Basic Algebra

$$\begin{aligned} (a+b)^2 &= a^2+2ab+b^2 \\ (a-b)^2 &= a^2-2ab+b^2 \\ (a+b)(a-b) &= a^2-b^2 \\ a^b a^c &= a^{b+c} \\ \frac{a^b}{a^c} &= a^{b-c} \\ (a^b)^c &= a^{bc} = (a^c)^b \end{aligned}$$

$$\begin{aligned} a^{1/2} &= \sqrt{a} \quad (\text{valid if } a \geq 0) \\ \sqrt{ab} &= \sqrt{a}\sqrt{b} \\ \sqrt{\frac{a}{b}} &= \frac{\sqrt{a}}{\sqrt{b}} \\ a^{1/q} &= \sqrt[q]{a} \\ a^{p/q} &= \left(a^{1/q}\right)^p = (a^p)^{1/q} = \left(\sqrt[q]{a^p}\right) \end{aligned}$$

$$\sum_{i=1}^{n}(a_i+b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} i = 1+2+...+n = \frac{1}{2}n(n+1)$$

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+...+n^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{i=1}^{n} i^3 = 1^3+2^3+...+n^3 = \left(\frac{1}{2}n(n+1)\right)^2 = \left(\sum_{i=1}^{n} i\right)^2$$

$$\sum_{i=0}^{n} a^i = \frac{1-a^{n+1}}{1-a}$$

$$ax^2+bx+c=0 \qquad \Leftrightarrow \qquad x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x^2+px+q=0 \qquad \Leftrightarrow \qquad x = -\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}$$

The vertex of $f(x)=ax^2+bx+c$ is at $\left(-\frac{b}{2a}, -\frac{b_2-4ac}{4a}\right)$

A quantity K that increases by $p\%$ per year will have increased after t years to

$$f(t) = K\left(1+\frac{p}{100}\right)^t$$

A quantity K that decreases by $p\%$ per year will have decreased after t years to

$$f(t) = K\left(1-\frac{p}{100}\right)^t$$

$$\begin{aligned} \ln(xy) &= \ln x + \ln y \\ \ln\frac{x}{y} &= \ln x - \ln y \\ \ln(x^p) &= p\ln x \end{aligned}$$

$$\begin{aligned} \ln 1 &= 0 \\ \ln e &= 1 \\ e^{\ln x} &= x \\ \ln e^x &= x \end{aligned}$$

Differentiation

- Rule 1: $f(x) = A \qquad \Rightarrow \qquad f'(x) = 0$

- Rule 2: $y = A + f(x) \qquad \Rightarrow \qquad y' = f'(x)$

- Rule 3: $y = Af(x) \qquad \Rightarrow \qquad y' = Af'(x)$

- Rule 4 (power rule): $\qquad f(x) = x^a \qquad \Rightarrow \qquad f'(x) = ax^{a-1}$

  with a being an arbitrary constant.

- Rule 5 (sums): If both f and g are differentiable at x, then the sum $f+g$ and the difference $f-g$ are both differentiable at x, and

$$h(x) = f(x) \pm g(x) \qquad \Rightarrow \qquad h'(x) = f'(x) \pm g'(x)$$

- Rule 6 (products): If both f and g are differentiable at x, then so is $h = f \cdot g$, and

$$h(x) = f(x) \cdot g(x) \qquad \Rightarrow \qquad h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

- Rule 7 (quotient): If both f and g are differentiable at x and $g(x) \neq 0$, then $h = f/g$ is differentiable at x, and

$$h(x) = \frac{f(x)}{g(x)} \qquad \Rightarrow \qquad h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

- Rule 8 (chain rule): If g is differentiable at x and f is differentiable at $u = g(x)$, then the composite function $h(x) = f(g(x))$ is differentiable at x, and

$$h'(x) = f'(u) \cdot g'(x) = f'(g(x)) \cdot g'(x)$$

- Rule 9:

$$f(x) = e^x \qquad \Rightarrow \qquad f'(x) = e^x$$

  The derivative of $f(x) = e^x$ is equal to the function itself.

