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Multilateral index approaches in the presence of product-specific price trends and data gaps*

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Abstract

The prices of some products respond more strongly to changes in the general price level than others. This study explains why such product-specific price level elasticities lead to biased time-product-dummy (TPD) estimates of price levels. Other popular multilateral index approaches such as the Gini-Éltető-Köves-Szulc (GEKS) and Geary-Khamis (GK) methods also fail to address this source of bias in inflation measurement. Therefore, this paper introduces the NLTPD regression – a nonlinear generalization of the TPD regression. By estimating product-specific price-level elasticities, the NLTPD regression effectively addresses this source of bias. A simulation study and an application to real-world scanner data compare the performance of the four multilateral index approaches. Except for the rather theoretical case of complete data, the NLTPD regression outperforms the other three approaches.

Keywords: inflation · measurement bias · price index · scanner data

JEL classification: C43 · E31 · E52

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1 Introduction

If all prices in an economy changed proportionally at all times, measuring inflation would be a straightforward task. In reality, however, different products follow different price trajectories. As a result, statistical offices monitor the prices of a representative sample of products. Given limited resources for price collection and processing, statistical offices must prioritize the selection of products, outlets, and regions to be surveyed. This prioritization can lead to the exclusion of relevant products from the sample and to missing observations even for the included products. Such data gaps may occur, for instance, when a price collector is unable to locate a sampled product in a visited store. Non-response is another common source of missing data, particularly in survey-based indices such as rental price indexes. Electronic datasets may also exhibit such data gaps for a variety of reasons. For example, filtering methods can compromise the representativeness of the retained observations.[1] Furthermore, data providers often remove from the transmitted scanner dataset products that were available at a known price but were not transacted.

These examples show that missing observations are common in datasets used for inflation measurement. In this study, missing observations refer to existing price and quantity information that is absent from the dataset. If no price or quantity information exists because a product was not yet available or was no longer available on the market, this situation is not considered a missing observation.

Most bilateral price indices can be understood as a weighted average of the price changes of individual products. If, in this calculation, the products with falling prices were predominantly missing, the bilateral price index would be biased upward. Conversely, a downward bias would occur if the missing products were primarily those that experienced particularly strong price increases. This issue could be mitigated – or even avoided – if the current price trajectory of the missing products were known. In that case, the direction of the bias could be anticipated and, to some extent, corrected.

Therefore, understanding the price trends of the missing products is crucial. Addressing this challenge requires a shift from a two-period to a multi-period perspective. In other words, data from additional time periods must be incorporated and properly accounted for in the index calculation. This, in turn, necessitates the use of multilateral index approaches.

Most of these methods were originally developed for spatial price comparisons and later adapted for intertemporal price measurement, where they are typically used to tackle "chain drift bias". This bias frequently arises when bilateral price indices based on monthly (or even weekly) data are chained (e.g., Kokoski et al., 1999, p. 141). Among the most popular multilateral index approaches are the time-product-dummy (TPD) regression, the Gini-Éltető-Köves-Szulc (GEKS) method, and the Geary-Khamis (GK) method.[2] In a comprehensive simulation, Auer (2025, pp. 12–17) confirms that these approaches effectively address the chain drift problem. However, the present study shows that these same methods are inadequate for resolving the issue of products with heterogeneous price trends and patterns of missing observations.

The paper's contributions begin with a critical analysis of the TPD regression approach. Like other multilateral index approaches, the TPD regression aggregates the individual

[1] ONS (2023) presents an analysis of its outlier detection procedures for grocery scanner data.

[2] Recent surveys of the various methods include Chessa et al. (2017), Chessa (2019), de Haan and Krsinich (2014), or Diewert and Fox (2022).

price trends of a group of products into a general price level trend. Unlike the other methods, the TPD regression permits statistical inference. This advantage, however, relies on an implicit assumption: for each product and each period, the price-level elasticity of product prices – that is, the elasticity of a product's price with respect to the general price level – is equal to one. In reality, some products display greater price-level elasticity than others, leading to substantial variation in price trends across products. The first contribution of this study is to show that product-specific price-level elasticities undermine the validity of statistical inference.

The paper's second contribution is related to the issue of data gaps. It shows that data gaps exacerbate the problems of the TPD regression. More specifically, product-specific price-level elasticities in conjunction with missing observations usually lead to biased estimates of the price levels. In other words, the TPD regression is unable to correct for the bias introduced by data gaps. Only when products with high and low price-level elasticities are equally likely to be missing do the TPD estimates remain unbiased, albeit inefficient.

As its third contribution, the present paper examines whether the GEKS and GK methods are more immune to these issues. It demonstrates that they are not.

Thus, an alternative approach is required. To this end, this paper introduces the NLTPD regression, a nonlinear generalization of the TPD regression. An analogous idea was originally proposed and developed in the context of interregional price comparisons (Auer and Weinand, 2022, 2025) and is here extended to the intertemporal setting. This extension significantly enhances the regression's applicability and constitutes the fourth contribution of this study.

Unlike the TPD regression, the NLTPD regression provides an estimate of the price-level elasticity for each product. Consequently, the NLTPD regression remains applicable even in the presence of patterns of missing observations and product-specific price-level elasticities.

The final contribution of this paper consists of a simulation study and an empirical application using scanner data. The simulation captures key features of real-world purchasing behavior, such as stockpiling (triggered by sales) and consumption smoothing (driven by adjustment costs), and also incorporates patterns of missing observations and product-specific price-level elasticities. The scanner data similarly exhibit substantial data gaps. Both the simulation and the empirical application compare the performance of the NLTPD regression, the TPD regression, the GEKS method, and the GK method.

The remainder of the paper is organized as follows. Section 2 introduces the TPD regression, the GEKS method, and the GK method using a simple illustrative example. Building on the same example, Section 3 highlights the failure of the TPD regression and explains why the GEKS and GK methods encounter similar issues. Section 4 presents the NLTPD regression, which addresses these problems. Section 5 develops a comprehensive simulation study comparing the various approaches. An application to scanner data is given in Section 6. Section 7 concludes.

2 Multilateral index approaches

Table 1 shows the prices, p_i^t, and quantities, q_i^t, of three products (i = A, B, C) of some elementary aggregate (or product group) in four consecutive periods ($t = 1, 2, 3, 4$). From this information, one may attempt to compute the price levels of periods 1 to 4. To this end, various multilateral index approaches are available.

Period	p_i^1	p_i^2	p_i^3	p_i^4	q_i^1	q_i^2	q_i^3	q_i^4
Product A:	*22*	22	21	*21*	*75*	70	78	*85*
Product B:	30	32	35	36	50	48	40	35
Product C:	49	50	52	55	25	25	20	18

Table 1: Prices and quantities of three products during four periods (the grey numbers in italics may be missing from the data set).

One possible approach is to apply the TPD regression. Its spatial analogue, the country-product-dummy (CPD) regression, was introduced by Summers (1973), who argued that it offers a simple and transparent regression framework capable of handling data gaps and enabling statistical inference. Balk (1980) adapted this approach for intertemporal price comparisons of seasonal products.

The TPD regression assumes that each price can be explained by the linear relationship

$$\ln p_i^t = \ln \pi_i + \ln P^t + u_i^t \,, \tag{1}$$

where P^t is the price level of period t, π_i is the general value of product i, and $u_i^t \sim N(0, \sigma^2)$ is a normally distributed error term with zero mean and variance σ^2. The TPD model (1) can be transformed into a regression equation with two sets of dummy variables that represent the set of the consecutive periods $\mathcal{T} = \{1, \ldots, T\}$ and the set of products $\mathcal{N} = \{1, \ldots, N\}$ (e.g., de Haan et al., 2021, p. 397). Weighted least squares yield estimates of the values of $\ln P^t$ and $\ln \pi_i$, where the weights are defined by the products' expenditure shares in period t,

$$w_i^t = \frac{p_i^t\, q_i^t}{\sum_{j \in \mathcal{N}^t} p_j^t\, q_j^t} \,, \tag{2}$$

where $\mathcal{N}^t$ denotes the set of products available in period t. The weights w_i^t sum to unity in each period. To avoid multicollinearity, the logarithmic price level estimates, $\widehat{\ln P^t}$, must be expressed relative to some index reference period r. The TPD index can then be derived from the logarithmic price level estimates as

$$P_{\text{TPD}}^{r,t} = 100 \cdot \exp\left(\widehat{\ln P^t}\right) \,. \tag{3}$$

In the example presented in Table 1, the TPD regression yields estimates of the four periods' logarithmic price levels. Taking anti-logs and multiplying by 100 yields the TPD price index depicted by the blue line in the left panel of Figure 1, where period 1 is the index reference period ($\widehat{\ln P^1} = 0$ and, therefore, $P_{\text{TPD}}^{1,1} = 100$).

