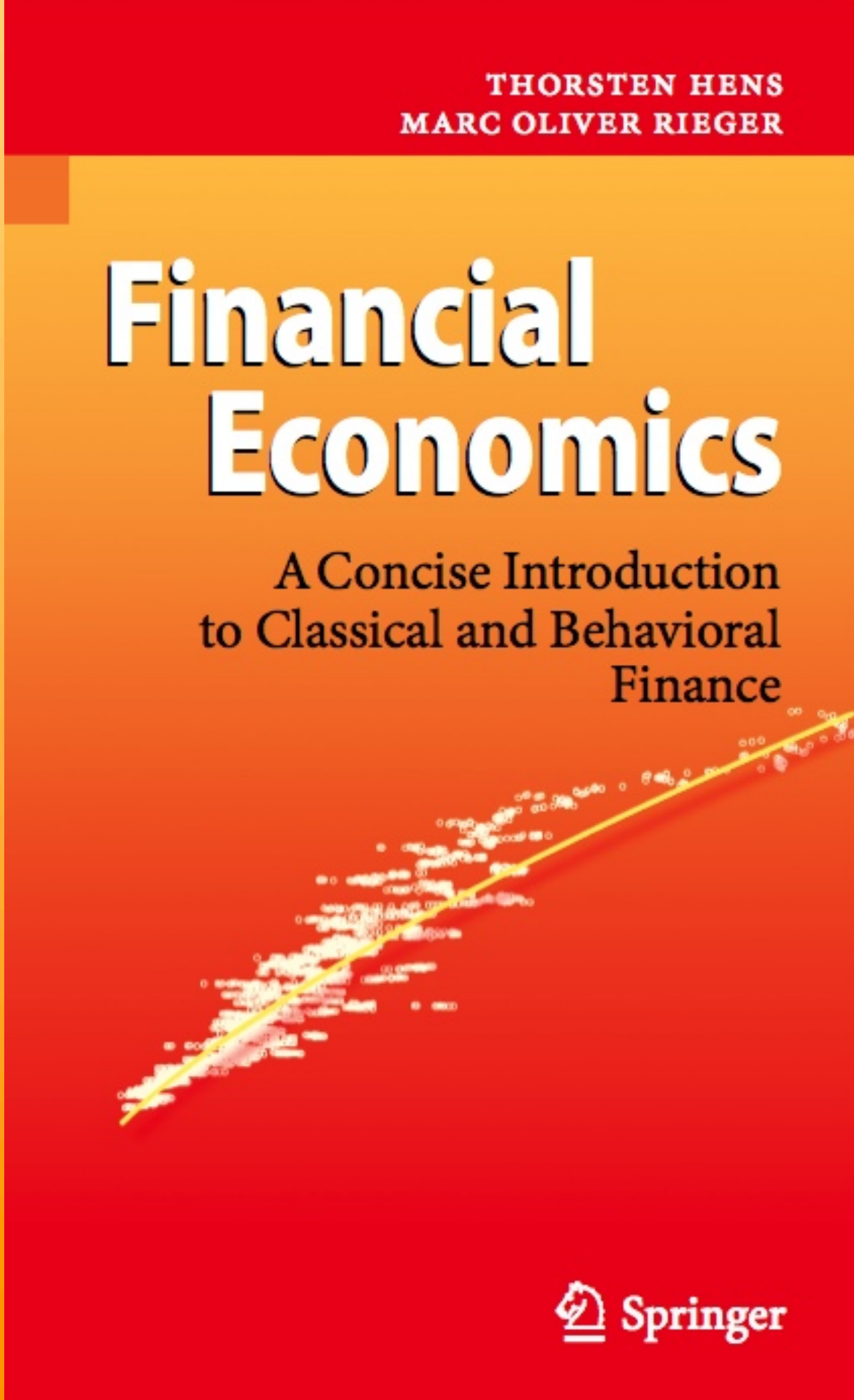

HENS · RIEGER

Financial Economics

Financial economics is a fascinating topic where ideas from economics, mathematics and, most recently, psychology are combined to understand financial markets. This book gives a concise introduction into this field and includes for the first time recent results from behavioral finance that help to understand many puzzles in traditional finance. The book is tailor made for master and PhD students and includes tests and exercises that enable the students to keep track of their progress. Parts of the book can also be used on a bachelor level. Researchers will find it particularly useful as a source for recent results in behavioral finance and decision theory.

The text book to this class is available at www.springer.com

On the book's homepage at www.financial-economics.de there is further material available to this lecture, e.g. corrections and updates.

Financial Economics

A Concise Introduction to Classical and Behavioral Finance
Chapter 5

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Multiple-Periods Model

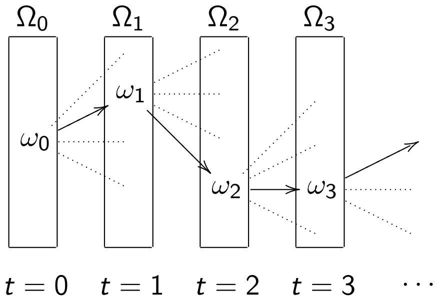
"It will fluctuate." John P. Morgan's reply, when asked what the stock market will do.

General Equilibrium Model

Lucas Tree Model

- $t = 0, 1, 2, \ldots, T$
- tree-like extension of two-period model
- $\omega_t \in \Omega_t$: finite set of realized *states* in each t
- $\omega^t = (\omega_0, \omega_1, \ldots, \omega_t)$: path of state realizations

Event Tree:



Model Setup (1)

- P: probability measure determining the occurrence of states
- P is defined over the set of paths
- We use P to model the exogenous dividends process
- If realizations are independent over time, P product of probabilities associated with building the vector $\omega^t = (\omega_0, \omega_1, \ldots, \omega_t)$.

