**Exercises for**
**“Mathematics for Economists”**
**Trier, Winter Semester 2023**

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## Exercises for Chapter 1:

### Simple Algebra

Simplify the following terms as much as possible:

1. $5(x-3)-2x(x+y-1)-x\,(7-2y)$
2. $\dfrac{x^{a+b}}{x^{2a}}$
3. $\dfrac{y^6+2\cdot y^5+6\cdot y^4}{y^4}$
4. $\dfrac{y^2+10\cdot y+50}{(y+5)^2+25}$
5. $(3\cdot y+2\cdot x+c)^k\cdot\left(\dfrac{3\cdot y}{4}+\dfrac{x}{2}+\dfrac{c}{4}\right)^{-k}$
6. $\dfrac{K^2}{L\cdot(L+K^2/L+2\cdot K)}+\dfrac{L\cdot(L+2\cdot K)}{(L+K)^2}$

Find all values for the variable $x$ that solve the equation (consider all other elements as constants):

7. $x^2=16x$
8. $\dfrac{3x+3}{6}+b=2a$
9. $4x^2+12x-18=0$
10. $P(x^2+\dfrac{R}{P}x)+C=0$

### Equations

1. Solve the following system of equations with *two* different algebraic methods:

$$\begin{aligned}8x+25y&=100\\x+10y&=-50\end{aligned}$$

2. Assume that $x_1$ and $x_2$ are solutions to the equation

$$a\cdot x^2+b\cdot x+c=0\ .$$

Find the most compact form for $x_1+x_2$. Hint: use the quadratic formula.

3. Assume again that $x_1$ and $x_2$ are solutions to the equation

$$a\cdot x^2+b\cdot x+c=0\ .$$

Find the most compact form for $x_1\cdot x_2$. Hint: Use the quadratic formula or the rule that

$$a\cdot x^2+b\cdot x+c=0\qquad\Leftrightarrow\qquad a\,(x-x_1)\,(x-x_2)=0\ .$$

**Exercises for Chapter 2:**

**Summation**

Find closed form terms (without summation signs) for the following sums and simplify them as far as possible.

1. $$\sum_{x=1}^{n+1} x^2$$
2. $$\sum_{x=1}^{10} x + 1$$
3. $$\sum_{L=1}^{10} (L+1)$$
4. $$\sum_{a=1}^{n} 2^a$$
5. $$\sum_{b=10}^{n+11} b^3$$

Rewrite the following expressions with summation sign(s):

6. $x_1(x_1 - 1) + 2x_2(x_2 - 1) + 3x_3(x_3 - 1)$
7. $x_1(x_{11} - 1) + x_1(x_{12} - 1) + 2x_2(x_{21} - 1) + 2x_2(x_{22} - 1) + 3x_3(x_{31} - 1) + 3x_3(x_{32} - 1)$

**Sets**

Given the sets $S_1 = \{2,4,6\}$, $S_2 = \{7,2,6\}$, $S_3 = \{4,2,6\}$, and $S_4 = \{2,4\}$, which of the following statements are true?

| | | |
|---|---|---|
| (a) $S_1 = S_3$ | (d) $3 \notin S_2$ | (g) $S_1 \supset S_4$ |
| (b) $S_1 = \mathbb{R}$ | (e) $4 \notin S_3$ | (h) $\varnothing \subset S_2$ |
| (c) $8 \in S_2$ | (f) $S_4 \subset \mathbb{R}$ | (i) $S_3 \supset \{1,2\}$ |

Referring to the four sets given, find

(j) $S_1 \cup S_2$ (k) $S_4 \cap S_2 \cap S_1$ (l) $S_3 \cup S_1 \cup S_4$

**Exercises for Chapter 3:**

**Two Functions With Two Unknowns**

Assume that supply and demand in a market follow the following equations (with quantities $Q$ and price $P$):

$$\begin{aligned} \text{demand} &: \quad Q = 10 - P \\ \text{supply} &: \quad Q = \frac{P^2}{5} \end{aligned}$$

Calculate the price $P$ and the quantity $Q$ that solve these two equations. This solution represents the market equilibrium. Sketch the two functions in one diagram and indicate the equilibrium.

