1 Theory of the Firm: Topics and Exercises

Firms maximize profits, i.e. the difference between revenues and costs, subject to technological (and other, here not considered) constraints.

1.1 Technology

Technology describes technically feasible input-output combinations. We consider single-output firms only, hence a real-valued production function $y = f(x)$ is appropriate. Note that $x = (x_1, ..., x_n)$ is an input vector while y is a scalar.

Exercise 1: Assume a two-inputs Cobb-Douglas technology, $y = x_1^\alpha x_2^{1-\alpha}$, with $\alpha \in (0, 1)$. Show that this function (1) exhibits constant returns to scale, and (2) positive but diminishing marginal products. Also demonstrate that the MRTS does depend on x_2/x_1, but not on the absolute values of factor inputs.

1.2 Costs

Input prices $w = (w_1, ..., w_n)$ are given. The firm solves:

$$c(w, y) \equiv \min wx \text{ s.t. } f(x) = y, \tag{1}$$

where y is considered as a parameter (i.e. fixed while solving (1)). Assume x^* solves (1). These inputs are called *conditional demand functions* as they depend on w and y . Technically, they are identified by the Lagrangian approach. This problem, as well as the properties of costs and conditional demand, is exactly the same as the expenditure minimizing problem on the consumers' side.

Exercise 2: Assume again a Cobb-Douglas technology, $y = x_1^\alpha x_2^{1-\alpha}$, with $\alpha \in (0, 1)$.

1. Give the first order optimality conditions using a Lagrangian approach.
2. Derive the conditional demand functions. Show that these functions are homogeneous of degree one in input prices and output.
3. Derive the cost function. Show that it is strictly concave in w.

Exercise 3: Prove Shepard's Lemma, using the same argument as we did with the expenditure function. Also prove that the cost function is concave in w.

2 Profit Maximization

There are two ways to approach the profit maximization problem: First, by use of the cost function:

$$\pi(p, w) \equiv \max (py - c(w, y)) \Rightarrow p = \frac{\partial c(w, y)}{\partial y}. \tag{2}$$

Second, by use of the production function:

$$\pi(p, w) \equiv \max (pf(x) - wx) \Rightarrow \forall i : p\frac{\partial f(x)}{\partial x_i} = w_i. \tag{3}$$

For constant returns to scale (CRS) profits are either zero or undefined. To see the argument, assume $\pi = pf(x^*) - wx^* > 0$, where x^* maximize profits. But since we assumed constant returns to scale, $\pi = pf(tx^*) - wtx^* = tpf(x^*) - wtx^* = t(pf(x^*) - wx^*) = t\pi > \pi$ for $t > 1$, which contradicts that x^* maximizes profits. In other words, if strictly positive profits are feasible, then the firm would expand production beyond any limits. In this case, capacity constraints have to be imposed to get reasonable results.

Zero-profits are also evident considering (2). With CRS, $c(w, y) = c(w, 1)y$. Hence, both marginal and average costs are given with $c(w, 1)$. But then we have $p = c(w, 1)$ as first order condition. This condition looks strange as it contains no variables under the firms control. Instead it is a condition on the parameters and prices such that (2) has an interior solution.

Since the maximum profit is zero again when prices and average costs coincide for all y, the firm is indifferent in producing anything between 0 and ∞.

Exercise 4: Assume $y = x_1^{\alpha} x_2^{1-\alpha}$, with $\alpha \in (0,1)$. Show that the profit maximization $\max(pf(x) - wx)$ does not yield a unique solution.

Solution to exercise 2:

The Lagrangian is

$$\mathfrak{L} = wx + \lambda(y - x_1^\alpha x_2^{1-\alpha}),$$

yielding the first order conditions:

$$w_1 - \lambda\alpha x_1^{\alpha-1} x_2^{1-\alpha} = 0, \text{ and } w_2 - \lambda(1-\alpha)x_1^\alpha x_2^{-\alpha} = 0,$$

and the production function. To derive the demand functions, first divide the first order conditions, such that

$$\frac{w_1}{w_2} = \frac{\lambda}{\lambda}\frac{\alpha}{(1-\alpha)}\frac{x_1^{\alpha-1}x_2^{1-\alpha}}{x_1^\alpha x_2^{-\alpha}} \Rightarrow \frac{w_1}{w_2} = \frac{\alpha}{(1-\alpha)}\frac{x_2}{x_1}. \tag{4}$$

From (4) we get

$$x_1 = \frac{w_2}{w_1}\frac{\alpha}{(1-\alpha)}x_2, \text{ and } x_2 = \frac{w_1}{w_2}\frac{(1-\alpha)}{\alpha}x_1. \tag{5}$$

We use both equations (5) to substitute for in the production function. Taking the second one (x_2) gives

$$y = x_1^\alpha \left(\frac{w_1}{w_2}\frac{(1-\alpha)}{\alpha}\right)^{1-\alpha} x_1^{1-\alpha}, \text{ hence } y = \left(\frac{w_1}{w_2}\frac{(1-\alpha)}{\alpha}\right)^{1-\alpha} x_1, \tag{6}$$

which finally gives

$$x_1 = \left(\frac{w_2}{w_1}\frac{\alpha}{(1-\alpha)}\right)^{1-\alpha} y, \tag{7}$$

and

$$x_2 = \left(\frac{w_2}{w_1}\frac{\alpha}{(1-\alpha)}\right)^{-\alpha} y. \tag{8}$$

These are the conditional demand functions. Finally, we multiply these with input prices and add up to arrive at the cost function:

$$c(w,y) = \left(w_1 \left(\frac{w_2}{w_1} \frac{\alpha}{(1-\alpha)} \right)^{1-\alpha} + w_2 \left(\frac{w_2}{w_1} \frac{\alpha}{(1-\alpha)} \right)^{-\alpha} \right) y. \tag{9}$$

This can be simplified to

$$c(w,y) = w_1^{\alpha} w_2^{1-\alpha} \left(\left(\frac{\alpha}{(1-\alpha)} \right)^{1-\alpha} + \left(\frac{\alpha}{(1-\alpha)} \right)^{-\alpha} \right) y, \tag{10}$$

and finally

$$c(w,y) = w_1^{\alpha} w_2^{1-\alpha} \left(\left(\frac{1}{1-\alpha} \right) \left(\frac{\alpha}{(1-\alpha)} \right)^{-\alpha} \right) y. \tag{11}$$

By use of the envelope theorem, we get Shephard's Lemma:

$$\frac{\partial c(w,y)}{\partial w_i} = x_i(w,y). \tag{12}$$

For a linear homogeneous production function, the cost function is linear in y, with

$$c(w,y) = c(w,1)y. \tag{13}$$

In this case, average and marginal costs coincide for all y and w.

Solution to exercise 4:

Taking derivatives gives

$$p\frac{\partial f(x)}{\partial x_1} = p\alpha x_1^{\alpha-1} x_2^{1-\alpha} = p\alpha \left(\frac{x_2}{x_1}\right)^{1-\alpha} = w_1, \tag{14}$$

and

$$p\frac{\partial f(x)}{\partial x_2} = p(1-\alpha) x_1^{\alpha} x_2^{-\alpha} = p(1-\alpha) \left(\frac{x_2}{x_1}\right)^{-\alpha} = w_2. \tag{15}$$

Dividing these optimality conditions gives the well-known result that the MRTS has to be equal to the factor price ratio:

$$\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{x_2}{x_1}\right) = \frac{w_1}{w_2}. \tag{16}$$