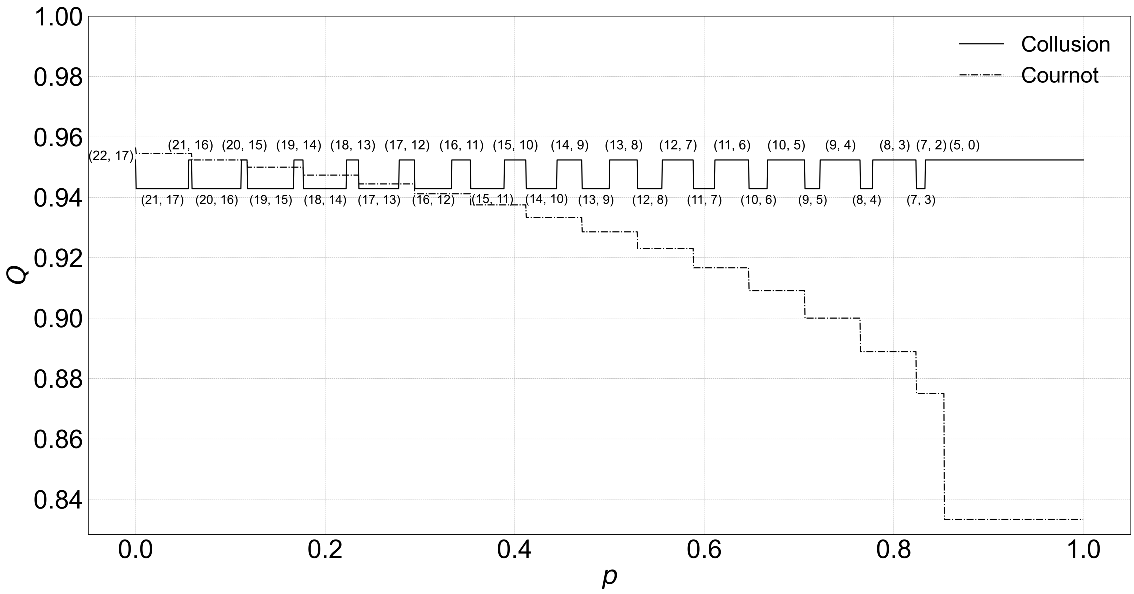

Figure 4: The Total Output, Q, and the Long-Run Equilibria, (n^{**}, k^{**}), Corresponding to Different Levels of Rigour, p.

The dotted line in Figure 4 indicates the output that would arise in Cournot competition with n^{**} firms. For most p-values, total output, Q, is above this dotted line and collusion and the courts would consider the cartel's collusion as legal. Thus, these p-values are not feasible.

Figure 4 illustrates the results related to the fixed cost $f = 0.002$. When the fixed cost is considerably higher (e.g., $f = 0.05$), an increase in p can cause a perceptible increase in Q. However, these events are of no practical relevance because they occur only in the range of infeasible p-values.

In sum, also the effectiveness of the third antitrust policy instrument (prosecution and punishment) is limited. It certainly does not reach the effectiveness of the first instrument (ensuring free market access).

5 Concluding Remarks

In the present paper, we developed a general oligopoly model of imperfect collusion. All standard oligopoly models are special cases of this general model. We derived the equilibrium number of firms operating on the market, the associated size of the cartel, and the associated total output. The model addresses the interplay between the four core issues of cartel theory: antitrust policy, cartel sustainability, cartel stability, and market stability.

For example, an antitrust policy of obstructing collusion does not lead to the desired goal. Although the deteriorated collusion leads to a higher cartel output, it harms the profit of the fringe firms. As a consequence, some of the fringe firms either join the cartel or leave the market. The associated reduction in fringe output overcompensates the increase in cartel output. Therefore, a lower level of collusion results in lower total output and, therefore, welfare.

An increase of the antitrust policy's rigour causes only minor changes in total output. A larger increase can arise only for high levels of rigour. However, these higher levels are unattainable because the associated total output is larger than Cournot output and the cartel's collusion could not be classified as illegal.

According to our model, the most effective antitrust policy is the removal of entry and exit barriers. This recommendation echoes a key result of the theory of contestable markets. However, that theory has been criticized for its assumption that incumbent firms can only react to market entries with delay. Our own model reveals that this controversial assumption is not necessary to emphasize the relevance of free market access for a successful competition policy.