- Rule 10: $\qquad f(x) = \ln x \qquad \Rightarrow \qquad f'(x) = \frac{1}{x}$

- When $z = F(x_1, ..., x_n)$ with

$$x_1 = f_1(t_1, ..., t_m), \quad ... \quad , x_N = f_n(t_1, ..., t_m)$$

  then

$$\frac{\partial z}{\partial t_j} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_j} + ... + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_j} \qquad j = 1, 2, ..., m$$

Optimization

Suppose $f(x, y)$ is a twice differentiable function in a domain S, and let (x_0, y_0) be an interior stationary point of S.

(a) If

$$\frac{\partial^2 f}{\partial x^2} < 0 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$$

then (x_0, y_0) is a (strict) local maximum point.

(b) If

$$\frac{\partial^2 f}{\partial x^2} > 0 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$$

then (x_0, y_0) is a (strict) local minimum point.

(c) If

$$\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 < 0$$

then (x_0, y_0) is a saddle point.

(d) If

$$\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0$$

then (x_0, y_0) could be a local maximum, a local minimum, or a saddle point.

Matrix Algebra

- $$\begin{aligned} (\mathbf{A}')' &= \mathbf{A} \\ \mathbf{A} + \mathbf{0} &= \mathbf{A} \\ \mathbf{A} + \mathbf{B} &= \mathbf{B} + \mathbf{A} \\ \mathbf{A}' + \mathbf{B}' &= (\mathbf{A} + \mathbf{B})' \end{aligned}$$

  Consider an $(Z \times S)$-Matrix $\mathbf{A}$. Then

  $$\begin{aligned} \mathbf{A}\mathbf{I}_S &= \mathbf{A} \\ \mathbf{I}_Z\mathbf{A} &= \mathbf{A} \\ \mathbf{A}\mathbf{0}_S &= \mathbf{0} \\ \mathbf{0}_Z\mathbf{A} &= \mathbf{0} \end{aligned}$$

  If for the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ the respective computations are admissable, then

  $$\begin{aligned} (\mathbf{AB})\,\mathbf{C} &= \mathbf{A}\,(\mathbf{BC}) \\ (\mathbf{A} + \mathbf{B})\,\mathbf{C} &= \mathbf{AC} + \mathbf{BC} \\ \mathbf{A}\,(\mathbf{B} + \mathbf{C}) &= \mathbf{AB} + \mathbf{AC} \\ (\mathbf{A} + \mathbf{B})\,(\mathbf{C} + \mathbf{D}) &= \mathbf{AC} + \mathbf{AD} + \mathbf{BC} + \mathbf{BD} \\ (\mathbf{AB})' &= \mathbf{B}'\mathbf{A}' \\ (\mathbf{ABC})' &= \mathbf{C}'\mathbf{B}'\mathbf{A}' \end{aligned}$$

- Let λ denote a scalar. Then,

$$\lambda \mathbf{AB} = \mathbf{A}\lambda\mathbf{B} = \mathbf{AB}\lambda$$

$$\begin{aligned}\text{rank}(\mathbf{A}) \leq \min(Z, S)\\ \text{rank}(\mathbf{A}') &= \text{rank}(\mathbf{A})\\ \text{rank}(\mathbf{A}'\mathbf{A}) &= \text{rank}(\mathbf{AA}') = \text{rank}(\mathbf{A})\\ \text{rank}(\mathbf{I}_Z) &= Z\end{aligned}$$

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

Computational rules for inverse matrices:

$$\left(\mathbf{A}^{-1}\right)' = (\mathbf{A}')^{-1}$$
$$(\lambda\mathbf{A})^{-1} = \lambda^{-1}\mathbf{A}^{-1}$$

Suppose that **A**, **B**, and **C** are three arbitrary regular $(Z \times Z)$-matrices. In such a case:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \qquad \text{and} \qquad (\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$$

The *quadratic form* of the quadratic $(S \times S)$-matrix **A** is

$$\mathbf{b}'\mathbf{Ab} = \sum_{i=1}^{S}\sum_{j=1}^{S} a_{ij}b_ib_j$$

where $\mathbf{b}' = [b_1\ b_2\ ...\ b_S]$.

- Let **A** be an arbitrary $(Z \times S)$-matrix with $\text{rank}(\mathbf{A}) = S$:

$\mathbf{A}'\mathbf{A}$ is always positive definite

- Let **A** be a positive definite matrix. Then

$\mathbf{A}^{-1}$ is also positive definite

- For every positive definite $(S \times S)$-matrix **C**:

$$\text{rank}(\mathbf{C}) = S$$