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The Inverse Relation of Addition and Subtraction: A Knowledge Integration Perspective

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A number of mathematical concepts and computational procedures are linked to the inverse relation of addition and subtraction on an abstract mathematical level. In this discussion article for the special issue on subtraction-related principles, we suggest that the mainstream of research on inversion is conducted from a *Knowledge Dissociation Perspective* in which researchers show that children often fail to see abstract relations in the domain. Implicit rationale of the studies is that seeing these inter-relations occurs naturally and that we have to find the cognitive processes that can explain exceptions from this rule. Based on a review of key findings from cognitive research on knowledge acquisition we argue that a *Knowledge Integration Perspective* would be more adequate in which children acquire different concepts and procedures in different situations. Only experts but not children who are new in a domain can see their abstract mathematical inter-relations. Thus, research should shift its focus from merely describing dissociations between children's concepts or procedures to looking for causal mechanisms and instructional approaches that help children to integrate their knowledge by seeing the underlying deep structures of mathematical principles and problems.

By taking the viewpoint of cognitive psychology, we discuss the five studies included in this special issue on *Young Children's Understanding and Application of Subtraction-Related Principles* together with other landmark studies in this field. Our line of argumentation is (a) mainstream research points out that there are a multitude of concepts and procedures that are related to the inverse relation of addition and subtraction on a theoretical (e.g., mathematical, abstract) level. (b) However, empirically, researchers focus on showing that children fail to see these abstract inter-relations. (c) We call this prevailing viewpoint in the literature the *Knowledge Dissociation Perspective*—children are expected to discover abstract relations between different concepts or procedures on their own. Empirical research should uncover exceptions from this rule. (d) However, studies on the cognitive processes underlying knowledge acquisition paint a different picture. Learners frequently fail to see abstract relations in a domain. This is the rule rather than the exception. The causes for this are already well understood. (e) Therefore, future research on inversion should focus on what we call the *Knowledge Integration Perspective*. We discuss the theoretical, methodological, and educational implications of this new viewpoint.

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THE KNOWLEDGE DISSOCIATION PERSPECTIVE

Inversion as Highly Relational Content Domain

The relations between addition and subtraction are a content domain, which involves a variety of concepts, procedures, and everyday life experiences. Subsequently, we use the generic term *pieces of knowledge* (cf. diSessa, 1988) to refer to these different concepts, procedures, and everyday life experiences. An example of a concept in this domain is the *inversion principle* ($a + b - b = a$, because addition and subtraction of the same number cancel each other out). An example of a procedure is the *shortcut strategy*. The problem $7 + 2 - 2$ can be solved by stating 7 as answer without carrying out any calculations. An example for a related everyday life experience could be recognizing that putting 2 cookies in a jar and later taking out 2 other cookies leaves the initial amount of cookies unchanged.

At an abstract mathematical level, the pieces of knowledge in the domain of inversion are not independent of each other but are highly inter-related. In case a child sees these abstract inter-relations, holding one piece of knowledge might help the child to infer or to better understand other pieces of knowledge. The literature on inversion is full of hypotheses about what piece of knowledge might aid the acquisition of what other pieces.

For example, intuitions about approximate numerical magnitudes might help children to solve inversion problems (Gilmore & Spelke, 2008). Experiences with inversion in everyday life might help children to later grasp inversion in mathematics education. A qualitative understanding of inversion (i.e., making initially clean clothes dirty and then cleaning them puts them back into their original state) might be the source of a subsequent quantitative understanding (Sherman & Bisanz, 2007). Understanding inversion on the level of concrete objects (e.g., numbers of cookies) could be the foundation for understanding inversion on the level of symbolic arithmetic (e.g., $5 + 3 - 3 = 5$), which, in turn, might help to later gain the algebraic insight that the shortcut strategy is valid independently of numerical values (i.e., $a + b - b = a$; cf. Bisanz et al., this issue). The inversion principle could help to acquire related principles and vice versa (Baroody, Lai, & Baroody, this issue), for instance the *subtractive negation principle* ($a - a = 0$), the *subtractive identity principle* ($a - 0 = a$), and the *complement principle* ($a + b = c$ is equivalent to $a = c - b$). Children's conceptual understanding of the inversion principle might also help them to acquire the shortcut strategy and, thus, improve their general arithmetic competence (cf., Gilmore & Papadatou-Pastou, this issue).