A popular alternative to the TPD regression is the GEKS method. The acronym GEKS honours the publications of Gini (1924, 1931), Eltetö and Köves (1964), and Szulc (1964). They, too, developed this approach for interregional price comparisons. Balk (1981,

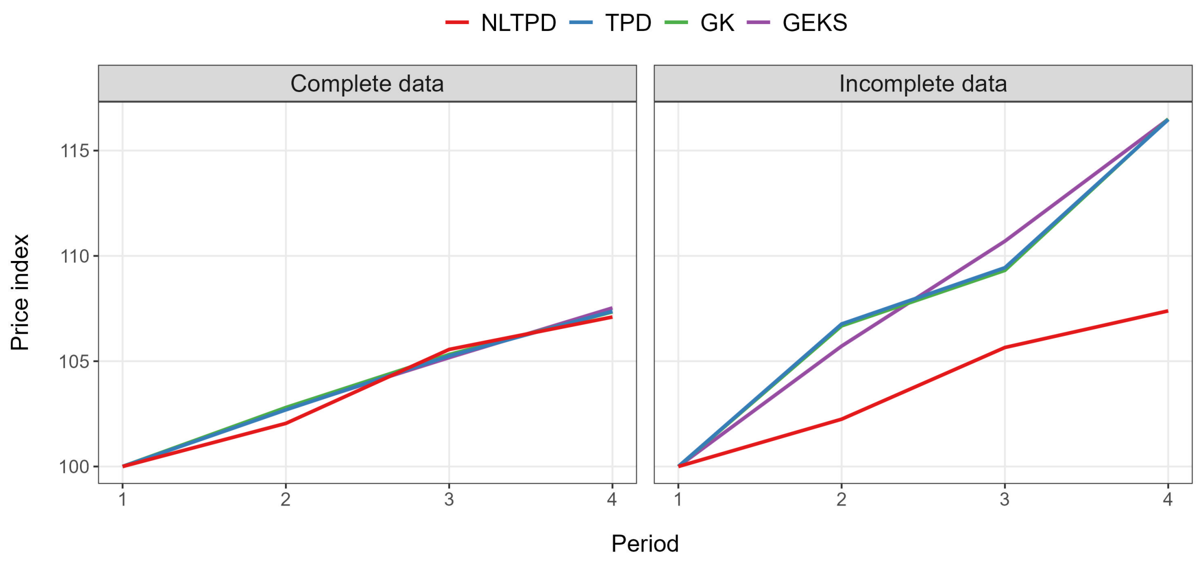

Figure 1: Price level estimates based on the complete price data from Table 1 (left panel) and the same data with two missing observations for Product A (right panel), using four different approaches: NLTPD, TPD, GK, and GEKS.

pp. 73-74) adapted this approach to an intertemporal context. Ivancic et al. (2011, p. 33) introduced a rolling-window variant of the GEKS method. Since then, many variants and refinements have been proposed (e.g., de Haan and Krsinich, 2014; Melser, 2018; Białek, 2023).

The central building block of the GEKS method is some bilateral index formula. Various options are available. The bilateral Törnqvist index for the price reference period s and the price comparison period t, $P_{\text{Tö}}^{s,t}$, is defined by

$$P_{\text{Tö}}^{s,t} = \prod_{i \in \mathcal{N}^{s,t}} \left(\frac{p_i^t}{p_i^s} \right)^{0.5\,\left(w_i^s + w_i^t\right)} , \tag{4}$$

where $\mathcal{N}^{s,t}$ denotes the set of products for which price observations are available in both periods s and t. The weights w_i^s and w_i^t are defined as in Eq. (2). The GEKS method computes the price change between two periods by the geometric average of all possible pairs of chained Törnqvist indices linking the two periods by some third period.[3] Thus, the GEKS index in period t relative to period r, $P_{\text{GEKS}}^{r,t}$, is computed by the formula

$$P_{\text{GEKS}}^{r,t} = 100 \cdot \prod_{s \in \mathcal{T}} \left(P_{\text{Tö}}^{r,s}\, P_{\text{Tö}}^{s,t} \right)^{1/T} . \tag{5}$$

Choosing period 1 as the index reference period ($P_{\text{GEKS}}^{1,1} = 100$), the GEKS method computes the values of $P_{\text{GEKS}}^{1,2}$, $P_{\text{GEKS}}^{1,3}$, and $P_{\text{GEKS}}^{1,4}$. For the data set in Table 1, these values are very similar to that of the TPD regression; see the purple line in the left panel of Figure 1.

Another popular multilateral index approach is the GK method. It was introduced by Geary (1958) and Khamis (1972) – again in an interregional price comparison context. The

[3] Some authors denote this method as CCDI method (Caves et al., 1982) because the original proposal used the bilateral Fisher index instead of the Törnqvist index.

GK method is based on the following two formulas:

$$\pi_i = \sum_{t \in \mathcal{T}_i} \frac{q_i^t}{\sum_{s \in \mathcal{T}_i} q_i^t} \left(p_i^t \Big/ P_{\text{GK}}^{r,t} \right) \quad \text{and} \quad P_{\text{GK}}^{r,t} = 100 \cdot \frac{\sum_{i \in \mathcal{N}^t} p_i^t q_i^t \Big/ \sum_{i \in \mathcal{N}^t} \pi_i q_i^t}{\sum_{i \in \mathcal{N}^r} p_i^r q_i^r \Big/ \sum_{i \in \mathcal{N}^r} \pi_i q_i^r}, \tag{6}$$

where $\mathcal{T}_i$ defines the set of periods in which product i is priced. The left equation estimates the general value of product i, while the right equation estimates the price level for period t relative to some period r. These two equations must be solved simultaneously. To this end, an iterative solution algorithm can be employed. The green line in the left panel of Figure 1 displays the GK price index for the data set in Table 1, with period 1 as the index reference period ($P_{\text{GK}}^{1,1} = 100$). Again, the results are very similar to those obtained from the TPD regression.

3 Bias caused by data gaps

Different products usually exhibit different price trends. The task of the price statistician is to derive from the individual price trends the average price development of the aggregated products. Table 1 listed the prices of three products (i = A, B, C) in four consecutive periods (t = 1, 2, 3, 4). The prices of Products B and C exhibit positive but unequal price-level elasticities, while Product A shows a moderately negative price-level elasticity. The price levels computed by the multilateral index approaches represent some average of the individual price trends of Products A, B, and C. Accordingly, the general price level increases quite in line with the price trend of Product C, though at a slower pace than Product B and a faster pace than Product A. This is true for all three multilateral index approaches (see left panel of Figure 1). However, how reliable are the price-level estimates obtained from the three approaches when observations are missing from the data?

The TPD regression implicitly assumes that the price-level elasticity of product prices is equal to one for all products. However, the prices in Table 1 violate this assumption. In addition, the TPD regression assumes that the conditional expected value of the error term is zero: $\mathrm{E}(u_i^t | \mathbf{x}_i^t) = 0$, where $\mathbf{x}_i^t$ represents the values of the two sets of dummy variables underlying the ordinary least squares regression. As long as the data set is complete, the latter assumption is satisfied – even when the products exhibit heterogeneous price-level elasticities. Thus, the TPD regression produces unbiased price level estimates.

However, a systematic pattern of missing observations leads to $\mathrm{E}(u_i^t | \mathbf{x}_i^t) \neq 0$, violating the TPD regression's basic assumption. In conjunction with heterogeneous price-level elasticities, this leads to biased price level estimates.

To see this, suppose that in Table 1 the price and quantity information printed in gray is missing. In other words, the product with the lowest price-level elasticity is missing in the first and last periods. Therefore, one gets $\mathrm{E}(u_i^1 | \mathbf{x}_i^1) < 0$ and $\mathrm{E}(u_i^4 | \mathbf{x}_i^4) > 0$, respectively. As a consequence, the TPD regression's estimated price level of the first period is below the level with complete data, while the TPD regression's estimated price level of the last period is above the level with complete data. Thus, the increase in the price levels is overestimated, that is, upward bias arises.[4]

The blue line in the right panel of Figure 1 shows the TPD results based on incomplete data, with Product A missing in Periods 1 and 4. As anticipated in the theoretical

[4] For a graphical exposition of these theoretical considerations, see Auer and Weinand (2025, Section 2).

discussion, the upward trend in price levels is overestimated relative to the situation with complete data.

There are two exceptional cases in which the TPD regression remains unbiased: when the dataset is complete (left panel of Figure 1) or when the pattern of missing observations is unrelated to the products' price-level elasticities. However, even in these two cases, the TPD regression is inefficient, and inference is invalid due to residuals that are both autocorrelated and heteroskedastic.

To see the autocorrelation and heteroskedasticity, return to Table 1. It shows that the general price level increases over time, while the prices of Product A exhibit a slight downward trend. As a result, the fitted TPD price path for Product A lies on one side of its observed price path at the beginning of the sample and on the other side toward the end of the sample. Hence, the residuals of Product A tend to have the same sign over consecutive periods before gradually changing sign. The opposite pattern arises for Product B, whose price trend is steeper than the general price trend. Its residuals also tend to change sign only gradually and therefore exhibit positive autocorrelation. Product C, by contrast, follows the general price trend more closely, so its residuals vary less systematically over time. Consequently, the TPD residuals are both autocorrelated and heteroskedastic.