Model Setup (2)

- Payoffs are determined by dividend payments and capital gains in every period
- $i = 1, \dots, I$: investors
- $k = 1, \dots, K$: long-lived assets in unit supply
  $k = 0$: consumption good
- Not every node in the tree has to result in a payoff

Example

Company paying out dividends once a year, but having quarterly earning reports

Perfect Foresight

- Assume always perfect foresight:
  Conditionally on the events, all investors agree on the prices
- Model is still flexible enough to accomodate different opinions: just split states into sub-states whenever some investors disagree about the prices in the original state.

Competitive Equilibrium (1)

In a competitive equilibrium with perfect foresight every investor decides about his portfolio strategy according to his consumption preferences over time.

Competitive Equilibrium (2)

Definition

A competitive equilibrium with perfect foresight is a list of portfolio strategies θ^i_t, $i = 1, \ldots, I$ *and a sequence of prices* q^k_t, $t = 0, 1, \ldots, T$ *s.t. for all* $i = 1, \ldots, I$

$$(\theta^i_0, \ldots, \theta^i_T) \in \underset{\substack{\theta^i_t \in \mathbb{R}^{K+1} \\ t=0,\ldots,T}}{\arg\max}\, U^i\left(\theta^{cons}\right) \qquad s.t.$$

$$\theta^{cons}_t + \sum_{k=1}^{K} q^k_t \theta^k \overset{!}{=} \sum_{k=1}^{K} \left(D^k_t + q^k_t\right) \theta^{i,k}_{t-1} + w^i_t, \quad \theta^{cons}_t \geq 0,$$

for all $t = 0, \ldots, T$, *where* D^k_t *are the total dividend payments of asset* k, *and markets clear:*

$$\sum_{i=1}^{I} \theta^{i,k}_t \overset{!}{=} 1, \qquad for\ all\ k\ and\ all\ t.$$

Competitive Equilibrium (3)

- $\theta_t^{i,k}(\omega^t) \in \mathbb{R}$: number of assets k that agent i has in period t given the path ω^t
- $\left(\theta_t^{i,k}(\omega^t)\right)_{k=0,\ldots,K} \in \mathbb{R}^{K+1}$: portfolio of assets that agent i has in period t given the path ω^t
- $\left(\theta_t^{i,k}(\omega_t)\right)_{\omega_t \in \Omega_t} \in \mathbb{R}^{|\Omega_t|}$: vector of asset k holdings across the states $\omega_t \in \Omega_t$
- $\theta_t^{i,k}$, $t = 0, 1, \ldots$: portfolio strategy along the set of paths

Competitive Equilibrium (4)

- Initial endowment of assets θ^i_{-1} such that $\sum_{i=1}^I \theta^i_{-1} = 1$
- Budget constraint at the beginning:

$$\theta_0^{i,\text{cons}} + \sum_{k=1}^K q_0^k \theta_0^{i,k} = \sum_{k=1}^K (q_0^k + D_0^k)\theta_{-1}^{i,k} + w_0^i.$$

Reformulation (1)

Rewrite in terms of asset allocation: $\theta_t^{i,k} = \lambda_t^{i,k} W_t^i / q_t^k$.
Equalizing demand with supply, i.e.,

$$\sum_{i=1}^{I} \frac{\lambda_t^{i,k} W_t^i}{q_t^k} \overset{!}{=} 1, \qquad \text{for all } k \text{ and all } t,$$

gives

Proposition

The price of asset k is the average wealth of the traders' asset allocation for asset k, i.e.

$$q_t^k = \sum_{i=1}^{I} \lambda_t^{i,k} W_t^i.$$

Reformulation (2)

Definition

A competitive equilibrium with perfect foresight is a list of portfolio strategies λ_t^i, and a sequence of prices q_t^k for all $t = 0, \ldots, T$, such that for all $i = 1, \ldots, I$

$$\lambda^i \in \arg\max_{\substack{\lambda_t^i \in \Delta^{K+1} \\ t=0,\ldots,T}} U^i(\lambda^{cons} W^i) \qquad \textit{such that}$$

$$W_t^i = \left(\sum_{k=1}^{K} \frac{D_t^k + q_t^k}{q_{t-1}} \lambda_{t-1}^{i,k} \right) W_{t-1}^i + w_t^i, \quad \textit{for all } t = 0, \ldots, T$$

and markets clear:

$$\sum_{i=1}^{I} \lambda_t^{i,k} w_t^i = q_t^k, \qquad \textit{for all } k \textit{ and all } t.$$

Complete and Incomplete Markets

Complete and Incomplete Markets (1)

Definition

A financial market (D, q) is said to be complete if any consumption stream $\{\theta^{cons}\}$ can be attained with at least one initial wealth w_0, i.e., it is possible to find some trading strategy θ such that for all periods $t = 1, 2, \ldots, T$,

$$\theta_t^{cons} + \sum_k q_t^k \theta_t^k = \sum_k (D_t^k + q_t^k)\theta_{t-1}^k, \quad and \quad \theta_0^{cons} + \sum_k q_0^k \theta_0^k = w_0.$$

A financial market is said to be incomplete if there are some consumption streams that cannot be achieved whatever the initial wealth is.