In all, the domain of inversion comprises everyday life experiences, intuitions about magnitudes, computational procedures, and mathematical principles. All these can be expressed by using objects, pictures of objects, concrete numbers, or algebraic variables. Holding some of these pieces of knowledge might help children to acquire related pieces of knowledge. Ideally, this process would help children to develop a general sense for mathematics as a system of abstract inter-relations (Baroody et al., this issue; Nunes, Bryant, Bell, Evans, & Hallett, this issue) and facilitate learners' flexible and adaptive use of alternative problem-solving strategies (Torbeyns, De Smedt, Stassens, Ghesquière, & Verschaffel, this issue).

Empirical Findings Indicating Knowledge Fragmentation

However, the actual empirical findings do not live up to these expectations. As described, many authors discuss *inter-relations* between different pieces of knowledge. However, their empirical findings comprise a long list of *dissociations* (i.e., significant mean differences, low inter-correlations, or an unexpected developmental ordering) of tasks assessing different pieces of knowledge in the domain. Often children hold one piece of knowledge without demonstrating any knowledge about closely related pieces. Even when children hold pieces of knowledge simultaneously, they may fail to see how they relate to each other on an abstract mathematical level. In this sense, children's knowledge about inversion can be characterized as *fragmented* (cf. diSessa, Gillespie, & Esterly, 2004) to some degree. In the following subsection we give a short overview of empirical findings that indicate this partial fragmentation of children's knowledge.

Dissociations Between Concepts and Procedures

In their meta-analysis of 14 studies with children aged 3 to 13 Gilmore and Papadatou-Pastou (this issue) show that there are groups of children with (a) a bad understanding of inversion and low arithmetic skill, (b) a good understanding of inversion and high arithmetic skill, or (c) good understanding of inversion and bad arithmetic skill. Two additional studies (Canobi, 2005; Watchorn et al., 2007, cited by Bisanz et al., this issue) suggest that even a fourth group exists: children with a bad understanding of inversion but good arithmetic skills. A related line of research found no evidence for relations between elementary school children's understanding of inversion and their arithmetic skills in factor analyses (Bryant, Christie, & Rendu, 1999). Same-aged children who apparently understand the inversion principle do not use this knowledge to adaptively choose between alternative subtraction strategies (Torbeyns et al., this issue). Taken together, these findings suggest that children's conceptual understanding of inversion and procedural skill in arithmetic are largely unrelated at least at some points during their development.

Dissociations Between Concepts

Some studies show that children acquire mathematically inter-related concepts independently of each other and do not link them in their mind. Developmental studies with 3 and 4 year olds yielded no evidence that children's quantitative understanding of the concept of inversion is based on a precursory qualitative understanding of inversion (Rasmussen, Ho, & Bisanz, 2003; Sherman & Bisanz, 2007). Baroody et al. (this issue) demonstrated a developmental precedence of understanding the subtractive negation principle and the subtractive identity principle (reliably mastered at an age of 4 years) relative to understanding the inversion principle (reliably mastered at an age of 6 years, when Baroody's method is used), although all these principles are closely related mathematically (e.g., $a + b - b = a$ implies that $b - b = 0$).

Dissociations Between Procedures

Neither object-counting procedures nor abstract counting procedures contributed to performance on inversion problems during the preschool years (Rasmussen et al., 2003; Sherman & Bisanz, 2007). Knowledge of the shortcut strategy for inversion problems and computational competence on other arithmetic problems is largely uncorrelated for 6 year olds (Bryant et al., 1999). Changing the position of the missing number or the order of operands in inversion problems significantly changed solution rates in a sample of 9 year olds, suggesting that these very similar versions of the task each tap different pieces of children's procedural knowledge (Gilmore, 2006). Children who just discovered the shortcut strategy often use it inconsistently (i.e., only on some of all inversion tasks) for a while. Only after using the strategy behaviorally consistently for some time do the children start to report their knowledge of the strategy verbally, suggesting that inconsistent use of the shortcut strategy, consistent use of the shortcut strategy, and explicit (i.e., verbalizable) knowledge of the strategy each reflect different pieces of children's procedural knowledge (Siegler & Stern, 1998).