The presence of autocorrelation and heteroskedasticity in the residuals renders the conventional covariance matrix of the TPD estimator inconsistent, thereby invalidating the usual t- and F-tests. In principle, this issue can be addressed using heteroskedasticity- and autocorrelation-consistent (HAC) covariance estimators. For example, Crompton (2000, p. 368) recommends White's heteroskedasticity-robust covariance estimator for the TPD regression. However, such estimators do not eliminate the bias in the estimated price levels that arises when heterogeneous price-level elasticities coincide with systematic patterns of missing observations.[5]

The GEKS method is not intended for valid inference, but does it avoid the bias of the TPD regression? According to the GEKS formula (5), the price change between periods 1 and 4 is the geometric mean of the four chain indices $P_{\text{Tö}}^{1,1}P_{\text{Tö}}^{1,4}$, $P_{\text{Tö}}^{1,2}P_{\text{Tö}}^{2,4}$, $P_{\text{Tö}}^{1,3}P_{\text{Tö}}^{3,4}$, and $P_{\text{Tö}}^{1,4}P_{\text{Tö}}^{4,4}$. With the complete data of Table 1, the stable prices of Product A dampen the price increases captured by the bilateral Törnqvist indices $P_{\text{Tö}}^{1,2}$, $P_{\text{Tö}}^{1,3}$, $P_{\text{Tö}}^{1,4}$, $P_{\text{Tö}}^{2,4}$, and $P_{\text{Tö}}^{3,4}$. When observations for Product A are missing in Periods 1 and 4, the five reported Törnqvist indices process solely the prices of Products B and C. As a result, the index values are higher than those calculated with complete data. Consequently, the geometric mean of the four chain indices – that is, the GEKS index – is also elevated relative to the complete-data case. In other words, an upward bias arises. Conversely, if Product B were missing instead of Product A, a downward bias would occur.

This intuition is confirmed by the numerical results derived from Table 1. These results are depicted by the purple line in the right panel of Figure 1. It is close to the blue line obtained from the TPD regression.

There are similar reservations about the GK method because it often approximates the numerical results of a TPD regression. The green line in the right panel of Figure 1, along

[5] Time-dummy hedonic regressions replace the product dummy variables of the TPD regression by variables that specify the individual characteristics of the products. Thus, the estimates of the general values, π_i, are replaced by estimates of the parameters relating to the products' characteristics. The estimates of the price levels, P^t, are not directly affected. Therefore, time-dummy hedonic regressions are also unlikely to be a solution.

with the simulation results in Section 5 and the empirical application in Section 6, support this conjecture. In Appendix A, the close formal relationship between the two approaches is demonstrated.

When the prices of Product A in Periods 1 and 4 are missing in Table 1, each of the multilateral index approaches discussed – TPD, GEKS, and GK – produces biased estimates of the price levels. This bias arises despite the fact that all three approaches incorporate the prices of Product A in Periods 2 and 3. However, they fail to identify the price-level elasticities of individual products, which are the underlying source of the bias. Therefore, this paper proposes an alternative multilateral index approach.

4 NLTPD regression

The multilateral NLTPD regression addresses the aforementioned bias by including the price-level elasticities of the individual products in the estimation. Section 4.1 introduces the NLTPD regression model and a necessary restriction on the price-level elasticities. Section 4.2 elaborates on the estimation of the NLTPD regression model. Section 4.3 compares the NLTPD price index to those of the other three multilateral index approaches for the stylized example in Table 1. Limitations and possible variations of the NLTPD method are discussed in Section 4.4. In Section 4.5, the NLTPD regression is related to seemingly similar proposals from the existing literature.

4.1 NLTPD model and its identification

The TPD regression implicitly assumes that all products respond identically to changes in the overall price level. Consequently, if price-level elasticities differ across products, the TPD regression model is misspecified. This issue could be addressed by allowing the elasticities to vary in the estimation. Thus, the regression model

$$\ln p_i^t = \ln \pi_i + \delta_i \ln P^t + u_i^t \tag{7}$$

generalizes the TPD model (1) by the products' price-level elasticities δ_i. The elasticities must be estimated, too. In the following, Eq. (7) is denoted as the nonlinear time-product-dummy (NLTPD) regression model.

The price-level elasticities, δ_i, measure the percentage change in the period t price of product i associated with a one percent change in the general price level P^t.[6] Products with elasticities larger than one exhibit a stronger upward trend than the price levels, P^t. In Table 1, this applies to the prices of Product B: $\delta_{\mathrm{B}} > 1$. Since the prices of Product A are decreasing over time, δ_{A} appears to be negative.

For the TPD model (1), a normalization such as $\ln P^1 = 0$ was sufficient to identify the model. For the NLTPD model (7), an additional restriction on the δ_i-values is needed because $\delta_i \ln P^t = (\delta_i/\lambda) \ln \tilde{P}^t$, with $\tilde{P}^t = (P^t)^\lambda$. Therefore, $\tilde{P}^t/\tilde{P}^s = (P^t/P^s)^\lambda$. This relationship implies that, without an appropriate restriction, the price level ratios (and

[6] The estimated price-level elasticities should be interpreted as measures of association rather than causal effects. Since the general price level is itself estimated from the product prices, the relationship is inherently one of correlation. When the number of products is large, the influence of any individual product on the estimated price level is typically negligible, making this distinction practically unimportant.

the values of δ_i) can be arbitrarily scaled up or down by the parameter λ, generating an infinite number of solutions.

The necessary restriction on the δ_i-values can be derived directly from the NLTPD model (7). Taking expectations, solving for $\delta_i \ln P^t$, multiplying by the average expenditure share $w_i = (1/T)\sum_{t\in\mathcal{T}} w_i^t$, and summing over all products in $\mathcal{N}$ yields the logarithm of the price level in period t:[7]

$$\ln P^t = \frac{1}{\sum_{i\in\mathcal{N}} w_i\delta_i} \sum_{i\in\mathcal{N}} w_i \left(\mathbb{E}\left(\ln p_i^t\right) - \ln \pi_i\right) .$$

Carrying out the same steps for period s, subtracting $\ln P^s$ from $\ln P^t$, and multiplying through by $\sum_{i\in\mathcal{N}} w_i\delta_i$, yields

$$\ln\left[\left(\frac{P^t}{P^s}\right)^{\sum_{i\in\mathcal{N}} w_i\delta_i}\right] = \sum_{i\in\mathcal{N}} w_i \mathbb{E}\left(\ln\frac{p_i^t}{p_i^s}\right) .$$

The right-hand side is the weighted mean of the expected values of the logarithms of the individual price ratios, $\ln(p_i^t/p_i^s)$. This weighted mean should coincide with the logarithm of the price level ratio, $\ln\left(P^t/P^s\right)$. Thus, the exponent on the left-hand side of the equation must be unity:

$$\sum_{i\in\mathcal{N}} w_i\delta_i = 1 . \tag{8}$$

This is the necessary restriction on the δ_i-values for the NLTPD model.

The TPD model (1) implicitly assumes that all price-level elasticities, δ_i, are equal to one. Thus, it automatically satisfies this restriction.

4.2 Estimation

The NLTPD model (7) generalizes the linear TPD model (1) by allowing for heterogeneous price-level elasticities. Nevertheless, the estimation of the NLTPD model uses the same set of dummy variables and weights as the estimation of the TPD model (1). No additional information is required. Since the model function is nonlinear in its parameters, parameter estimates must be derived by nonlinear regression. As with nonlinear estimators in general, finite samples may give rise to a small estimation bias.