**Graphical Representation of Functions**

Sketch the following functions:

$$\begin{aligned} f(x) &= 20 - 2x^{0.5} \\ g(x) &= -0.25x^2 + x - 10 \end{aligned}$$

**Inversion of Functions**

Sketch the function

$$y = f(x) = \frac{2}{x-1}$$

Find the inverse of this function and denote it by $g(y)$.

**Computing With Functions**

**1.** Find the range and domain of the following two functions:

$$f(u) = e^u \qquad \text{and} \qquad g(x) = \frac{x}{x-50}$$

Find the combinations $f \circ g$ and $g \circ f$.

**2.** Find the range and domain of the following two functions:

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = 2x^2 + 2x + 2$$

**Exercises for Chapter 4**

**Differentiation**

Calculate the first and the second order derivatives of the following functions:

1. $f(x) = 20x^4 + 10x$
2. $f(x) = e^{4x} + \ln(6x)$
3. $f(x) = e^{4x} \cdot \ln(6x) + \ln(a)$
4. $f(x) = e^{\ln(4x)}$
5. $f(x) = r(k - x^2)^r$
6. $f(x) = ax^4 + x^{-1}e^{2x}$

**Exercises for Chapter 5:**

**Extreme Values**

Consider the following three functions:

$$\begin{aligned} f(x) &= (x-6)^2 - 10 \\ g(x) &= \frac{1}{3}x^3 + 0.7x^2 - x \\ h(x) &= 2x^3 - \frac{3}{4}x^4 \end{aligned}$$

For each of these functions,

**a)** calculate the first, second and third order derivatives (if possible),

**b)** calculate the function value at $x = 0$ and find out if there are points at which the dependent variable is equal to zero,

**c)** find all extreme and inflection points (both local and global), and

**d)** sketch the graph.

**Exercises for Chapter 6:**

**Partial Derivatives**

Find every first- and second-order partial derivative (including cross second-order partial derivatives) of the following functions:

$$\begin{aligned} f(x,y) &= 2x^2 + e^{4y} \\ g(x,y) &= 2y\ln(x) \\ h(x,y) &= 2xy^4 - e^{xy} \end{aligned}$$

**Production Functions As Examples for Functions With Many Variables**

**1.** The production function is given by:

$$Q(K,L) = \left(K - \frac{L}{4}\right)^{\frac{2}{3}} \cdot L^{\frac{1}{3}}$$

where $Q$ is the quantity produced while $K$ and $L$ denote the amount of labour and capital that are used in the production of $Q$. What is the degree of homogeneity?

**2.** Consider the production function:

$$Q(K,L) = a \cdot K^b L^c$$

where $Q$ is the quantity produced while $K$ and $L$ denote the amount of labour and capital that are used in the production of $Q$. Furthermore, $a$, $b$, and $c$ are parameters with $a, b, c > 0$.

**a)** Find the degree of homogeneity.

**b)** For which positive values of the parameters $(a, b, c)$ does the production function exhibit positive first-order partial derivatives but negative second-order partial derivatives?

**Exercises for Chapter 7:**

Consider the function

$$f(x, y) = x^3 + 2xy - 5x - y^2$$

Calculate the first and second order partial derivatives (if possible). Also find and classify the stationary points.

**Exercises for Chapter 8:**

## Constrained Optimization

Solve the following optimization problems with the Lagrange multiplier method. When you find more than one stationary point, compute the corresponding function values and check which of them is a maximum.