Complete and Incomplete Markets (2)

A financial market is complete iff

$$\operatorname{rank} A_t(\omega^{t-1}\omega_t) = \left|\Omega_t(\omega^{t-1})\right| \quad \text{for all } \omega^t,\ t = 1, 2, \ldots, T.$$

where

$$A_t(\omega^{t-1}\omega_t) := \left[D_t^k(\omega^{t-1}\omega_t) + q_t^k(\omega^{t-1}\omega_t)\right]_{\omega_t \in \Omega_t}^{k=1,\ldots,K}.$$

Hence, if $K < |\Omega_t(\omega^t)|$ for some ω^t, then markets are incomplete.

An example can be found in the text book on page 226ff.

Redundancy

An asset is redundant if it has payoffs $\{D_t^{K+1}(\omega)\}_{\omega \in \Omega_t}$ which are a (positive) linear combination of the existing assets $k = 1, 2, \ldots, K$, i.e., for some α^k:

$$D_t^{K+1}(\omega) = \sum_{k=1}^{K} \alpha^k(t) D_t^k(\omega) \qquad \text{for all } \omega \in \Omega_t.$$

Choosing prices according to the linear rule $q^{K+1} = \sum_k \alpha^k q^k$ in every event ω^t we have:

$$\text{rank}\left[A_t(\omega^{t-1}\omega_t) \,\middle|\, D_t^{K+1}(\omega^{t-1}\omega_t) + q_t^{K+1}(\omega^{t-1}\omega_t)\right] = \text{rank}\, A_t(\omega^{t-1}\omega_t),$$

i.e., the rank of the payoff matrix does not change.
If there are non-redundant assets the market cannot have been complete.

Term Structure of Interest

Term Structure of Interest

- Want to apply the multi-period model to fixed-income markets.
- r_{t_0,t_n}: annual interest rate applied for borrowing and lending money between t_0 and t_n.
- The collection of interest rates $r_{t_0,t_1},\ldots,r_{t_0,t_T}$ is called the spot rate curve or the term structure of interest rates, usually increasing and concave.
- The forward rate f_{t_0,t_1,t_2} is the (annual) interest rate between t_1 and t_2 that is agreed today for the borrowing and lending between t_1 and t_2.
- The forward rate can be determined by the No-arbitrage Principle.

Term Structure of Interest

- Forward rate:

$$1 + f_{t_0,t_1,t_2} = \frac{(1 + r_{t_0,t_2})^{\frac{t_2}{t_2-t_1}}}{(1 + r_{t_0,t_1})^{\frac{t_1}{t_2-t_1}}}.$$

- Forward rate often seen as the interest rate we expect today for tomorrow, i.e. the realized interest rate between t_1 and t_2 should be on average the same as the forward rate.
- But: Empirical data show, that the forward rate bias (difference between the forward rate and realized interest rate) is quite persistent.

Term Structure without Risk

- Consider a three-period economy with $t = 0, 1, 2$ and a representative investor with the utility function

$$U(c) = \ln(c_0) + \frac{1}{1+\delta}\mathbb{E}(\ln(c_1)) + \frac{1}{(1+\delta)^2}\mathbb{E}(\ln(c_2)).$$

- The forward rate is exactly the realized interest rate and there is no forward rate bias.
- The model is too simple to capture the effects causing a forward rate bias.

Term Structure without Risk

- Consider quasi-hyperbolic instead of exponential time discounting (compare Sec. 2.7).
- Utility of the representative investor:

$$U^H(c) = \ln(c_0) + \frac{1}{1+\beta}\left(\frac{1}{1+\delta}\mathbb{E}(\ln(c_1)) + \frac{1}{(1+\delta)^2}\mathbb{E}(\ln(c_2))\right),$$

  where $\beta > 0$ describes the degree of quasi-hyperbolic discounting.
- Hyperbolical discounting implies a negative forward rate bias, but in reality positive and negative forward rate biases can be observed.
- Hyperbolic discounting would lead to a decreasing term structure, contrary to what we usually observe on the market.

Term Structure with Risk

- Include risk into our model so that in $t = 1$ the economy can develop better or worse, i.e. we have two states, an up and a down state.
- Utility maximization problem of the representative investor with beliefs for the occurrence of the up state:

$$\max_{\substack{c_0, c_{1u}, c_{1d}, c_{2u}, c_{2d} \\ s_{01}, s_{02}, s_{12}}} \ln(c_0) + \text{prob}\left(\frac{1}{1+\delta}\ln(c_{1u}) + \frac{1}{(1+\delta)^2}\ln(c_{2u})\right) + (1-\text{prob})\left(\frac{1}{1+\delta}\ln(c_{1d}) + \frac{1}{(1+\delta)^2}\ln(c_{2u})\right),$$

$$\begin{aligned} \text{s.t.} \quad p_0 c_0 + s_{01} + s_{02} &= p_0 w_0 \,, \\ p_{1u} c_{1u} + s_{12} &= p_{1u} w_{1u} + (1 + r_{01}) s_{01} \,, \\ p_1 c_{1d} + s_{12} &= p_{1d} w_{1d} + (1 + r_{01}) s_{01} \,, \\ p_{2u} c_{2u} &= p_{2u} w_{2u} + (1 + f_{12}) s_{12} + (1 + r_{02})^2 s_{02} \,, \\ p_{2d} c_{2d} &= p_{2d} w_{2d} + (1 + f_{12}) s_{12} + (1 + r_{02})^2 s_{02} \,, \end{aligned}$$