The residuals $\hat{u}_i^t$ of the NLTPD regression model (7) are defined by $\hat{u}_i^t = \ln p_i^t - \hat{\delta}_i\widehat{\ln P^t} - \widehat{\ln \pi_i}$. Accordingly, the weighted sum of squared residuals, $S_{\hat{u}_i^t\hat{u}_i^t}$, can be written as

$$S_{\hat{u}_i^t\hat{u}_i^t} = \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}^t} w_i^t\left(\ln p_i^t - \hat{\delta}_i\widehat{\ln P^t} - \widehat{\ln \pi_i}\right)^2 = \sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}_i} w_i^t\left(\ln p_i^t - \hat{\delta}_i\widehat{\ln P^t} - \widehat{\ln \pi_i}\right)^2 . \tag{9}$$

The NLTPD-estimators can be derived by minimizing $S_{\hat{u}_i^t\hat{u}_i^t}$ with respect to $\widehat{\ln \pi_i}$, $\hat{\delta}_i$, and $\widehat{\ln P^t}$. For that purpose, the common normalization on the logarithmic price levels,

$$\widehat{\ln P^1} = 0 , \tag{10}$$

[7] In this derivation, the expenditure shares, w_i^t, are treated as fixed weights. Thus, the expectation operator is taken conditional on the observed expenditure shares. For $T = 2$, the average expenditure shares, w_i, coincide with the Törnqvist weights defined in Eq. (4).

is applied. Moreover, restriction (8) must be appropriately implemented in the estimation procedure. Accordingly, the NLTPD regression must be complemented by the restriction

$$\sum_{i \in \mathcal{N}} w_i \widehat{\delta}_i = 1 \; , \tag{11}$$

where $\widehat{\delta}_i$ is the estimator of δ_i. One $\widehat{\delta}_i$ value is obtained from restriction (11) rather than estimated directly. This choice does not affect any of the parameter estimates. Throughout the following, $\widehat{\delta}_1$ is obtained from this restriction:

$$\widehat{\delta}_1 = \frac{1 - \sum_{i \in \mathcal{N} \setminus \{1\}} w_i \widehat{\delta}_i}{w_1} \; . \tag{12}$$

Using normalization (10) and restriction (12), the minimization of $S_{\hat{u}_i^t \hat{u}_i^t}$ gives the following first-order conditions:

$$\widehat{\ln P^t} = \frac{\sum\limits_{i \in \mathcal{N}_t} w_i^t \widehat{\delta}_i \left(\ln p_i^t - \widehat{\ln \pi_i} \right)}{\sum\limits_{i \in \mathcal{N}_t} w_i^t \left(\widehat{\delta}_i \right)^2} \qquad \text{for } t = 2, \ldots, T \tag{13a}$$

$$\widehat{\ln \pi_i} = \sum_{t \in \mathcal{T}_i} \frac{w_i^t}{\sum_{s \in \mathcal{T}_i} w_i^s} \left(\ln p_i^t - \widehat{\delta}_i \widehat{\ln P^t} \right) \qquad \text{for } i = 1, \ldots, N \tag{13b}$$

$$\widehat{\delta}_i = \frac{\sum_{t \in \mathcal{T}_i} w_i^t \widehat{\ln P^t} \left(\ln p_i^t - \widehat{\ln \pi_i} \right)}{\sum_{t \in \mathcal{T}_i} w_i^t \left(\widehat{\ln P^t} \right)^2} \qquad \text{for } i = 2, \ldots, N \; . \tag{13c}$$

These first-order conditions do not admit explicit closed-form solutions. Instead, they can be solved by an iterative numerical procedure that minimizes $S_{\hat{u}_i^t \hat{u}_i^t}$. To this end, the TPD estimates $\widehat{\ln \pi_i}$ and $\widehat{\ln P^t}$ are derived from model (1). Their insertion into formula (13c) provides start values for $\widehat{\delta}_i$. The values of $\widehat{\delta}_i$ and $\widehat{\ln P^t}$ can then be used to calculate new $\widehat{\ln \pi_i}$ values according to formula (13b). The values of $\widehat{\delta}_i$ together with the updated $\widehat{\ln \pi_i}$ values are subsequently used in the computation of new $\widehat{\ln P^t}$ values using formula (13a). The new $\widehat{\ln \pi_i}$- and $\widehat{\ln P^t}$-values are used in the calculation of new $\widehat{\delta}_i$-values, and so on. This cyclical iteration is repeated until the coefficients $\widehat{\ln \pi_i}$, $\widehat{\ln P^t}$, and $\widehat{\delta}_i$ have converged to stable values.[8]

If all coefficients $\widehat{\delta}_i$ were equal to unity, the NLTPD regression would be unnecessary. The TPD regression would suffice and permit valid statistical inference. However, if at least one coefficient $\widehat{\delta}_i$ significantly deviates from one, the TPD model is misspecified. Thus, one may want to verify whether, for the dataset at hand, the TPD regression is inappropriate. Since the TPD model is nested in the NLTPD model, the joint null hypothesis $H_0 : \delta_1 = \delta_2 = \ldots = \delta_N = 1$ can be tested by a Likelihood Ratio (LR) test.

4.3 Stylized example

Figure 1 shows the estimated price levels for the NLTPD regression. The red line in the left panel depicts the NLTPD price index for the complete price data listed in Table 1. Since

[8] For a comprehensive survey of more sophisticated nonlinear least squares algorithms see Kelley (1999).

no prices are missing, the index is similar to those obtained from the TPD regression, the GEKS method, and the GK method.

When the prices of Product A are missing in periods 1 and 4, this results in only minor changes in the estimated price levels under the NLTPD regression. In contrast, the other multilateral index approaches show a markedly steeper increase in price levels; see the right panel of Figure 1. In other words, their price trends are upward biased. The NLTPD regression substantially reduces this bias by estimating the product-specific price-level elasticities. In the example, the NLTPD estimates of the price-level elasticities are $\widehat{\delta}_{\mathrm{A}} = -1.42$, $\widehat{\delta}_{\mathrm{B}} = 2.58$, and $\widehat{\delta}_{\mathrm{C}} = 1.56$.

4.4 Limitations and variations

The reliability of the NLTPD regression hinges upon its ability to produce accurate estimates of the price-level elasticities, δ_i. Problems may arise when there are only a few observations per product, when the elasticities are similar, when they change over time, when sales are concentrated either at the end or at the beginning of the time window covered by the regression, or when periods exist in which only products are offered that exhibit constant prices over time.

In such cases, the NLTPD regression may not converge. As a solution, one may consider a refined version in which some δ_i-values are predetermined (in TPD regressions, all of them are predetermined) and the remaining ones are estimated. For example, products with fixed prices over time are invariant to the general price level. They have a price-level elasticity of zero. Instead, of estimating this value, one can impose it as a parameter restriction in the NLTPD regression by setting $\delta_i = 0$ for all products i with fixed prices. Similarly, for products with only one observation in the whole data set, the parameter restriction $\delta_i = 1$ should be used.[9] No price-level elasticity can be estimated for such products. These refinements would also improve the computational performance of the NLTPD regression, as fewer parameters need to be estimated. Nevertheless, owing to its nonlinear specification, the NLTPD regression is generally slower than the TPD regression as well as the GK and GEKS methods. This is particularly true for large scanner datasets containing a high number of individual products.

Other refinements of the NLTPD regression are also conceivable. For example, when the δ_i-value of some product changes, this can be captured by an additional dummy variable or by more sophisticated approaches such as spline regression.

Some index approaches exist that replace missing prices by imputed prices. The NLTPD regression can provide such imputed prices. For example, de Haan (2022) explains why clearance sales distort the results of the GEKS method. He proposes to use the TPD regression to derive imputed prices for the products that have vanished from the market. These imputed prices can be used in the GEKS method. When products exhibit individual price-level elasticities, the imputation could be improved by using the NLTPD regression instead of the TPD regression.

An alternative strategy would be to compare multilateral index methods based on imputed prices with the direct application of the NLTPD regression. Such a comparison

[9] Ideally, such products should be removed from the dataset. While they contribute no information to temporal price comparisons, they nevertheless distort the calculation of expenditure shares.

would shed light on the trade-off between the imputation error introduced by completing the price matrix and the estimation error associated with the nonlinear estimation of the NLTPD model.

4.5 Related stochastic models

Consider the following stochastic relationship:

$$\ln \frac{p_i^t}{p_i^s} = \ln \frac{P^t}{P^s} + u_i^{s,t} , \tag{14}$$

where $u_i^{s,t}$ is a random error with expected value 0 and variance σ^2.[10] Summing over all periods s and rearranging leads to the TPD regression model in Eq. (1), with $\ln \pi_i = (1/T) \sum_{s \in \mathcal{T}} \ln(p_i^s / P^s)$ and $u_i^t = (1/T) \sum_{s \in \mathcal{T}} u_i^{s,t}$ (Weinand, 2022, pp. 121-123). When $\sigma^2 = 0$, the stochastic model (14) and, therefore, also the TPD regression model gives

$$\frac{p_i^t / p_j^t}{p_i^s / p_j^s} = 1 ,$$

which led Keynes (1930, p. 87) to criticize that the model assumes fixed relative prices.

In contrast, the NLTPD regression accommodates relative prices that evolve over time as a function of the price levels, P^t and P^s, and the price-level elasticities, δ_i and δ_j:

$$\frac{p_i^t / p_j^t}{p_i^s / p_j^s} = \left(\frac{P^t}{P^s} \right)^{\delta_i - \delta_j} .$$

However, the NLTPD regression is not the first approach that allows for greater flexibility in relative prices. Clements and Izan (1987, p. 341) consider the case $s = t - 1$ and propose to extend the stochastic model (14) by an additive constant δ_i:

$$\ln \frac{p_i^t}{p_i^s} = \ln \frac{P^t}{P^s} + \delta_i + u_i^{s,t} , \tag{15}$$

where $u_i^{s,t}$ is a random error with expected value 0 and variance proportional to $1/w_i$. To identify this model, the restriction $\sum_{i \in \mathcal{N}} w_i \delta_i = 0$ is imposed. Note the similarity of this restriction to the NLTPD regression's restriction $\sum_{i \in \mathcal{N}} w_i \delta_i = 1$. In the stochastic model (15), the parameter δ_i represents the systematic part in the deviation between the price ratio p_i^t / p_i^s and the price level ratio P^t / P^s. When $\sigma^2 = 0$, the stochastic model (15) yields

$$\frac{p_i^t / p_j^t}{p_i^s / p_j^s} = e^{(\delta_i - \delta_j)} .$$

The right-hand side of this equation is time-invariant. Consequently, relative prices evolve in a deterministic manner. In other words, whether the general price level remains constant or doubles, the evolution of relative prices remains unaffected.