Theoretical Rationale of the Knowledge Dissociation Perspective

Many authors suggest, based on theoretical analyses, that the domain of inversion is highly relational and, thus, a valuable field of learning for students. However, on the empirical level the authors find more and more dissociations between pieces of knowledge. At first glance, this seems paradoxical: Is knowledge on inversion highly relational or is it fragmented?

The apparent paradox dissolves when the argumentation in most studies on inversion is taken into account. In our impression, almost all studies are based on the following argumentation. There are two tasks that are both related to inversion on a theoretical level. Children can be expected to see the relations between our tasks. We show on the empirical level that children have high knowledge about one task but not about the other or that children fail to see relations between the two tasks. This is a noteworthy finding because it points to cognitive mechanisms that prevent children from seeing the relations between the tasks and, thus, between the underlying concepts, principles, and everyday life experiences.

We call this approach to research on inversion the *Knowledge Dissociation Perspective*. It is the mainstream perspective in research on inversion. Two articles included in this special issue seem to back up this hypothesis. Gilmore and Papadatou-Pastou (this issue) conducted a meta-analysis with 14 studies that all investigate dissociations between understanding of the inversion principle and arithmetic competence. Both the number of the single studies and the fact that the only meta-analysis in research on inversion focuses on this topic suggest that many researchers seem to see this dissociation as one of the central findings in the field so far.

Even more compelling evidence comes from the theoretical framework provided by Bisanz and colleagues (this issue). After reviewing the literature, they point out that an understanding of inversion is not an all-or-none process. Many studies show that children frequently demonstrate understanding of inversion on some tasks but not on others. These patterns of dissociations change over time. They conclude that a matrix is needed to characterize, for each child and each point in time, what pieces of knowledge in this domain the child holds. The matrix suggested by Bisanz and colleagues has 15 cells organized along different dimensions. Each cell represents a piece of knowledge (although Bisanz does not use this term), which has been shown empirically to be partially independent of other pieces of knowledge in this domain.

This illustrates our point in this article. So far, mainstream research on inversion was implicitly conducted from a Knowledge Dissociation Perspective. Researchers were looking for pieces of knowledge that are partly independently of each other in the children's minds. Therefore, the empirical research results so far can best be summarized in a table that lists a partly independent piece of knowledge in each cell. In the following sections we suggest an alternative approach to educational research about inversion.

THE KNOWLEDGE INTEGRATION PERSPECTIVE

Cognitive Mechanisms of Knowledge Acquisition: Empirical Findings

Different lines of cognitive research on knowledge acquisition converge in showing that the fragmentation of knowledge (i.e., not seeing abstract inter-relation between pieces of knowledge) is a normal phenomenon that occurs frequently and for good reasons. We review findings from four areas of research: (a) representation of conceptual knowledge, (b) representation of procedural knowledge, (c) knowledge transfer, and (d) differences between experts and novices.

Representation of Conceptual Knowledge

Conceptual knowledge of experts is highly relational. Accordingly, cognitive theories assume that concepts are mentally represented in a network of inter-related nodes (e.g., Anderson & Schunn, 2000). In this view, learning can consist in changing the content of a node, add a new node, add a new edge to the network, or modify an existing edge (diSessa & Sherin, 1998). The network is stored in long-term memory, where it is not conscious. Only the very limited number of nodes necessary to solve a task is activated and, thus, becomes consciously accessible in working memory (Anderson & Schunn, 2000).

Working memory has a limited capacity and can only hold a few nodes at a time (Kane et al., 2004). This is a very useful mechanism. If a learner would consciously access all concepts and other pieces of knowledge stored in his or her long term memory at the same time, he or she would drown in a flood of—largely task-irrelevant—information.

However, the mechanism also has a drawback. When a learner accesses a concept, he or she can only see how it relates to the other pieces of knowledge active in working memory but not to all pieces of knowledge stored in long-term memory. This explains why a child can hold two closely related concepts or even two contradictory concepts in long-term memory without noticing their inter-relations. When a child acquires a new concept and the respective node is inserted into long-term memory, the node is not automatically connected to all relevant nodes in the network. These edges only emerge when a child loads both pieces of information into working memory and actively reasons about their inter-relation (diSessa, 1988, 1993).