Appendix A

Proof of Lemma 2: The first- and second-order derivatives of $E[\pi_K(k)]$ with respect to k are

$$\frac{\partial E[\pi_K(k)]}{\partial k} = \frac{\lambda(1-p)}{(1+\lambda)^2}\,\frac{2k-(n+1)}{(n-k+1)^2\,k^2} \tag{21}$$

$$\frac{\partial^2 E[\pi_K(k)]}{\partial k^2} = \frac{\lambda(1-p)}{(1+\lambda)^2}\,\frac{2(n^2+2n-3nk+3k^2-3k+1)}{(n-k+1)^3\,k^3}\,. \tag{22}$$

The denominators of the second quotients on the right-hand side are positive. The numerator of the second quotient on the right hand side of (22) can be expressed in the form

$$2\left(n-\frac{3k}{2}\right)^2+\frac{3}{2}\,(k-2)^2+4(n-1)>0\,.$$

Therefore, $\partial^2 E[\pi_K(k)]/\partial k^2 > 0$. At $k=(n+1)/2$, the numerator of (21) is equal to 0. Therefore, $\partial E[\pi_K(k)]/\partial k > 0$, for all $k > (n+1)/2$. The first- and second-order derivatives of $\pi_F(k)$ with respect to k are

$$\frac{\partial \pi_F(k)}{\partial k} = \left(\frac{\lambda}{1+\lambda}\right)^2 \frac{2}{(n-k+1)^3} > 0$$
$$\frac{\partial^2 \pi_F(k)}{\partial k^2} = \left(\frac{\lambda}{1+\lambda}\right)^2 \frac{6}{(n-k+1)^4} > 0 \; .$$

q.e.d.

Appendix B

Proof of Theorem 2: External stability requires that $E[\pi_K(k+1)] \leq \pi_F(k)$. Using the profit functions (12) and (11), this condition yields

$$\frac{(n-k)(k+1)}{(n-k+1)^2} \geq \Omega \; .$$

Simplifying this inequality yields the following convex quadratic function:

$$(\Omega+1)\,k^2 - (\Omega 2\,(n+1) - 1 + n)\,k + \Omega\,(n+1)^2 - n \leq 0 \; .$$

Setting the left hand side equal to 0, gives the two solutions

$$k_1^{ext} = \frac{n-1+2\Omega\,(n+1)}{2\,(\Omega+1)} - z \qquad \text{and} \tag{23}$$
$$k_2^{ext} = \frac{n-1+2\Omega\,(n+1)}{2\,(\Omega+1)} + z \; , \tag{24}$$

with z being defined by (17):

$$z = \frac{\sqrt{(n+1)^2 - 4\Omega\,(n+2)}}{2\,(\Omega+1)} \; .$$

The numerator of z is defined, if and only if

$$n^2 + 2n + 1 \geq 4\Omega n + 8\Omega \; . \tag{25}$$

Since $n \geq n' = 4\Omega$ (see Theorem 1), we have $n^2 > 4\Omega n$ und $2n > 8\Omega$. As a consequence, for all $n \geq n'$ the numerator of z is defined and external stability holds for all $k \in [k_1^{ext}, k_2^{ext}]$. Expressions (23) and (24) imply that $k_2^{ext} = k_1^{ext} + 2z$.

For $k_1^{ext} \leq 1$, the smallest integer k^* satisfying $k_1^{ext} \leq k^*$ is 1. However, a cartel with only one member is not defined. Thus, by definition, $k_1^{ext} \leq 1$ yields $k^* = 0$.

Next, we show that $k_2^{ext} \leq n$. Using (17) and (24), this inequality can be rearranged to

$$\sqrt{(n+1)^2 - 4\Omega\,(n+2)} \leq n - 2\Omega + 1 \; . \tag{26}$$

Since we consider the case $n \geq n' = 4\Omega$, both sides of (26) are defined and positive. Taking squares on both sides of (26) and simplifying yields

$$0 \leq 4\Omega\,(\Omega+1)$$

which is always satisfied.

Internal stability requires that $E[\pi_K(k)] > \pi_F(k-1)$. Using the profit functions (12) and (11), this condition yields

$$\frac{(n-k+1)\,k}{(n-k+2)^2} < \Omega\ . \tag{27}$$

For $p = 1$, we get $\Omega = 0$ and internal stability is always violated. Simplifying inequality (27) gives the following convex quadratic function:

$$(\Omega+1)\,k^2 - (1+4\Omega+n+2n\Omega)\,k + \Omega\left(4n+n^2+4\right) > 0\ .$$

Setting the left hand side equal to 0, gives the two solutions

$$k_1^{int} = \frac{n+1+2\Omega\,(n+2)}{2\,(\Omega+1)} - z \tag{28}$$

$$k_2^{int} = \frac{n+1+2\Omega\,(n+2)}{2\,(\Omega+1)} + z\ , \tag{29}$$

where $k_2^{int} = k_1^{int} + 2z$.