[10] For discussions of the historical origins of this relationship, see, for example, the surveys of the stochastic approach by Clements et al. (2006), Diewert (2010), and Selvanathan and Rao (1994).

Another stochastic model has been proposed by Selvanathan and Rao (1994, p. 62):

$$\frac{p_i^t}{p_i^s} = \frac{P^t}{P^s} + \delta_i + u_i^{s,t} , \tag{16}$$

where s is a fixed base period and the values of δ_i are restricted by the condition $\sum_{i \in \mathcal{N}} w_i \delta_i = 0$. Assuming $\sigma^2 = 0$, the stochastic model (16) leads to

$$\frac{p_i^t / p_j^t}{p_i^s / p_j^s} = \frac{P^t / P^s + \delta_i}{P^t / P^s + \delta_j} .$$

This would imply that for sufficiently strong increases in the general price level P^t, the relative prices, p_i^t / p_j^t, tend towards their values of the base period s. Analogous concerns are expressed by Diewert (2010, p. 342, footnote 15).

5 Simulation

The illustrative example introduced in Section 3 revealed that, in the presence of data gaps, the TPD regression is usually biased. It was argued that this also applies to the GEKS and GK methods. It would be desirable to know whether the magnitude of the bias differs between the various approaches. A sound empirical analysis of this question requires a meaningful benchmark for quantifying the bias of the various multilateral index approaches. This benchmark is introduced in Section 5.1. The simulation framework is described in Section 5.2 and the results are discussed in Section 5.3.

5.1 Benchmark

Simply *defining* the results of the NLTPD regression (or of any other multilateral index approach) as the benchmark, would be a prejudiced approach. It is preferable to have a benchmark against which all multilateral index approaches, including the NLTPD regression, can be compared. Thus, a dataset with a known general price level change is required. Any deviation between this benchmark and the general price change computed by the investigated multilateral price index can be interpreted as bias.

In practice, the available price and quantity data are generally characterized by individual price trends and shifts in expenditure weights. Consequently, such data do not possess a known general price level change and, therefore, no unassailable benchmark. A promising alternative to such data are carefully designed simulation studies where the purchased quantities of the various products reflect the typical features of actual purchasing behaviour.

While microeconomic textbooks typically assume that households adjust their purchased quantities instantaneously and completely in response to price changes, real-world households often require time to fully adjust their purchasing behavior. For example, during sales, households not only increase current consumption but also build up inventories. Thus, the purchased quantities exceed the level of consumption. After the sale, the households start to reduce their inventories. Thus, during the after-sales period, the purchased quantities fall below the level of consumption. A second feature of actual purchasing behaviour is delayed adjustments to price changes. Adjustment costs are an important cause for such consumption smoothing. Auer (2025, pp. 6-7) denotes the quantity effects caused by stockpiling as "overshooting quantities" and the quantity effects caused by consumption smoothing as "sticky quantities".

The present study develops a simulation framework that accommodates the textbook case of "ordinary quantities" due to instantaneous and complete adjustments, overshooting quantities due to stockpiling, and sticky quantities due to consumption smoothing. To get an unassailable benchmark, household behaviour is represented by two CES subutility functions (one representing consumption and the other stockpiling) that are embedded in a Cobb-Douglas utility function with exponents adding to one. The details are provided in Appendix B. From the perspective of utility maximizing households, the products are symmetric in the sense that the purchased quantities of any pair of products equalize when their prices are identical and constant. Given this symmetric appreciation of the available products, it is possible to construct a natural benchmark for the performance of the various multilateral index approaches.

As an illustration, Table 2 lists the prices and purchased quantities of three products during periods r and t. In addition, the table lists a hypothetical price-quantity scenario s with prices p_i^s and quantities q_i^s. The period r prices of Products A to C prevailed also during the previous periods. In other words, the quantities q_i^r represent the households' optimal "long-run" reaction to the prices p_i^r. The hypothetical price-quantity scenario s is merely a permutation of the period r scenario. Thus, for households with a symmetric appreciation of the available products, any multilateral price index should indicate that no price change occurred between period r and the hypothetical price-quantity scenario s.[11]

	p_i^r	q_i^r	p_i^s	q_i^s	p_i^t	q_i^t
Product A:	20	6	*40*	*3*	44	3
Product B:	40	3	*10*	*9*	11	9
Product C:	10	9	*20*	*6*	22	6

Table 2: Prices and quantities of three products during periods r and t.

Multiplying the prices p_i^s by the factor 1.1, yields the prices p_i^t. In analogy to period r, it is assumed that the period t prices prevailed also in the preceding periods. Thus, the purchased quantities q_i^t are the households' optimal "long-run" reaction to the prices p_i^t. They are identical to the purchased quantities during period s. Since from period s to period t all prices increase by 10% and all quantities remain constant, the change in the average price level is also 10%.[12]

In sum, the price change between periods r and t can be decomposed into the price change between period r and the hypothetical price-quantity scenario s and the price change between the hypothetical price-quantity scenario s and period t. Between period r and scenario s, the price level remains constant, while between scenario s and period t, the price level increases by 10%. Thus, the price level change between periods r and t is also 10%. This number is the benchmark for any multilateral index approach comparing the prices of periods r and t, completely independent of the prices and quantities that

[11] Krtscha (1979, p. 69) and Auer (2002, p. 534) formulate an analogous postulate in the context of bilateral axiomatic index theory.

[12] In the context of bilateral axiomatic index theory, the Proportionality axiom makes the same postulate even without imposing the restriction of constant quantities.

occurred in the intervening periods. If a multilateral index approach yields a price change that deviates from 10%, this deviation can be considered as bias. This is the rationale of the simulation approach developed in the present study.

5.2 Framework

In the simulation study, $N = 40$ different products exist and the number of periods is $T = 25$. The exogenously given periodic income of the households increases over time. In period $t = 25$, it is 10% larger than in period $t = 1$. Thus, the periodic income increases each period by 0.3979%. In each period, the households spent their complete income.

The prices of period $t = 1$ are randomly drawn from the interval $[15, 25]$. The corresponding quantities are derived from the Cobb-Douglas utility function introduced in Section 5.1 and described in detail in Appendix B. More specifically, each simulated price vector is inserted into the utility-maximization framework of Appendix B, yielding the corresponding optimal purchased quantities. The basic pattern of the prices and quantities of periods $t = 1$ and $t = 25$ is like in Table 2. More specifically, the quantities in period $t = 25$ are a permutation of the quantities of period $t = 1$. The prices represent the same permutation, the only difference being that in period $t = 25$ all prices are 10% higher than in period $t = 1$. Thus, total expenditures during period $t = 25$ are 10% larger than during period $t = 1$.

The price trend of each product i is log-linear with an additive stochastic component. The slopes of the price trends vary between the products. Some products have downward trending prices. The linear trends can be interrupted by stochastic sales. On average, 10 randomly selected products are on sale at least once during the 25 periods (except the first and last period). The discounts randomly range from 5% to 30%.

As an illustration, the upper panel of Figure 2 presents a typical price trajectory for a representative product. It resembles a random process with drift, where an overall upward trend is intermittently interrupted by sales (indicated by the gray-shaded areas). The second panel displays the corresponding ordinary quantities purchased by households that adjust instantaneously and completely to the price changes shown above (the simple textbook case). The third panel illustrates the overshooting quantities of stockpiling households, while the bottom panel depicts the sticky quantities of households that smooth their consumption over time.

In the simulations, the price level of period $t = 1$ is set to 100: $P^1 = 100$. Thus, the benchmark for the estimated price level $\widehat{P}^{25}$ of the various multilateral index approaches is the value $P^{25} = 110$. Any deviation from this value represents bias.

Four distinct cases are considered. Case 1 assumes that the data are complete. Thus, all multilateral index approaches should yield unbiased $\widehat{P}^{25}$-values. To examine this conjecture, for each multilateral index approach (NLTPD, TPD, GK, and GEKS) and each type of quantity adjustment (ordinary, overshooting, and sticky), 2000 samples are computed and the average value of the 2000 $\widehat{P}^{25}$-values is calculated. An average value of 110 would say that the multilateral index approach is unbiased.