Term Structure with Risk

- Interest rate in the up state:

$$1 + r_{12u} = (1 + \delta)(1 + g_{12u})$$

- Interest rate in the down state:

$$1 + r_{12d} = (1 + \delta)(1 + g_{12d})$$

- Expected return:

$$\mathbb{E}(1 + r_{12}) = (1 + \delta)\mathbb{E}(1 + g_{12}).$$

Example

- Nominal growth rate: $g_{t,t+1,s} = (w_{t+1,s} p_{t+1,s})/(w_{t,s} p_{t,s}) - 1$
- $\delta = 0.1$, prob$= 0.5$, $g_{0,1,u}(u) = g_{1,2,u} = 1/9$, $g_{0,1,d} = -1/21$ and $g_{0,2,d} = 0$.
- Interest rates in $t = 0, 1$:

$$1 + r_{01} = 1.128, \qquad 1 + r_{02} = 1.156, \qquad 1 + f_{12} = 1.185.$$

- Interest rates realized in $t = 1$ depend on the state:

$$1 + r_{12u} = 1.222, \qquad 1 + r_{12d} = 1.155, \qquad \mathbb{E}(1 + r_{12}) = 1.189.$$

- In the upper state the realized interest rate rises and we have a negative forward rate bias. in the down state the opposite happens.

Arbitrage in the Multi-Period Model

Arbitrage

Definition

An arbitrage is a self-financing trading strategy, i.e., there is some strategy θ_t *with* $\theta_{-1} = 0$ *such that for all* $t = 0, 1, 2, \ldots, T$,

$$\theta_t^{cons} + \sum_{k=1}^{K} q_t^k \theta_t^k = \sum_{k=1}^{K} (D_t^k + q_t^k)\theta_{t-1}^k$$

and the resulting consumption is positive: $\theta_t^{cons} > 0$, *i.e.,* $\theta_t^{cons}(\omega^t) \geq 0$ *for all* ω^t *and all* $t = 0, 1, 2, \ldots, T$, *and* $\theta^{cons} \neq 0$.

(We give here only the strict monotonic variant of arbitrage – compare Chapter 4.)

Fundamental Theorem of Asset Pricing

Theorem (FTAP)

There is no arbitrage opportunity if and only if there is a state price process $\pi_{t=1,2,\dots,T} \gg 0$ *such that for all* ω^t

$$q_{t-1}^k(\omega^{t-1}) = \frac{1}{\pi_{t-1}(\omega^{t-1})} \sum_{\omega_t \in \Omega_t} \pi_t(\omega^t)(D_t^k(\omega^t) + q_t^k(\omega^t)) \qquad (1)$$

where $\omega^t = \omega^{t-1}\omega_t$.

The proof can be found in the text book on page 235.

Consequences of No-Arbitrage

As in the Two-Period case, no-arbitrage has two immediate consequences:

- Law of one Price
- Linear Pricing

Law of One Price (1)

Corollary

If from period t onward two assets have identical dividend processes, then in period $t-1$ they must have the same price.

The proof can be found in the text book on page 236.

Linear Pricing

Corollary

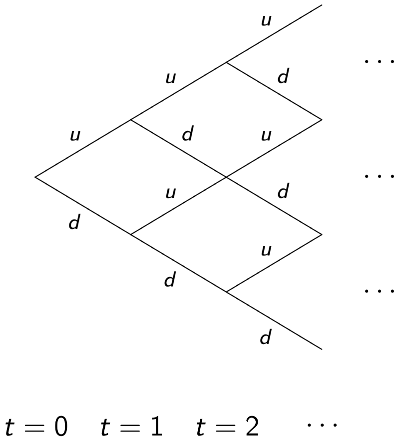
If in period $t-1$ one buys and holds a portfolio $\hat{\theta}_{t-1}$ then in $t-1$ the price of the portfolio must be a linear combination of its components:

$$q_{t-1}(\hat{\theta}) = \hat{\theta}_{t-1} q_{t-1} = \sum_{k=1}^{K} \hat{\theta}^k_{t-1} q^k_{t-1}.$$

The proof can be found in the text book on page 236.

Option Pricing

Binomial lattice model:



Risk-Neutral Probability

The risk-neutral probability is stationary, i.e. it remains the same at every node.

To see this, suppose that the stock price is S. Then, its expected value after one period is:

$$\mathbb{E}_{\pi^*}(S) = \pi^* Su + (1 - \pi^*)Sd.$$

In a risk-less world this value must be equal to SR:

$$\mathbb{E}_{\pi^*}(S) = \pi^* Su + (1 - \pi^*)Sd = SR.$$

Thus, the risk-neutral probability is constant over the time and depends only on the size and the frequency of "up" and "down" movements.

Example Call Option

- Periods $t = 0, 1, 2$
- Value in $t = 1$ depends on the realized state, i.e.:

$$C_u := \frac{1}{R}\left(\pi^* C_{uu} + (1 - \pi^*) C_{ud}\right),$$
$$C_d := \frac{1}{R}\left(\pi^* C_{ud} + (1 - \pi^*) C_{dd}\right).$$

Example Call Option

- Value of the call at $t = 0$ is

$$C = \frac{1}{R^2}\left((\pi^*)^2 C_{uu} + 2\pi^*(1-\pi^*)C_{ud} + (1-\pi^*)^2 C_{dd}\right),$$

i.e.,

$$C = \frac{1}{R^2}\Big((\pi^*)^2 \max\{0, u^2S - K\} + 2\pi^*(1-\pi^*)\max\{0, udS - K\} + (1-\pi^*)^2 \max\{0, d^2S - K\}\Big).$$

- For $t \to \infty$ this gives the normal distribution

Stock Prices as Discounted Expected Payoffs (1)

Suppose we have two assets:

- short-lived and risk-free
- risky

Then

$$q^1_{t-1}(\omega^{t-1}) = \frac{1}{\pi_{t-1}(\omega^{t-1})} \sum_{\omega_t \in \Omega_t} \pi_t(\omega^{t-1}\omega_t)(\underbrace{D^1_t(\omega^{t-1}\omega_t)}_{1} + \underbrace{q^1_t(\omega^{t-1}\omega_t)}_{0})$$

$$= \frac{1}{1 + r_{ft-1}(\omega^{t-1})}.$$

Stock Prices as Discounted Expected Payoffs (2)

Using this we get

$$q^k_{t-1}(\omega^{t-1}) = \frac{1}{1 + r_{f t-1}(\omega^{t-1})} \sum_{\omega_t \in \Omega_t} \pi^*_t(\omega^{t-1}\omega_t)(D^k_t(\omega^{t-1}\omega_t) + q^k_t(\omega^{t-1}\omega_t)),$$

where

$$\pi^*_t(\omega^t) = \frac{\pi_t(\omega^t)}{\sum_{\omega_t \in \Omega_t} \pi_t(\omega_t)} > 0$$

is a (risk-neutral) probability measure based on the information of period $t-1$. Hence, asset prices can be presented as discounted expected payoffs.

Stock Prices as Discounted Expected Payoffs (3)

Forward iteration along paths yields the discounted dividends model:

$$q^k_{t-1}(\omega^{t-1}) = E_{\pi^\star_{t-1}(\omega^{t-1})} \sum_{\tau=t}^{\infty} \frac{D^k_\tau(\omega^\tau)}{\prod_{\tau'=t-1}^{\tau}(1 + r_{f\tau'}(\omega^{\tau'}))}$$

Price movements depend only on movements of the risk-free interest rate and the dividend payments.
If the dividend process follows a random walk, then anticipated prices must be random, i.e.,

$$\mathbb{E}_{\pi^*_t}(q^k_{t+1} - (1 + r_t)q^k_t) = -\mathbb{E}_{\pi^*_t}(D^k_{t+1}).$$

In terms of excess returns we get $\mathbb{E}_{\pi^*_t}(R^k_{t+1} - R_{ft}) = 0$.
The net present value of a strategy with respect to the risk-neutral probability must be equal to 0.

Equivalent Formulations of No-Arbitrage (1)

If a price process is arbitrage-free, there exists no strategy θ_t, $t = 0, 1, 2, \ldots$, that generates risk-free returns on q.
This is equivalent to the existence of a market expectation or a risk-neutral probability such that

$$q_t^k = \frac{1}{1 + r_t} \mathbb{E}_{\pi_t^*}(D_{t+1}^k + q_{t+1}^k).$$

Applying forward iteration we get the Dividend Discount Model (DDM):

$$q_t^k = \mathbb{E}_{\pi_t^*}\left(\sum_{\tau=t+1}^{\infty} \left(\frac{1}{1+r} \right)^{\tau - t} D_\tau^k \right), \qquad t = 0, 1, \ldots, T.$$

Equivalent Formulations of No-Arbitrage (2)

There are no expected gains $E_{\pi_t^\star}(G_{t+1} - G_t) = 0$, i.e. the gains process is a *martingale*, where $G_t = \sum_{\tau=1}^{t} g_\tau \theta_{t-1}$ for some portfolio strategy θ and

$$g_{t+1}^k = \left(\frac{1}{1+r}\right)^t \left[\frac{1}{1+r}(D_{t+1}^k + q_{t+1}^k) - q_t^k\right]$$

is the discounted gain from holding asset k from t till $t+1$.
Hence, the cumulative expected gains are zero:

$$E_{\pi_t^\star}\left(\sum_{\tau=t+1}^{\infty} g_\tau \theta_{\tau-1}\right) = 0$$

"Nobody can beat the market", i.e. you cannot beat a martingale.

Pareto efficiency

Pareto Efficiency (1)

As in the two-period case (see Sec. 4.1), we can prove that market equilibria in complete markets are Pareto-optimal.
We first have to generalize the necessary definitions:

Definition (Attainability)

An allocation of consumption streams $\{\{\theta_t^{i,cons}(\omega^t)\}_{t=0}^T\}_{i=1}^I$ *is attainable if each component is in the consumption set of the agent and it does not use more consumption than is available from the dividend process D and exogenous endowments:*

$$\sum_{i=1}^{I} \theta_t^{i,cons}(\omega^t) = \sum_{k=1}^{K} D_t^k(\omega^t) + \sum_{i=1}^{I} w_t^i(\omega^t) \quad \text{for every } \omega^t,\ t = 0, \ldots, T.$$

Pareto Efficiency (2)

Definition (Pareto efficiency)

In a financial market the allocation of consumption streams $\{\{\theta_t^{i,cons}(\omega^t)\}_{t=0}^T\}_{i=1}^I$ is Pareto-efficient if and only if it is attainable and there does not exist an alternative attainable allocation of consumption streams $\{\{\hat{\theta}_t^{i,cons}(\omega^t)\}_{t=0}^T\}_{i=1}^I$, such that no consumer is worse off and some consumer is better off:

$$U^i\left(\{\hat{\theta}_t^{i,cons}(\omega^t)\}_{t=0}^T\right) \geq U^i\left(\{\theta_t^{i,cons}(\omega^t)\}_{t=0}^T\right) \ \forall\, i \text{ and } > \text{ for some } i.$$

First Welfare Theorem (1)

Theorem (First Welfare Theorem)

In a complete financial market, the allocation of consumption streams in any market equilibrium is Pareto-efficient.

Market efficiency does not rule out that some agents consume much more than others. From the perspective of fairness, this might not be optimal.