Next, we show that $k_2^{int} > n$ which implies that the threshold k_2^{int} is irrelevant for any stability considerations and that internal stability holds for all $k < k_1^{int}$. Rearranging the inequality $k_2^{int} > n$ gives:

$$n - 4\Omega - 1 < \sqrt{(n+1)^2 - 4\Omega\,(n+2)}\ . \tag{30}$$

For $n \geq n' = 4\Omega$, the right-hand side of (30) is defined and positive. If also $n < 4\Omega + 1$, (30) is satisfied, because its left-hand side becomes negative. If $n \geq 4\Omega+1$, squaring both sides of (30) and simplifying yields $-4\,(\Omega+1)\,(n-4\Omega) < 0$, which is true.

Finally, we show that $k_1^{int} \leq n$. This inequality can be rearranged to

$$\sqrt{(n+1)^2 - 4\Omega\,(n+2)} \geq -n + 4\Omega + 1\ . \tag{31}$$

For $n \geq n' = 4\Omega$, the left-hand side of (31) is defined and positive. If also $n < 4\Omega+1$, the right-hand side of (31) is negative and the inequality is satisfied. If $n \geq 4\Omega + 1$, squaring both sides of (31) and simplifying yields $(n-4\Omega)\,(n+3) \geq 0$, which is true.

In sum, for $n \geq n'$, stability arises for the unique integer k^* in the interval $[k_1^{ext}, k_1^{int})$. q.e.d.

Appendix C

Proof of Corollary 1: The derivative of k_1^{int} with respect to λ is

$$\frac{\partial k_1^{int}}{\partial \lambda} = \frac{A+B}{2(1-p+\lambda)^2\sqrt{(n+1)^2\lambda^2 - 4(n+2)\lambda(1-p)}}\ .$$

where

$$\begin{aligned}
A &= -(1-p)(n+3)\sqrt{(n+1)^2\lambda^2 - 4(n+2)\lambda(1-p)} \qquad (<0) \\
B &= -(1-p+\lambda)[\lambda(n+1)^2 - 2(1-p)(n+2)] \qquad (<0 \quad \forall n > n'')
\end{aligned}$$

The denominator is positive and the numerator is negative. Therefore, $\partial k_1^{\text{int}}/\partial\lambda < 0$ is negative.

The derivative of k_1^{int} with respect to n is

$$\frac{\partial k_1^{int}}{\partial n} = \frac{(2(1-p)+\lambda)\sqrt{(n+1)^2\lambda^2 - 4(n+2)\lambda(1-p)} - (n+1)\lambda^2 + 2\lambda(1-p)}{2(1-p+\lambda)\sqrt{(n+1)^2\lambda^2 - 4(1-p)(n+2)\lambda}} \; .$$

The denominator is positive. Defining

$$\begin{aligned}
X &= (2(1-p)+\lambda)\sqrt{(n+1)^2\lambda^2 - 4(n+2)\lambda(1-p)} \qquad (>0) \\
Y &= (n+1)\lambda^2 - 2\lambda(1-p) \qquad (>0 \quad \forall n > n'') \; ,
\end{aligned}$$

the numerator can be expressed as

$$X - Y = \frac{(X-Y)\,(X+Y)}{X+Y} = \frac{X^2 - Y^2}{X+Y} \; .$$

The denominator of this quotient is positive. The numerator gives

$$\begin{aligned}
X^2 - Y^2 &= (2(1-p)+\lambda)^2\left[(n+1)^2\lambda^2 - 4(n+2)\lambda(1-p)\right] - \left[(n+1)\lambda^2 - 2\lambda(1-p)\right]^2 \\
&= 4(1-p)(1-p+\lambda)\left[(n+1)^2\lambda^2 - 4(n+2)(1-p)\lambda - \lambda^2\right] \\
&= 4\lambda(1-p)(1-p+\lambda)(n+2)[(n+1)\lambda - 4(1-p) - \lambda] \; .
\end{aligned}$$

The expression in square brackets is positive if and only if $n > 4(1-p)/\lambda (= n')$. Thus, for $n \geq n'$, we get $\partial k_1^{int}/\partial n > 0$.