Cases 2 to 4 allow for data gaps. A gap arises when in some period the price and the quantity of some product are missing. The gaps occur randomly. On average, the probability of a data gap is 30%. Regardless of the gaps, the price-quantity combinations of period $t = 1$ prevail in permuted form in period $t = 25$, the only difference being, that

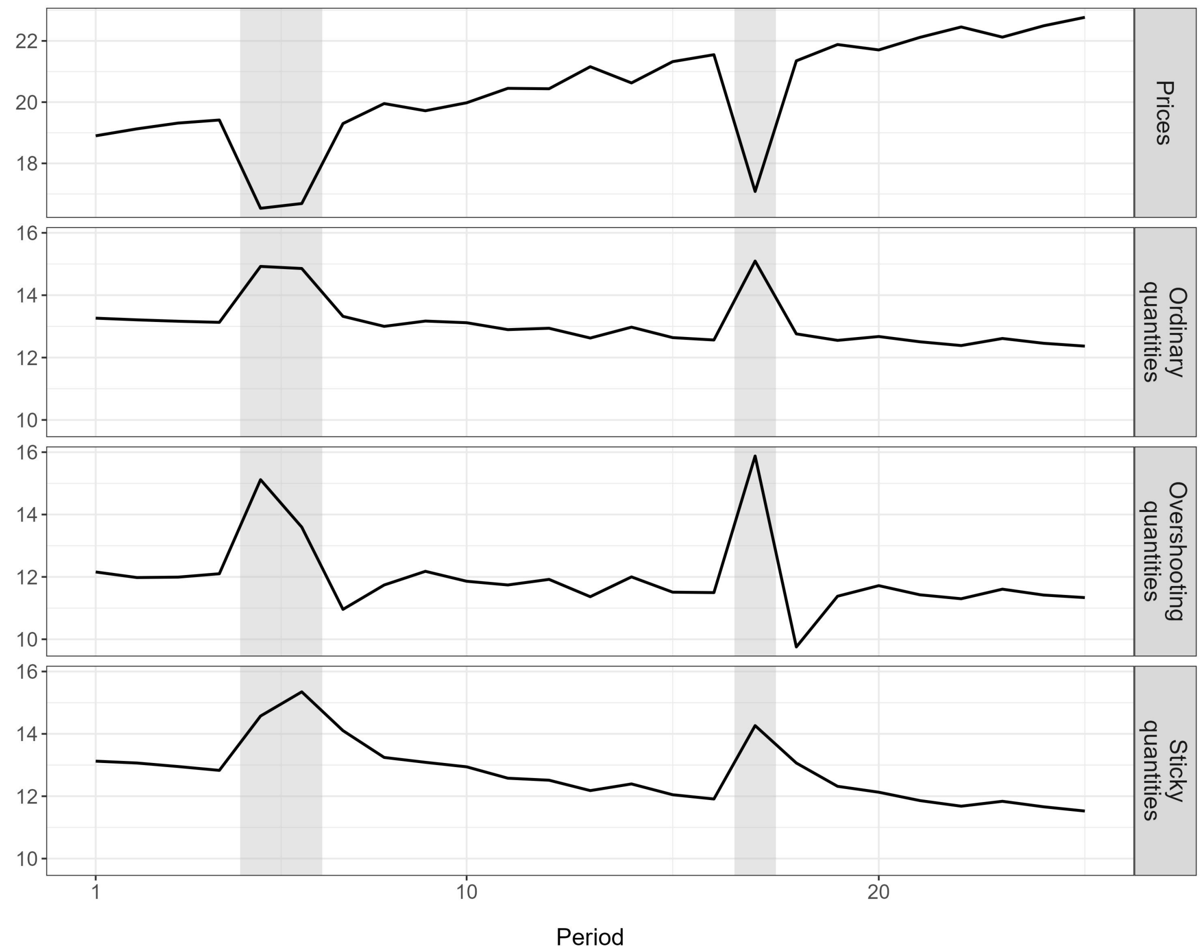

Figure 2: Simulated prices (upper panel) and ordinary, overshooting and sticky quantities (lower panels) of a representative product.

the prices during period $t = 25$ are 10% higher than in period $t = 1$.

In Case 2, the probability of a data gap of some product i is independent of its δ_i-value. Therefore, regardless of the applied multilateral index approach, the 2000 samples should yield $\widehat{P}^{25}$-values that average to 110. In Case 3, the δ_i-value of some product i and its number of data gaps are negatively correlated. Thus, except for the NLTPD method, the average of the $\widehat{P}^{25}$-values is expected to exceed 110. In Case 4, a positive correlation is considered. Thus, the opposite direction of bias should arise.

5.3 Results

Table 3 reports the bias of the estimated index numbers $\widehat{P}^{25}$. In Case 1, all multilateral index approaches perform well, although, for ordinary and overshooting quantities, the NLTPD regression performs slightly worse than the other three approaches. The modest residual bias of the NLTPD regression reflects the trade-off associated with the greater flexibility of the nonlinear estimation approach. In Case 2, the choice among the four approaches makes little difference. This, however, changes in Cases 3 and 4. Here, the NLTPD regression performs as good as before, while the other approaches exhibit substantial bias. In Case 3, the GEKS method shows an upward bias exceeding 11 index points, whereas both the TPD regression and the GK method display an upward bias of nearly 7 index points. In Case 4, the biases reverse: the TPD and GK methods show a downward bias of about 6 index points, and the GEKS method exceeds 10 index points in the same direction.

Case	Quantities	NLTPD	TPD	GK	GEKS
1	Ordinary	-0.110	0.000	-0.001	-0.000
	Overshooting	-0.267	0.000	-0.001	-0.063
	Sticky	0.088	-0.000	-0.000	0.106
2	Ordinary	-0.119	0.107	0.113	0.133
	Overshooting	-0.278	0.161	0.166	0.114
	Sticky	0.076	0.093	0.101	0.209
3	Ordinary	0.029	6.670	6.686	11.287
	Overshooting	-0.139	6.722	6.743	11.278
	Sticky	0.218	6.741	6.729	11.472
4	Ordinary	-0.155	-6.072	-6.086	-10.028
	Overshooting	-0.351	-6.139	-6.148	-10.135
	Sticky	0.035	-6.162	-6.141	-10.047

Table 3: Bias (in index points) of indices in the last simulation period.

A second relevant aspect for the performance of the four approaches is the root mean squared error (RMSE). Considering the bias observed for some of the cases, Table 4 reports the centered RMSE of the index values $\widehat{P}^{25}$.[13] Except for Case 1, the NLTPD regression's centered RMSE is far below those of the other three approaches. This is true for ordinary, overshooting, and sticky quantities.

Case	Quantities	NLTPD	TPD	GK	GEKS
1	Ordinary	0.382	0.013	0.047	0.001
	Overshooting	0.429	0.015	0.052	0.015
	Sticky	0.402	0.024	0.087	0.024
2	Ordinary	0.565	1.875	1.872	2.017
	Overshooting	0.629	1.855	1.857	2.006
	Sticky	0.588	1.844	1.836	1.999
3	Ordinary	0.550	2.172	2.160	2.482
	Overshooting	0.625	2.260	2.251	2.578
	Sticky	0.578	2.179	2.166	2.520
4	Ordinary	0.757	1.905	1.894	2.054
	Overshooting	0.801	1.952	1.943	2.099
	Sticky	0.775	1.991	1.979	2.092

Table 4: Centered RMSE (in index points) of indices in the last simulation period.

The results in Tables 3 and 4 indicate that the type of quantity adjustment (ordinary, overshooting, or sticky) appears to matter only marginally. Overall, despite its modest

[13] Since the true index values in this period are a constant, the centered RMSE is identical to the standard deviation of the estimated index values $\widehat{P}^{25}$.

finite-sample bias, the NLTPD regression demonstrates superior performance to the other approaches in the simulation, except in the case involving complete data.[14] To gain a more complete empirical picture, it is worth investigating whether similar deviations between the NLTPD regression and the other three approaches occur in real-world scanner data.

6 Application to scanner data

In this section, the NLTPD regression is applied to Dominick's Finer Food scanner data (James M. Kilts Center, University of Chicago Booth School of Business, 2018). For snack crackers, (unit value) prices of individual products were recorded on a weekly basis from 1989 to 1997. Since some weeks are not covered in some years, the analysis is restricted to the period from December 1989 to December 1991 ($t = 1, 2, \ldots, 25$). Following standard practice in statistical offices that process scanner data using multilateral index approaches (European Commission, Eurostat, 2022), the weekly transactions for each product and month are aggregated. As a result, there are roughly 3,500 monthly unit values and aggregated quantities for 185 products ($i = 1, 2, \ldots, 185$). Not all products were sold in all months. Hence, there are gaps in the data.

Snack crackers can be regarded as relatively homogeneous. Consequently, one might argue that their individual price-level elasticities, δ_i, are close to one. This assumption, in turn, could justify the use of the TPD regression, GEKS method, or GK method. The NLTPD regression is capable of empirically testing whether products share a price-level elasticity of one, as it is the only method that yields estimates of the price-level elasticities, $\widehat{\delta}_i$. The results are presented in Figure 3. It depicts the density curve of the estimated price-level elasticities of the individual products. The distribution is quite broad, with approximately 90% of the elasticities falling between -0.5 and 2.25. Only a small proportion of products exhibit an elasticity close to one.