Representation of Procedural Knowledge

As pointed out by Siegler (1996), procedural knowledge (i.e., problem-solving strategies) is not one-dimensional and homogeneous. People simultaneously hold alternative strategies for solving a problem. This leads to the intra-personal variability of strategy use—when the same person solves the same problem repeatedly, he or she may use different strategies on different trials. It is assumed that alternative problem-solving strategies are stored independently of each other in long-term memory so that they do not interfere with each other during execution (Anderson & Schunn, 2000). This implies that we cannot meaningfully characterize a child's procedural knowledge by only saying whether he uses a strategy. Instead, a number of different parameters are necessary to describe the breadth and efficiency of a person's strategy repertoire (Siegler & Lemaire, 1997). Siegler (1996) showed that this multifaceted nature of procedural knowledge decisively contributes to further learning and development. The more alternative strategies a person tries out and is experienced with, the more the person subsequently learns in a domain.

In sum, conceptual and procedural knowledge are each composed of different pieces of knowledge that are often not inter-related. There are good reasons for this partial knowledge fragmentation.

Problems with Transfer

Supporting evidence comes from research on transfer of knowledge across different contexts (i.e., problems, situations, content domains). Many studies investigated under what circumstances children apply a piece of knowledge acquired in one context to solve structurally similar problems in other contexts. The results were deflating (cf. Greeno & The Middle School Mathematics Through Applications Project Group, 1998). In his introduction to a book about transfer, Detterman (1993, p. 15) concluded: "First, most studies fail to find transfer. Second, those studies claiming transfer can only be said to have found transfer by the most generous of criteria and would not meet the classical definition of transfer. . . . In short, transfer is rare, and its likelihood of occurrence is directly related to the similarity between two situations." So knowledge is always acquired in a certain context (i.e., from a teacher, from friends, from books, from experiences during playing, etc.). Learners often fail to see abstract similarities between different contexts. This hampers transfer and can lead to partially fragmented knowledge (Wagner, 2006).

Expert-Novice Differences

Somewhat more encouraging results come from cognitive research on differences between experts (who are experienced and competent in a domain) and novices. Unlike novices, experts in a domain can abstract pieces of knowledge from the context in which they were acquired (Chi, Feltovich, & Glaser, 1981; Koedinger & Anderson, 1990). Through long processes of knowledge acquisition, abstraction, and restructuring they learned to distinguish between circumstantial superficial characteristics of a situation (the so-called surface structure) and abstract meaningful characteristics of a situation (the so-called deep-structure; e.g., Gentner & Toupin, 1986; Kercood, Zentall, & Lee, 2003). This helps experts to see abstract relations between pieces of knowledge acquired in different situations and to, thus, integrate their knowledge.

Theoretical Rationale of the Knowledge Integration Perspective

Basic Assumptions

The Knowledge Dissociation Perspective is based on the implicit assumption that it is the rule that children see abstract relations between pieces of knowledge and that we have to uncover the exceptions from this rule. As we have argued, this perspective is implausible for two reasons. First, empirical findings show that dissociations between pieces of children's knowledge about inversion are more frequently found than inter-relations. Second, cognitive theories of knowledge acquisition show that this is necessarily the case. The nature of human cognition implies that novices often do not see abstract relations between pieces of knowledge acquired in different situations.

For these reasons we suggest that an alternative approach for the investigation of children's learning about inversion might be more productive in the long run. Following Linn (2006) and drawing on Baroody (2003) and Izsák (2005), we call this approach the *Knowledge Integration Perspective*. It is based on three assumptions: First, children's knowledge is composed of different pieces that can be inter-related or fragmented with different degrees. Second, children (and novices in general) often fail to see the deep structure underlying different pieces of knowledge. This is what distinguishes novices from experts in a field. Third, although novices' knowledge is fragmented initially, they can learn to integrate their knowledge and, thus, become experts in a field. This requires focusing on the abstract deep structures underlying superficially different contexts.

Implications

This perspective has a direct implication for educational research: When it is already known that it is difficult for children to spontaneously see how different pieces of knowledge are inter-related, then it is of limited use to search for more and more pieces of this knowledge (concepts, procedures, everyday life experiences) and to simply list them. Instead, we should find out how we can help children to see these relations and to, thus, integrate their initially fragmented knowledge. Baroody (2003) already made a strong point for integrating children's concepts and procedures. We take this one step further by arguing that pieces of knowledge in general need to be integrated. Thus, important questions for further research are: What pieces of knowledge causally affect other pieces of knowledge and, thus, foster knowledge integration? What developmental mechanisms contribute to children's restructuring of knowledge? What educational interventions can help children to integrate their knowledge?