The proof can be found in the text book on page 244.

First Welfare Theorem (2)

If utility functions are smooth, we have for all investors i, j:

$$MRS^i_{s,z} = \frac{\partial_{\theta_s^{\text{cons}}} U^i(c^i)}{\partial_{\theta_z^{\text{cons}}} U^i(c^i)} = \frac{\partial_{\theta_s^{\text{cons}}} U^j(c^j)}{\partial_{\theta_z^{\text{cons}}} U^j(c^j)} = MRS^j_{s,z}.$$

Dynamics of Price Expectations

Dynamics of Price Expectations

- Consider two types of input data on which the price expectations are formed:
  - previous prices used by chartists
  - fundamental values used by fundamentalists
- Combine this with two types of expectations rules:
  - momentum
  - reversal
- Hence, we study the interaction of four types of expectation formations: momentum or reversal rules for chartists and fundamentalists.

Momentum Effect

- Efficient market hypothesis would say that information is always already incorporated into the current price – at least when we consider time scales above split-seconds.
- Momentum effect: Investing in assets that had previously outperformed the market leads on average to excess returns, even when controlling for risk.
- The momentum effect disappears on long time scales (years), where instead a reversal is observable.

Explanations

- [Barberis et al., 1998]:
  Underreaction explanation for short-term momentum, and an overreaction explanation for long term reversals.
- [Daniel et al., 1998]:
  Existence of private signals and overconfident reaction to them and afterwards a too slow adjustment to "reality".
- [Hong and Stein, 1999]:
  Market with two investor types, "newswatchers" (fundamentalists) and "momentum traders" (chartists).

Dynamical Model of Chartists and Fundamentalists

- Chartists: use previous price data
- Fundamentalists: rely on fundamental values.
- The steady state of the dynamical system is characterized by the discounted dividends rule, and the stability of it will depend on the relative proportions of investors with different types of price expectation.
- Assume that given his price expectations every trader maximizes a mean-variance utility for one period ahead.

Mean-Variance Utility Functions (1)

- Assume that the economy follows a discrete time tree model with a finite number of states in each period.
- Consider the utility of agent i:

$$U^i(c_1^i, \dots, c_S^i) = \mu(c_1^i, \dots, c_S^i) - \frac{\gamma^i}{2}\sigma^2(c_1^i, \dots, c_S^i)$$

- Note that his consumption in state s is given by

$$c_s^i = \lambda^{i,c} R_s^i \lambda^i w^{i,f}$$

and recall the budget constraint

$$\sum_{k=0}^{K} \hat{\lambda}^{i,k} = 1.$$

Mean-Variance Utility Functions (2)

- Hence, for any portfolio $\hat{\lambda}$ we get a utility from that portfolio:

$$U^i(\hat{\lambda}^{i,0},\ldots,\hat{\lambda}^{i,K}) = w^{i,f}\left(\mu(\lambda^{i,c}R^i\hat{\lambda}^i) - \frac{\gamma^i w^{i,f}}{2}\sigma^2(\lambda^{i,c}R^i\hat{\lambda}^i)\right).$$

- The solution is

$$\hat{\lambda}^i = \left(COV(R^i)\right)^{-1}\frac{\mu(R^i) - R_f}{\gamma^i\lambda^{i,c}w^{i,f}}.$$

- Written in economic terms:

$$\theta^i = \frac{1}{\gamma^i}\left(COV(A^i)\right)^{-1}\left(\mu(A^i) - R_f q\right).$$

Price Expectation Dynamics (1)

- Recall that in the multi-period model payoffs are given by dividends and resale prices, i.e.

$$A^i(\omega^t) = D^i(\omega^t) + q^i(\omega^t).$$

- Assume point expectations, i.e. q^i_t is independent of (ω^t).
- Then, the demand in period t given the expectations for the next period is

$$\hat{\lambda}^i_t = \frac{1}{\gamma^i \lambda^{i,c}_t w^{i,f}_t} \Lambda(q_t) \left(COV(D_{t+1})\right)^{-1} \left(\mu(D_{t+1}) + q^i_{t+1} - R_f q_t\right).$$

- Normalizing the supply of each asset to 1, assuming short-run equilibrium and defining $\bar{\gamma}^{-1} = \sum_{i=1}^{I} (\gamma^i)^{-1}$ gives

$$\sum_{i=1}^{I} \lambda^{i,c}_t w^{i,f}_t = q_t = \Lambda(q_t) \left(COV(D_{t+1})\right)^{-1} \left(\frac{\mu(D_{t+1})}{\bar{\gamma}} + \sum_{i=1}^{I} \frac{q^i_{t+1}}{\gamma^i} - \frac{R_f q_t}{\bar{\gamma}} \right).$$

Price Expectation Dynamics (2)

- Multiplying both sides by $\Lambda(q)_k^{-1}$ and $COV(D_{t+1})$ and defining $D^M = \sum_{k=1}^K D^k$, for any asset k gives

$$q_t^k = \frac{\mu(D_{t+1}^k) - \bar{\gamma}_t COV(D_{t+1}^k, D_{t+1}^M) + \sum_{i=1}^I \delta^i q_{t+1}^{i,k}}{R_f},$$

where

$$\delta^i = \frac{\bar{\gamma}}{\gamma^i}.$$

- Hence, the price of any asset k in period t is given by the discounted expected dividends minus the risk of those dividends relative to the market dividends plus the average expected price for the next period.