This visual impression is confirmed by a formal LR test. The LR statistic is based on the residual sums of squares of the nested TPD and NLTPD models. The null hypothesis that all products share a price-level elasticity of one is rejected at a 1% significance level. This finding alone does not imply biased price-level estimates. Bias arises only when heterogeneity in price-level elasticities is systematically associated with product-level missingness.

To further investigate this question, the correlation between the estimated price-level elasticities, $\widehat{\delta}_i$, and the number of missing observations is calculated. In the data, this correlation is -0.16. A similar conclusion is obtained using the gap pattern coefficient, G, proposed by Auer (2026). Unlike the correlation coefficient, G does not require a preceding NLTPD estimation. Instead, it directly measures the rank concordance between the price sensitivity of individual products and their number of missing observations. The coefficient ranges from -1 to 1. In the present application, it takes the value $G = -0.09$.

In view of these findings, it is interesting to examine the estimated price levels of snack crackers more closely. Figure 4 depicts the results. The price levels are normalized such that the resulting price index equals 100 in December 1989. The figure shows that

[14] Białek (2023) recently proposed the GEKS-Laspeyres and GEKS-Geometric Laspeyres methods as part of a class of GEKS indices that satisfy the identity test. In the simulation study, however, both methods yielded higher bias and centered RMSE than the traditional GEKS method.

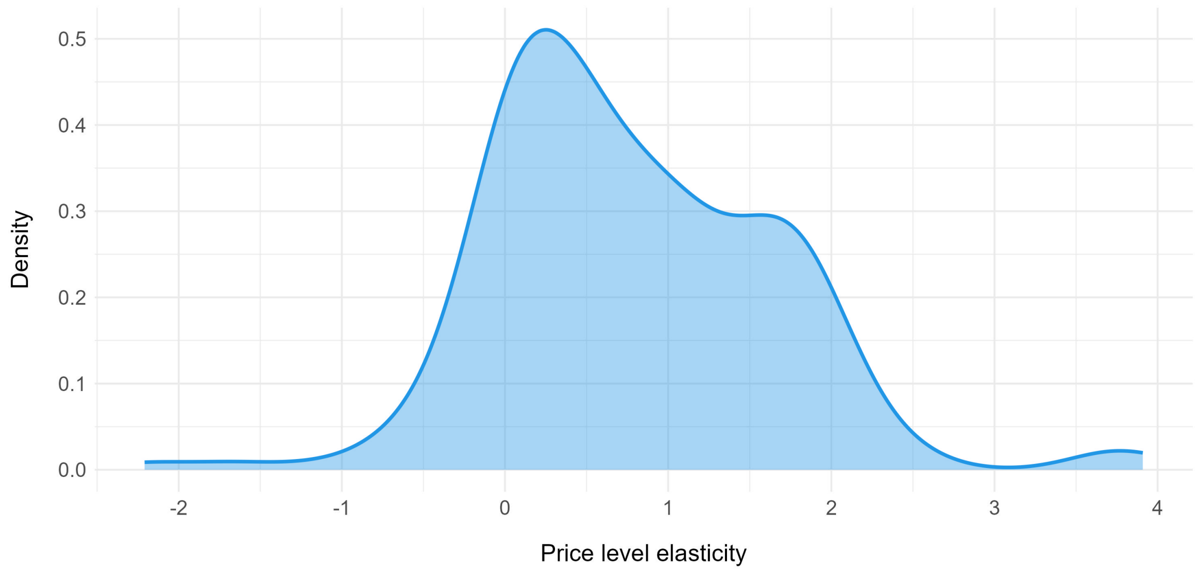

Figure 3: Density of NLTPD estimates for the price-level elasticities, $\hat{\delta}_i$, of snack crackers. Source: James M. Kilts Center, University of Chicago Booth School of Business (2018); authors' own computations.

prices moderately increased from December 1989 to December 1991. While the prices were raised in the beginning of a year, they kept relatively stable in between (except for some promotional sales) and finally declined again in November and December.[15] All four index approaches show this movement. The TPD and GK price indices are almost identical. The GEKS index shows smaller deviations in some months.

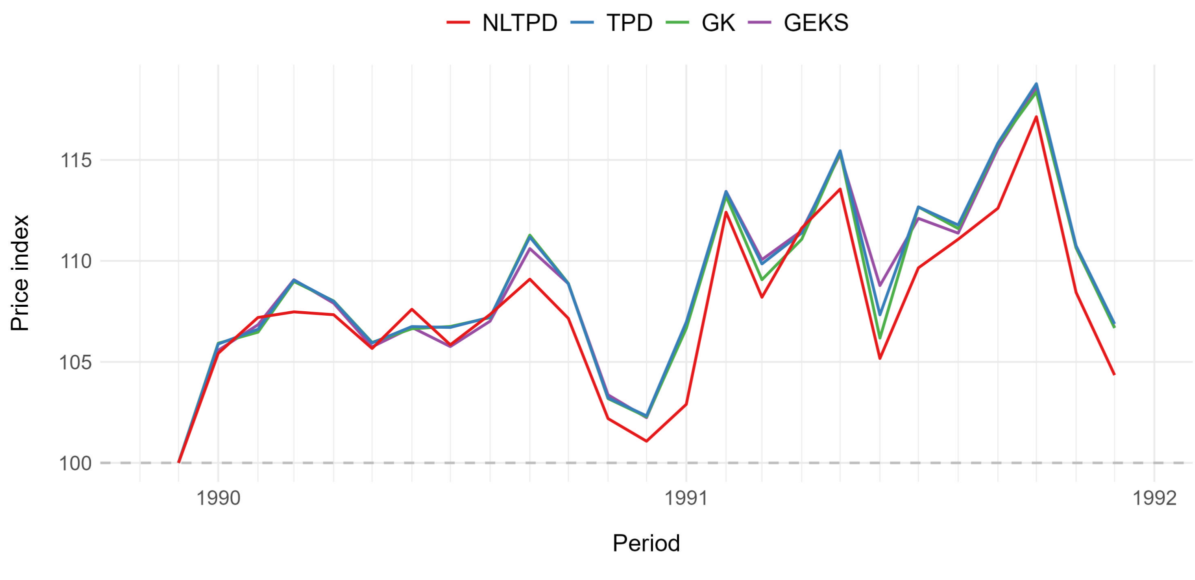

Figure 4: Price indices for snack crackers (December 1989=100). Source: James M. Kilts Center, University of Chicago Booth School of Business (2018); authors' own computations.

As discussed above, the negative relationship between price-level elasticity and number of missing observations is expected to induce an upward bias in the price levels estimated

[15] Due to the aggregation from weekly to monthly frequency, promotional sales with noticeable price reductions are not that pronounced in the indices.

by the TPD regression, the GEKS method, and the GK method, whereas the NLTPD regression is more robust to this type of missingness. Figure 4 is consistent with this expectation. By the end of the sample period, the price index estimated by the NLTPD regression lies more than two index points below those obtained from the other approaches.

When longer time intervals are analyzed, multilateral index approaches are typically supplemented with some form of splicing. In this process, rolling windows are connected using one or more linking periods (a simple graphical illustration is provided in Auer, 2025, Online Appendix B). Splicing could also be applied to the NLTPD regression. When applied to the TPD regression, the GEKS method, or the GK method, any bias present in the selected linking period is propagated into subsequent windows. Consequently, splicing does not eliminate the bias but instead causes it to accumulate over time. The same mechanism applies to the NLTPD regression. However, because the NLTPD regression is less susceptible to this source of bias, the accumulated bias is correspondingly smaller.

7 Concluding remarks

Multilateral price index approaches have been advocated as an effective remedy against chain drift bias. Do they have the potential to mitigate also other problems in intertemporal price comparisons? One such problem is data gaps in conjunction with products that exhibit different price-level elasticities, that is, different elasticities of the product prices with respect to the general price level. If gaps occur more frequently for products with small or negative price-level elasticities than for those with large positive elasticities, bilateral index numbers would exhibit an upward bias. Conversely, a downward bias would arise when the gaps are concentrated among products with large positive price-level elasticities.

The problem could be remedied if it were known whether the gaps are mostly related to products with large price-level elasticities or to products with small and negative price-level elasticities. Thus, information is required from periods in which the missing products are offered. The most widely used multilateral index approaches (GEKS, TPD, and GK) draw on such information. Nevertheless, they are unable to avoid the bias present in the bilateral index because they do not identify the products' price-level elasticities. For example, the GEKS method merely processes bilateral price indices and the latter are biased because of the data gaps. Therefore, the GEKS method yields biased results, too.