The prevailing Knowledge Dissociation Perspective led to results best summarized in a table with each cell standing for a partly independent piece of knowledge. In contrast, the newly suggested Knowledge Integration Perspective would ideally lead to a boxes-and-arrows model, with the boxes standing for cognitive structures and pieces of knowledge and the arrows standing for causal relations' contribution to the transformation and integration of this knowledge. Knowing which causal factors influence the content and structure of children's knowledge would be much more helpful for educators than a mere list of potentially independent pieces of knowledge.

Research conducted to date has not uncovered many of these relations. Causality can only be investigated by means of pretest-posttest designs with at least one treatment group, a control group, and a randomized assignment of children to the groups. Instead of conducting these studies, prior research largely made use of cross-sectional designs, which do not allow for tests of hypotheses about causation. So while both perspectives emphasize the importance of knowledge integration on a theoretical level, this integration has not been investigated empirically under the prevailing Knowledge Dissociation Perspective. In contrast, the Knowledge Integration Perspective suggests that this is one of the most important fields for empirical research.

Plausibility of the Perspective

At first glance, the Knowledge Integration Perspective might seem strange to mathematics educators who are well trained to see mathematics as a system of abstract inter-relations. However, as described above, it is important to keep in mind that the same domain might appear

very different to experts and novices. We have to be careful to not superimpose the view that we as researchers have on children who are just in the process of entering this domain. Getting clothes dirty and cleaning them again; putting 2 cookies in a jar while taking 2 others out; the fact that $7 - 7$ equals 0; the equation $a + b - b = a$; and so on—for us it is clear what they all have in common, but it is really not surprising that children do not see this as easily as we do.

This struggle for knowledge integration prevails not only in individual learning histories but also in the history of mathematics itself. For us it is obvious what the equation $y = a * x + b$ and a straight line in a coordinate system have to do with each other, but for many centuries many great mathematicians did not see this. Some of the greatest mathematical achievements of all times consisted in working out the abstract relations between two hitherto independent ideas or areas (Boyer & Merzbach, 1991).

Further support for the Knowledge Integration Perspective comes from research on conceptual change in science learning. Mathematics and natural sciences differ in content knowledge, and children's learning processes in the two subjects are surely not identical. On the other hand, children have only one brain, which they use to learn and process information in all content domains. If the reasons for children's initially fragmented and later progressively integrated knowledge really lie in the architecture of the human cognition (e.g., the different functions of long-term memory and working memory) they should be found independently of the content domain (Vosniadou & Verschaffel, 2004). In accordance with this, research on children's conceptual change in scientific domains found that children's initial knowledge in a domain can be fragmented and is likely to be integrated over time (cf. Clark, 2006; Linn, 2006), although there are also critical voices and open questions for further research (cf. Vosniadou, 1999).

CONCLUSION

The articles in this special issue are not only interesting from the prevailing Knowledge Dissociation Perspective but also from the long neglected Knowledge Integration Perspective. Gilmore and Papadatou-Pastou (this issue) and Bisanz et al. (this issue) integrate findings from a variety of different studies by discussion or meta-analysis; Baroody et al. (this issue) look at broad patterns of behavior across a variety of tasks; and Nunes et al. (this issue) conducted one of the critically needed intervention studies on inversion. Torbeyns et al. (this issue) report another intervention study and even compare the competencies of novices (elementary school children) and experts (young adults). The fact that the studies in this special issue often relate to the Knowledge Dissociation Perspective but are also highly interesting from a Knowledge Integration Perspective might indicate that research on inversion is gradually moving to a change in theoretical perspective.

Future empirical research should explicitly search for ways to integrate children's fragmented knowledge (Hammer, 1996; Linn, 2006). Instructional approaches that have been shown to foster knowledge transfer might be useful for this because they help children to see abstract similarities between superficially different contexts (cf. Bereiter, 1997; Novick & Holyoak, 1991; Wagner, 2006). Examples for this (from other content domains) are diagrams that illustrate the abstract structures underlying different contexts (Novick & Hmelo, 1994; Stern, Aprea, & Ebner, 2003) and curricula structuring children's knowledge by providing meaningfully structured learning environments (Hardy, Jonen, Möller, & Stern, 2006).

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