The derivative of k_1^{int} with respect to p is

$$\frac{\partial k_1^{int}}{\partial p} = \frac{-\lambda(n+3)}{2(1-p+\lambda)^2} - \frac{2\lambda^2(n+2) + [(n+1)^2\lambda^2 - 2\lambda(n+2)(1-p)]}{2(1-p+\lambda)^2\sqrt{(n+1)^2\lambda^2 - 4\lambda(n+2)(1-p)]}} \; .$$

Both, the numerator and the denominator of the second fraction are positive. Therefore, $\partial k_1^{int}/\partial p < 0$.

Since $k_1^{ext} = k_1^{int} - 1$, we get

$$\frac{\partial k_1^{ext}}{\partial\lambda} = \frac{\partial k_1^{int}}{\partial\lambda} < 0 \; , \qquad \frac{\partial k_1^{ext}}{\partial n} = \frac{\partial k_1^{int}}{\partial n} > 0 \; , \quad \text{and} \quad \frac{\partial k_1^{ext}}{\partial p} = \frac{\partial k_1^{int}}{\partial p} < 0 \; .$$

q.e.d.

Appendix D

Proof of Theorem 3: When the n firms of a complete cartel are in Cournot competition with $\lambda = 1/n$, their individual expected profits are $E[\pi_K(n,n)] =$

$(1-p)/(n+1)^2 - f$; see profit function (11). Let n_C denote the *unique* integer n satisfying the two conditions

$$\text{A:} \quad \frac{1-p}{(n_C+1)^2} - f \geq 0 \qquad \text{and} \qquad \text{B:} \quad \frac{1-p}{(n_C+2)^2} - f < 0\ .$$

Thus, n_C is the maximum integer n that ensures that the complete cartel with $\lambda = 1/n$ earns a non-negative expected profit.

When $k(\leq n)$ firms form a perfectly colluding cartel ($\lambda = 1$) that acts as Stackelberg leader and competes against $(n-k)$ Stackelberg followers, the expected profit of each member of the cartel is $E[\pi_K(n,k)] = (1-p)/[4(n-k+1)k] - f$; see profit function (11). Let (n^{**}, k^{**}) denote the corresponding long-run equilibrium. If a new firm entered the market and joined the cartel, output and price would not be affected, but expected profits would become negative (otherwise, the firm would have joined the cartel before). Then, n^{**} is the *unique* integer satisfying the following two conditions:

$$\text{C:} \quad \frac{1-p}{4(n^{**}-k^{**}+1)k^{**}} - f \geq 0 \quad \text{and} \quad \text{D:} \quad \frac{1-p}{4(n^{**}-k^{**}+1)(k^{**}+1)} - f < 0\ .$$

We have to prove that for given p and f the unique n_C-integer satisfying conditions A and B is always equal to the unique n^{**}-integer satisfying conditions C and D, and *vice versa*.

Because

$$(n+1)^2 - 4(n-k+1)k = (n+1-2k)^2 \geq 0\ ,$$

the denominator in A is at least as large as the denominator in C (when $n_C = n^{**}$). Therefore, the left hand-side of A is smaller than the left-hand side of C. Thus, all n-values that satisfy A also satisfy C (A $\Rightarrow$ C). Similarly,

$$(n+2)^2 - 4(n-k+1)(k+1) = (n-2k)^2 \geq 0\ .$$

Therefore, the left-hand side of B is always smaller than the left hand-side of D and all n-values that satisfy D also satisfy B (D $\Rightarrow$ B). Since n_C is the unique integer satisfying conditions A and B, and n^{**} is the unique integer satisfying conditions C and D, simultaneous satisfaction of the logical relationships A $\Rightarrow$ C and D $\Rightarrow$ B requires that $n_C = n^{**}$.

The total output corresponding to $(n_C, n_C) = (n^{**}, n^{**})$ and $\lambda = 1/n^{**}$ is $Q'' = 1 - 1/(n^{**}+1)$ while the total output corresponding to (n^{**}, k^{**}) and $\lambda = 1$ is $Q' = 1 - 1/[2(n^{**}-k^{**}+1)]$. The ratio of the two output levels is

$$\frac{Q''}{Q'} = 1 + \frac{2k-n-1}{(n+1)[n-(2k-n-1)]}\ . \tag{32}$$

Lemma 2 implies that $2k-n-1 > 0$. From Proposition 2 of Shaffer (1995, p. 746) we know that, for $\lambda = 1$ and $p = 0$, internal stability requires that $2k-n-1 < 2$. Our Corollary 1 implies that, for given n, the value of k_1^{int} falls as p increases. Therefore, the quotient on the right-hand side of Equation (32) is always positive, but smaller than $2/[(n+1)(n-2)]$. q.e.d.

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