Price and Quantity Indices as Role Models for Composite Indicators

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Types of measures

- Measuring change (intertemporal).
- Measuring difference (interspatial).
- Ratio-type measure (p/p’) will be called *index*.
- Difference-type measure (p-p’) will be called *indicator*.

Three approaches for measuring aggregate price or quantity change (1)

- *Economic approach*.
- Keywords: preference order (consumer) or technology (producer); optimization; decomposition of value change.
- Literature: CPI Manual (ILO, Geneva, 2004) or PPI Manual (IMF, Washington DC, 2004).

- *Stochastic approach*.
- Keywords: price change = common component + specific component + random component; estimation.
- Literature: Clements, Izan and Selvanathan, International Statistical Review 2006.

Three approaches for measuring aggregate price or quantity change (2)

- *Axiomatic approach*.
- Keywords: properties; tests; functional equations.
- Literature: Balk, International Statistical Review 1995.

Notation

- Commodities $n = 1,\ldots,N$
- Periods $t = 0,1,\ldots,T$
- Prices p_n^t
- Quantities x_n^t
- Single price indices p_n^1/p_n^0 $(n = 1,\ldots,N)$
- Single quantity indices x_n^1/x_n^0 $(n = 1,\ldots,N)$

Price index $P(p^1, x^1, p^0, x^0)$: axioms

- **A1: Monotonous in prices.**
- **A2: Linearly homogeneous in period 1 prices: $P(\lambda p^1, x^1, p^0, x^0) = \lambda P(p^1, x^1, p^0, x^0)$ $(\lambda>0)$.**
- **A3: Identity test: $P(p^0, x^1, p^0, x^0) = 1$.**
- **A4: Homogeneous of degree 0 in prices: $P(\lambda p^1, x^1, \lambda p^0, x^0) = P(p^1, x^1, p^0, x^0)$ $(\lambda>0)$.**
- **A5: Invariant to changes in units of measurement: $P(p^1\Lambda, x^1 \Lambda^{-1}, p^0 \Lambda, x^0 \Lambda^{-1}) = P(p^1, x^1, p^0, x^0)$ (Λ diagonal with positive elements).**

Quantity index $Q(p^1, x^1, p^0, x^0)$: axioms

- **A1': Monotonous in quantities.**
- **A2': Linearly homogeneous in period 1 quantities:** $Q(p^1, \lambda x^1, p^0, x^0) = \lambda Q(p^1, x^1, p^0, x^0)$ $(\lambda>0)$.
- **A3': Identity test:** $Q(p^1, x^0, p^0, x^0) = 1$.
- **A4': Homogeneous of degree 0 in quantities:** $Q(p^1, \lambda x^1, p^0, \lambda x^0) = Q(p^1, x^1, p^0, x^0)$ $(\lambda>0)$.
- **A5': Invariant to changes in units of measurement:** $Q(p^1 \Lambda, x^1\Lambda^{-1}, p^0\Lambda, x^0\Lambda^{-1}) = Q(p^1, x^1, p^0, x^0)$ **(Λ diagonal with positive elements).**

Some implications

- **A2 and A3 imply A6: Proportionality:**
  $P(\lambda p^0, x^1, p^0, x^0) = \lambda \quad (\lambda>0)$.

- **A1 and A6 imply A7: Mean value property:**
  $\min\{p_n^1/\ p_n^0\} \leq P(p^1, x^1, p^0, x^0) \leq \max\{p_n^1/\ p_n^0\}$.

- **A2' and A3' imply A6'.**

- **A1' and A6' imply A7'.**

Important tests

- T1: Circularity (transitivity) : $P(p^2, x^2, p^1, x^1) \times P(p^1, x^1, p^0, x^0) = P(p^2, x^2, p^0, x^0)$.
- T2: Time reversal: $P(p^1, x^1, p^0, x^0) = 1/P(p^0, x^0, p^1, x^1)$.
- T1' and T2' similarly.
- T3=T3': Product: $P(p^1, x^1, p^0, x^0) \times Q(p^1, x^1, p^0, x^0) = V^1/V^0$ (value ratio).
- T4=T4': Factor reversal: $P(p^1, x^1, p^0, x^0) \times P(x^1, p^1, x^0, p^0) = V^1/V^0$ or $Q(x^1, p^1, x^0, p^0) \times Q(p^1, x^1, p^0, x^0) = V^1/V^0$.