Stationary Equilibrium (1)

- Assume that the trading strategies $\hat{\lambda}^i$, the consumption rates $\lambda^{i,c}$, the expected dividends $\mu(D)$ and the covariance $COV(D)$ are all stationary.
- Price equation:

$$\bar{q}^k = \frac{\mu(D^k) - \bar{\gamma} COV(D^k, D^M)}{r_f},$$

which is equal to the discounted expected dividends of the constant payoff

$$\mu(D^k) - \bar{\gamma} COV(D^k, D^M)$$

discounted at $R_f = 1 + r_f$.

Stationary Equilibrium (2)

- Substitute

$$\bar{\gamma} = \frac{\mu(D^M) - r_f q^M}{\sigma^2(D^M)},$$

from summing the above formula over k.

- Then we obtain

$$\left(\bar{q}^k - \mu\left(\frac{D^k}{r_f}\right)\right) = \beta^k \left(\bar{q}^M - \mu\left(\frac{D^M}{r_f}\right)\right),$$

where

$$\beta^k = \frac{COV(D^k, D^M)}{\sigma^2(D^M)}.$$

- Hence, we have derived a Security Market Line formula similar to that of the static CAPM, but in terms of first principles: dividends and the risk-free rate!

Structure on Price Expectations (1)

- Two types of traders: chartists $i \in C$ and fundamentalists $i \in F$.
- Chartists form the price expectations

$$q_{t+1}^{i,k} = q_t^k + a^{i,k}(q_t^k - q_{t-1}^k)$$

with $a^{i,k} > 0$ being a momentum chartist and $a^{i,k} < 0$ being a reversal chartist.
- Fundamentalists form the price expectations

$$q_{t+1}^{i,k} = q_t^k + b^{i,k}(\bar{q}_t^k - q_t^k)$$

with $b^{i,k} > 0$ being value investors and $b^{i,k} < 0$ being growth investors.

Structure on Price Expectations (2)

- Note that the price dynamics developed above is an inhomogeneous first order difference equation.
- Such a dynamical system converges to its steady state $\bar{q}$ iff the absolute value of the coefficient in front of the price variable $\sum_{i=1}^{I} \delta^i q_{t+1}^{i,k}$ is smaller than one.
- Inserting the expectation functions we get:

$$R_f q_t^k = \sum_{i \in C} \delta^i \left(q_t^k + a^{i,k}(q_t^k - q_{t-1}^k) \right) + \sum_{i \in F} \delta^i \left(q_t^k + b^{i,k}(\bar{q}_t^k - q_t^k) \right).$$

- Rearranging while ignoring constant terms we get:

$$\left(\sum_{i \in C} \delta_t^i (1 + a^{i,k}) + \sum_{i \in F} \delta_t^i (1 - b^{i,k}) - R_f \right) q_t^k = \left(\sum_{i \in C} \delta_t^i a^{i,k} \right) q_{t-1}^k$$

or $$q_t^k = \frac{\left(\sum_{i \in C} \delta_t^i a^{i,k} \right)}{\left(\sum_{i \in C} \delta_t^i a^{i,k} - \sum_{i \in F} \delta_t^i b^{i,k} - r_f \right)} q_{t-1}^k =: \frac{\bar{a}_t}{\bar{a}_t - \bar{b}_t - R_f}.$$

Structure on Price Expectations (3)

For the stability analysis we get 4 cases:

- Case 1 (numerator and denominator positive)
  This happens, e.g. with strong momentum and weak value.
  Consequently stability occurs iff

$$\sum_{i \in F} \delta^i b^{i,k} + R_f < 0,$$

which is unlikely since $R_f > 1$.

- Case 2 (numerator positive and denominator negative)
  This happens, e.g. with medium momentum and strong growth.
  Consequently stability occurs iff

$$2 \sum_{i \in C} \delta^i a^{i,k} < \sum_{i \in F} \delta^i b^{i,k} + r_f,$$

which cannot be since in this case

$$\sum_{i \in C} \delta_t^i a^{i,k} > 0 \quad \text{and} \quad \sum_{i \in F} \delta_t^i b^{i,k} < 0.$$

Structure on Price Expectations (4)

- Case 3 (numerator negative and denominator positive)
  This happens, e.g. with reversal and strong growth. Consequently stability occurs iff

$$\sum_{i \in F} \delta^i b^{i,k} + r_f < 2 \sum_{i \in C} \delta^i a^{i,k},$$

  which is well possible.

- Case 4 (numerator and denominator negative)
  This happens, e.g. with reversal and value or weak growth. Stability occurs iff

$$-2 \sum_{i \in C} \delta^i a^{i,k} < \sum_{i \in F} \delta^i b^{i,k} + r_f,$$

  which is possible if the reversal is not too strong relative to value.

Survival of the Fittest on Wall Street

Survival of the Fittest on Wall Street

- We analyze the long term dynamics of our model, the evolution of wealth over time and uncertainty.
- Assuming complete markets, perfect foresight and intertemporal utility maximization, the wealth of investors with rational expectations will grow fastest in a financial market equilibrium.

Market Selection Hypothesis (1)

- Use Pareto efficiency property of competitive equilibria to formulate a Market Selection Hypothesis that determines which investor survives best in the dynamics of the market in terms of (relative) wealth over time.
- If every investor has some expected utility function with the same time preferences, but possibly different risk attitude, and it payoffs are stochastic, then investor i will dominate investor j if his beliefs on the occurrence of the states are more accurate.
- Note: the investor's dominance is not defined over his strategy, but on his ability to make good estimates.

Market Selection Hypothesis (1)

- The better the agents' beliefs, the more those agents get in the more likely states. Hence, their wealth will grow faster.
- Note: The only requirement for our fitness criteria to hold is decreasing marginal utilities as in the expected utility framework.
- Investors get more wealth in those states to which they assign a higher probability.