**1.** $\max f(x,y) = xy$ subject to $n(x,y) = x + 3y = 18$

**2.** $\max f(x,y) = x(y+12)$ subject to $n(x,y) = x + y = 8$

**3.** $\max f(x,y,z) = x + y + z^2$ subject to $n(x,y,z) = x^2 + y^2 + z^2 = 1$

## Revenue Maximization With Given Budget (Cost)

Suppose that you are running a factory, producing some sort of widget that requires steel as raw material. Your costs are human labour, which is € 20 per hour of labour ($h$), and the steel itself, which runs for € 160 per ton of steel ($s$). Suppose that you can sell each unit of the widget at a price of € 200 and that the production function is $q(h,s) = h^{2/3}s^{1/3}$. What is the resulting revenue function, $R(h,s)$? If your budget is € 19,200, what is the maximum possible revenue? Use the Lagrange approach.

## Utility Maximization With Two Goods

A household lives on water ($W$) and bread ($B$). The household's utility is given by the following function:

$$U(W,B) = 100WB + W + 2B$$

**(a)** Using the Lagrange multiplier method, calculate the optimal consumption plan if the household has a budget of 1000 € and water costs 2 € per unit while bread costs 4 € per unit (assume that all money not spend on water or bread is lost).

**(b)** What would happen to the consumption plan if all prices doubled?

**(c)** What would be the impact of an increase of the price of water to 8 € whereas the price of bread remains at 4 €?

**(d)** Assume that another household has the same budget, but its utility function is

$$U(W,B) = 2W + 3B$$

What would be the optimal consumption? Assume that the original price structure is still in place.

**Exercises for Chapter 9:**

**Basic Operations**

**1.** The following matrices are given:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 5 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & 4 & 4 \\ 1 & 2 & 2 \\ 5 & 3 & 2 \end{bmatrix} \qquad \mathbf{C} = [2]$$

Calculate:

(a) $\mathbf{A} + \mathbf{B}$

(b) $\mathbf{A} - \mathbf{B}$

(c) $\mathbf{A}' + \mathbf{B}'$

(d) $\mathbf{CA}$

(e) $\mathbf{BC}$

**2.** The following two matrices (vectors) are given:

$$\mathbf{A} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Calculate $\mathbf{C} = \mathbf{AB}$.

**3.** The following two matrices are given:

$$\mathbf{A} = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 7 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 7 \\ 5 & 8 \\ 6 & 9 \end{bmatrix}$$

Calculate $\mathbf{C} = \mathbf{AB}$.

**Rank of a Matrix**

Show without the use of determinants that the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 1 & -2 & 0 \end{bmatrix}$$

is a singular matrix.

**Definiteness of a Matrix**

Examine whether the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 3 \\ -1 & -2 \end{bmatrix}$$

is negative definite.

## Determinant and Inverse of a Matrix

**1.** Calculate the following determinants:

$$\text{(a)} \quad \begin{vmatrix} 5 & -2 \\ 3 & -2 \end{vmatrix} \qquad\qquad \text{(b)} \quad \begin{vmatrix} 2 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 4 & 6 \end{vmatrix}$$

**2.** Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & t \\ 2 & 1 & t \\ 0 & 1 & 1 \end{bmatrix} \qquad\qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

For which value of $t$ is matrix $\mathbf{A}$ singular? Is matrix $\mathbf{B}$ singular or regular? Suppose that $t$ is such that $\mathbf{A}$ is regular. Find a matrix $\mathbf{X}$ such that $\mathbf{B} + \mathbf{X}\mathbf{A}^{-1} = \mathbf{A}^{-1}$. Hint: "Solve" the equation for $\mathbf{X}$.

## Cramer's Rule

For what values of $t$ does the system of equations

$$\begin{aligned} -2x + 4y - tz &= t - 4 \\ -3x + y + tz &= 3 - 4t \\ (t-2)\,x - 7y + 4z &= 23 \end{aligned}$$

have a unique solution for the three variables $x$, $y$, and $z$? To answer this question, use Cramer's rule and the rule of Sarrus. Also derive the solution for $x$ (not $y$ and $z$). Hint: $5\left(t^2 - 9t + 8\right) = 5\,(t-1)\,(t-8)$ and $-21t^2 + 176t - 64 = -\,(21t-8)\,(t-8)$.