Other tests

- **T5 and T5’: Value dependence.**
- **T6 and T6’: Consistency-in-aggregation.**
- **T7 and T7’: Equality.**
- **….**

Main areas of research in the axiomatic approach

- **Consistency of combinations of axioms and tests.**

- **Characterization of index formulas by axioms and tests.**

An important example (1)

- T1 (transitivity) implies that $P(p^1, x^1, p^0, x^0) = g(p^1, x^1)/ g(p^0, x^0)$.
- A3 (identity) implies that $g(p, x) = g(p)$.
- A5 (invariance to u-o-m) implies that $g(p)/g(1)$ is a multiplicative function.
- A1 (monotonicity) implies (Aczél 1966) that $g(p)/g(1) = \prod_n p_n^{\alpha_n}$ for some $\alpha_n > 0$.
- A2 (linear homogeneity) implies that $\sum_n \alpha_n = 1$.

An important example (2)

- **Conclusion: A1, A2, A3, A5 and T1 imply that $P(p^1, x^1, p^0, x^0) = \prod_n (p_n^1 / p_n^0)^{\alpha_n}$ with $\alpha_n > 0$ and $\sum \alpha_n = 1$.**

- **This is known as the Cobb-Douglas (price) index.**

- **The α_n are constants.**

- **In practice one chooses α_n to be value share of period 0 or 1 or average → T1 violated.**

An important example (3)

- **Similarly, A1', A2', A3', A5' and T1' imply that $Q(p^1, x^1, p^0, x^0) = \prod_n (x_n^{\,1}/\, x_n^{\,0})^{\beta n}$ with $\beta_n > 0$ and $\sum \beta_n = 1$.**
- **The β_n are (other) constants.**
- **In general $\prod_n (p_n^{\,1}/\, p_n^{\,0})^{\alpha n} \times \prod_n (x_n^{\,1}/\, x_n^{\,0})^{\beta n} \neq V^1/V^0$; that is, T4 violated.**

Application (1)

- Re-interpret (p, x) as vector of attributes, with p measured (as positive real variables) and x un-measured.
- Composite index compares situation 1 to situation 0: $I(p^1, x^1, p^0, x^0)$.
- Invariance to units of measurement: $(p\Lambda, x)$ describes the same situation as (p, x).
- Or, $I(p^1\Lambda, x^1, p^0\Lambda, x^0) = I(p^1, x^1, p^0, x^0)$.

Application (2)

- *Transitivity*: $I(p^2, x^2, p^1, x^1) \times I(p^1, x^1, p^0, x^0) = I(p^2, x^2, p^0, x^0)$. This implies that $I(p^1, x^1, p^0, x^0) = g(p^1, x^1) / g(p^0, x^0)$.
- *Identity*: $I(p^0, x^1, p^0, x^0) = 1$. This means that unmeasured variables are not relevant for comparing situations.
- Transitivity + Identity implies that $g(p^0, x^1) = g(p^0, x^0)$, thus $g(p, x) = g(p)$.
- Or, $I(p^1, x^1, p^0, x^0) = g(p^1) / g(p^0)$.

Application (3)

- *Invariance* to u-o-m then implies that $g(p^1\Lambda) / g(p^0\Lambda) = g(p^1) / g(p^0)$; that is, $g(p) / g(1)$ is a multiplicative function.
- If $g(p)$ is strictly *increasing*, which is a natural assumption, then Aczél's (1966) result implies that $I(p^1, x^1, p^0, x^0) = \prod_n (p_n^1 / p_n^0)^{\alpha_n}$ for some $\alpha_n > 0$.
- Imposing *linear homogeneity* means imposing that $\sum_n \alpha_n = 1$.

Conclusion

- **Requirements like transitivity, identity, and invariance to u-o-m have implications for the form of the composite index.**

- **Theory, however, does not tell us how to choose the weights α_n of the single indices p_n^1/p_n^0 .**