Evolutionary Portfolio Model (1)

- We base our evolutionary model on the Lucas (1978) asset pricing model.
- As in the traditional model, we start from the fundamental equation of wealth dynamics:

$$W_{t+1}^i = \sum_{k=1}^{K} \frac{D_{t+1}^k + q_{t+1}^k}{q_t^k} \lambda_t^{i,k} W_t^i$$

with $\sum_{k=1}^{K} \lambda_t^{i,k} 1 - \lambda_t^{i,c}$ for all i and t.

Evolutionary Portfolio Model (2)

We restrict attention to relative wealth, relative dividends and relative prices.

$$r^i_{t+1} = \sum_k \frac{\lambda^c d^k_{t+1} + \hat{q}^k_{t+1}}{\hat{q}^k_{t+1}} \lambda^{i,k}_t r^i_t,$$

where

$$\hat{q}^k_{t+1} = \frac{q^k_t}{\sum_i W^i_t}, \quad d^k_{t+1} = \frac{D^k_{t+1}}{\sum_{k'} D^{k'}_{t+1}},$$

$$r^i_t = \frac{W^i_t}{\sum_i W^i_t}, \qquad \lambda^c \sum_{i=1}^{I} W^i_t = \sum_k D^k_t$$

Evolutionary Portfolio Model (3)

- Important assumption: All strategies have the same consumption rate λ^c.
- The relative asset prices are the convex combination of the strategies in the market:

$$\hat{q}_t^k = \sum_{i=1}^{I} \lambda_t^{i,k} r_t^i.$$

- Strategies are "playing the field" i.e. one strategy has an impact on any other strategy only via the average of the strategies.

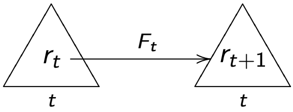
Evolutionary Portfolio Model (4)

- $\Lambda_{t+1} := (\hat{\lambda}^{i,k}_{t+1})_{i,k}$ is the matrix of portfolio strategies

$$r_{t+1} = \lambda^c \left(\mathrm{Id} - \left[\frac{\hat{\lambda}^{i,k}_t r^i_t}{\hat{\lambda}^{\cdot,k}_t r^{\cdot}_t} \right]_{i,k} \Lambda^T_{t+1} \right)^{-1} \left[\sum_k d^k_{t+1} \frac{\hat{\lambda}^{i,k}_t r^i_t}{\hat{\lambda}^{\cdot,k}_t r^{\cdot}_t} \right]_i$$

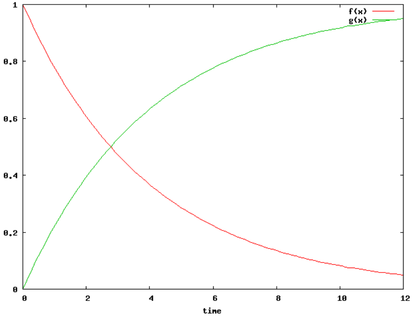
- This equation is a first order stochastic difference equation describing a mapping from the simplex $\triangle$ into itself.

$$r_{t+1}(\omega^{t+1}) = F_t(\omega^{t+1}, r_t)$$



Simulation Analysis (1)

Simulation with two strategies



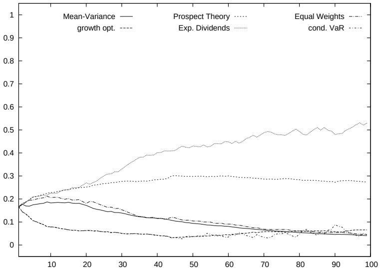
- red line: mean-variance analysis
- green line: equal weight.

Simulation Analysis (2)

- Even though initially the wealth of the mean-variance rule accounts for 90% of the market wealth after a few iterations the situation has reversed and the $1/n$ rule has 90% of the market wealth.
- This wealth dynamics is reflected in the asset prices: they initially reflect the mean-variance rule but rapidly converge to the $1/n$ rule.
- Seemingly rational portfolio rules like mean-variance can do quite poorly against seemingly irrational rules like $1/n$.

Simulation Analysis (3)

- Including the expected relative dividends portfolio $\lambda^{*,k} = \mathbb{E}_P d^k$, the process always converges to the strategy λ^*.
- Our conjecture from these simulations is: *Starting from any initial distribution of wealth, on P-almost all paths the market selection process converges to λ^*, if the dividend process d is i.i.d.*



Simulation Analysis (4)

- It can be shown analytically, that if dividends d follow an i.i.d. process and we only consider stationary adapted strategies, then

$$\hat{\lambda}^{\star,k} = (1-\lambda^c)\mathbb{E}_p d^k_{(\omega)}$$

is the unique evolutionary stable strategy.

- Expected growth rate of wealth of any strategy $\hat{\lambda}$ in a market governed by strategy λ^M:

$$g(\hat{\lambda}, \hat{\lambda}^M) = \mathbb{E}_p \ln\left(1 - \lambda^c + \lambda^c \sum_{k=1}^{K} \frac{\mathrm{d}^k \hat{\lambda}^k}{\hat{\lambda}^{M,k}}\right).$$

Simulation Analysis (5)

- Note: $g(\hat{\lambda}^M, \hat{\lambda}^M) = 0$, i.e. if all strategies are identical the none can grow at the cost of others.
- Evolutionary stability property: $g(\hat{\lambda}, \hat{\lambda}^\star) < 0$, hence all strategies die out in a market governed by $\lambda^\star$.
- It is apparent that under-diversified strategies have no chance to survive.

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