The paper introduced the NLTPD regression, a nonlinear extension of the TPD regression that estimates product-specific price-level elasticities. The theoretical analysis showed that allowing for heterogeneous price-level elasticities substantially mitigates the bias affecting standard multilateral index methods. The simulation study demonstrated that, except for the theoretical case of complete data, the NLTPD regression consistently outperforms the standard multilateral index approaches (TPD, GEKS, and GK). Finally, the empirical application illustrated the practical applicability of the proposed method to real-world scanner data.

These findings have clear practical implications. The simulation results indicate that, whenever data gaps are present, the NLTPD regression is generally the preferred approach for multilateral price measurement. Nevertheless, practitioners may prefer the standard multilateral index approaches because of their simpler specification, lower computational burden, and greater numerical robustness. In such cases, the question is not whether the NLTPD regression performs better, but whether its additional flexibility is warranted for the dataset at hand. To assess this, the LR test proposed in this paper provides a practical diagnostic for evaluating the adequacy of the TPD specification. Furthermore, practitioners may complement this assessment by using the recently proposed gap pattern coefficient (Auer, 2026), which evaluates whether the observed pattern of missing observations is likely to induce a meaningful bias. Together, these diagnostic tools provide practical guidance on whether the computational simplicity of the standard multilateral index approaches justifies foregoing the additional flexibility of the NLTPD regression.

Appendix A Similarity between the TPD regression and the GK method

This appendix explains why the TPD regression and the GK method produce similar index values in the illustrative example, the simulation study, and the empirical application.

The TPD regression model defined by Eq. (1) assumes that, on average, the logarithms of the deflated prices of product i, $\ln(p_i^t/P^t)$, coincide with the logarithm of the general value of product i, $\ln\pi_i$:

$$\ln(p_i^t/P^t) = \ln\pi_i + u_i^t \ ,$$

where $u_i^t \sim N\left(0,\sigma^2\right)$ is the error term. If the logarithms were deleted, one would obtain the following relationship:

$$p_i^t/P^t = \pi_i + \epsilon_i^t \ ,$$

where $\epsilon_i^t \sim N\left(0,\sigma^2\right)$ is the error term. Rearranging gives

$$\epsilon_i^t = p_i^t/P^t - \pi_i \ .$$

Therefore, using as weights the quantities q_i^t, the weighted sum of squared residuals is given by the following expression:

$$S_{\hat{\epsilon}_i^t\hat{\epsilon}_i^t} = \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{N}_t} q_i^t\left(p_i^t/\widehat{P^t} - \widehat{\pi}_i\right)^2 \ .$$

This expression can be minimized with respect to the unknowns $\widehat{\pi}_i$ ($i = 1,\ldots,N$) and $\widehat{P^t}$ ($t = 2,\ldots,T$). The first-order conditions of this minimization problem are

$$\frac{\partial S_{\hat{\epsilon}_i^t\hat{\epsilon}_i^t}}{\partial\widehat{\pi}_i} = \sum_{t\in\mathcal{T}_i} q_i^t\left(p_i^t/\widehat{P^t} - \widehat{\pi}_i\right) = 0 \tag{17a}$$

$$\frac{\partial S_{\hat{\epsilon}_i^t\hat{\epsilon}_i^t}}{\partial\widehat{P^t}} = \sum_{i\in\mathcal{N}^r} q_i^t\left(p_i^t/\widehat{P^t} - \widehat{\pi}_i\right)\frac{1}{\widehat{P^t}^2} = 0 \ . \tag{17b}$$

Solving these two equations for $\widehat{\pi}_i$ and $\widehat{P^t}$, respectively, gives the definitions of the GK method (6). Hence, the difference between the (weighted) TPD regression and the GK method boils down to the use of the logarithmic variables $\ln(p_i^t/P^t)$ and $\ln\pi_i$ instead of the variables p_i^t/P^t and π_i and to the use of different regression weights (expenditure shares instead of quantities).

Appendix B Utility function of simulation

In the simulation, three types of households are considered. They differ with respect to their response to price changes (instantaneous quantity adjustments, overshooting quantity adjustments due to stockpiling, and sticky quantity adjustments due to consumption smoothing). All of these types can be accomodated by a Cobb–Douglas utility function that consists of two CES subutility functions, one for consumption and one for stockpiling.

Consider a representative household with income m and inherited stocks $\bar{s}_i$ ($i = 1, \ldots, N$). Time subscripts are suppressed, and a bar denotes previous-period values. The household's current budget constraint is

$$m + \sum_{i=1}^{N} p_i \bar{s}_i = \sum_{i=1}^{N} p_i c_i + \sum_{i=1}^{N} p_i (1+\mu) s_i \ , \tag{18}$$

where p_i is the price, c_i consumption, s_i end-of-period stocks, and $\mu p_i s_i$ ($\mu \geq 0$) storage costs.[16] The left-hand side defines net endowment $\tilde{m} = m + \sum_{i=1}^{N} p_i \bar{s}_i$. Purchased quantity x_i equals consumption plus the change in stocks,

$$x_i = c_i + s_i - \bar{s}_i \ . \tag{19}$$

Subutilities from consumption and stockpiling are

$$C = \left[\sum_{i=1}^{N} \left(\frac{c_i}{\bar{c}_i^{\gamma}} \right)^{\theta} \right]^{1/\theta} , \qquad S = \left[\sum_{i=1}^{N} s_i^{\phi} \right]^{1/\phi} , \tag{20}$$

with $\theta, \phi \neq 0$. The term $c_i / \bar{c}_i^{\gamma}$ captures the effect of past consumption $\bar{c}_i$ on current utility. A positive γ reflects aspiration effects that reduce the utility of current consumption. The household's utility is

$$U = C^{\alpha} S^{1-\alpha} \ , \tag{21}$$

where α measures the preference for current consumption. Parameter restrictions are $0 < \alpha \leq 1$, $\theta < 1$ and $\phi < 1$.

If $\alpha = 1$, the household never stockpiles. Auer (2024, Appendix C, pp. 24-29) shows that $-1/(1-\theta)$ and $-1/(1-\phi)$ approximate the price elasticities of consumption and stockpiling. Hence, negative θ and ϕ imply inelastic demands and limited substitution across items.

Changes in c_i affect future optimisation through $\bar{c}_i$, inducing further adjustment even without new price changes. The term $-\gamma\theta/(1-\theta)$ approximates the elasticity of current consumption with respect to past consumption. Stability requires $|-\gamma\theta/(1-\theta)| < 1$. Consumption smoothing occurs when $0 < -\gamma\theta/(1-\theta) < 1$, while $-1 < -\gamma\theta/(1-\theta) < 0$ implies oscillatory adjustment. Only if $\gamma\theta = 0$ does adjustment occur instantaneously.

As shown in Auer (2024, Appendix C, pp. 24-29), utility maximisation subject to constraint (18) yields optimal purchases:

$$x_i^* = \underbrace{\alpha \tilde{m} \bar{c}_i^{\frac{-\gamma\theta}{1-\theta}} p_i^{\frac{-1}{1-\theta}} P_C^{\frac{\theta}{1-\theta}}}_{c_i^*} + \underbrace{(1-\alpha) \tilde{m} \tilde{p}_i^{\frac{-1}{1-\phi}} P_S^{\frac{\phi}{1-\phi}}}_{s_i^*} - \bar{s}_i \ , \tag{22}$$

[16] Households may sell inherited stocks $\bar{s}_i$ at price p_i. In the simulations, this never occurs.

where $\tilde{p}_i = p_i(1+\mu)$. The expressions

$$P_C = \left(\sum_{i=1}^{N} \bar{c}_i^{\frac{-\gamma\theta}{1-\theta}} p_i^{\frac{-\theta}{1-\theta}}\right)^{-\frac{1-\theta}{\theta}} \qquad \text{and} \qquad P_S = \left(\sum_{i=1}^{N} \tilde{p}_i^{\frac{-\phi}{1-\phi}}\right)^{-\frac{1-\phi}{\phi}} \tag{23}$$

represent "price indices". The variables x_i^*, c_i^*, and s_i^* are the optimal counterparts of the variables in Eq. (19).

Given prices p_i, income m, inherited stocks $\bar{s}_i$, past consumption $\bar{c}_i$, and parameters $(\alpha, \theta, \phi, \mu, \gamma)$, Eq. (22) determines optimal purchases. Stockpiling without smoothing requires $0 < \alpha < 1$ and $\gamma = 0$, yielding

$$x_i^* = \tilde{m}\left(\alpha p_i^{\frac{-1}{1-\theta}} P_C^{\frac{\theta}{1-\theta}} + (1-\alpha)\tilde{p}_i^{\frac{-1}{1-\phi}} P_S^{\frac{\phi}{1-\phi}}\right) - \bar{s}_i \ .$$

Smoothing without stockpiling arises for $\alpha = 1$ and $\gamma \neq 0$, with

$$x_i^* = \tilde{m}\bar{c}_i^{\frac{-\gamma\theta}{1-\theta}} p_i^{\frac{-1}{1-\theta}} P_C^{\frac{\theta}{1-\theta}},$$

where $\tilde{m} = m$.

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