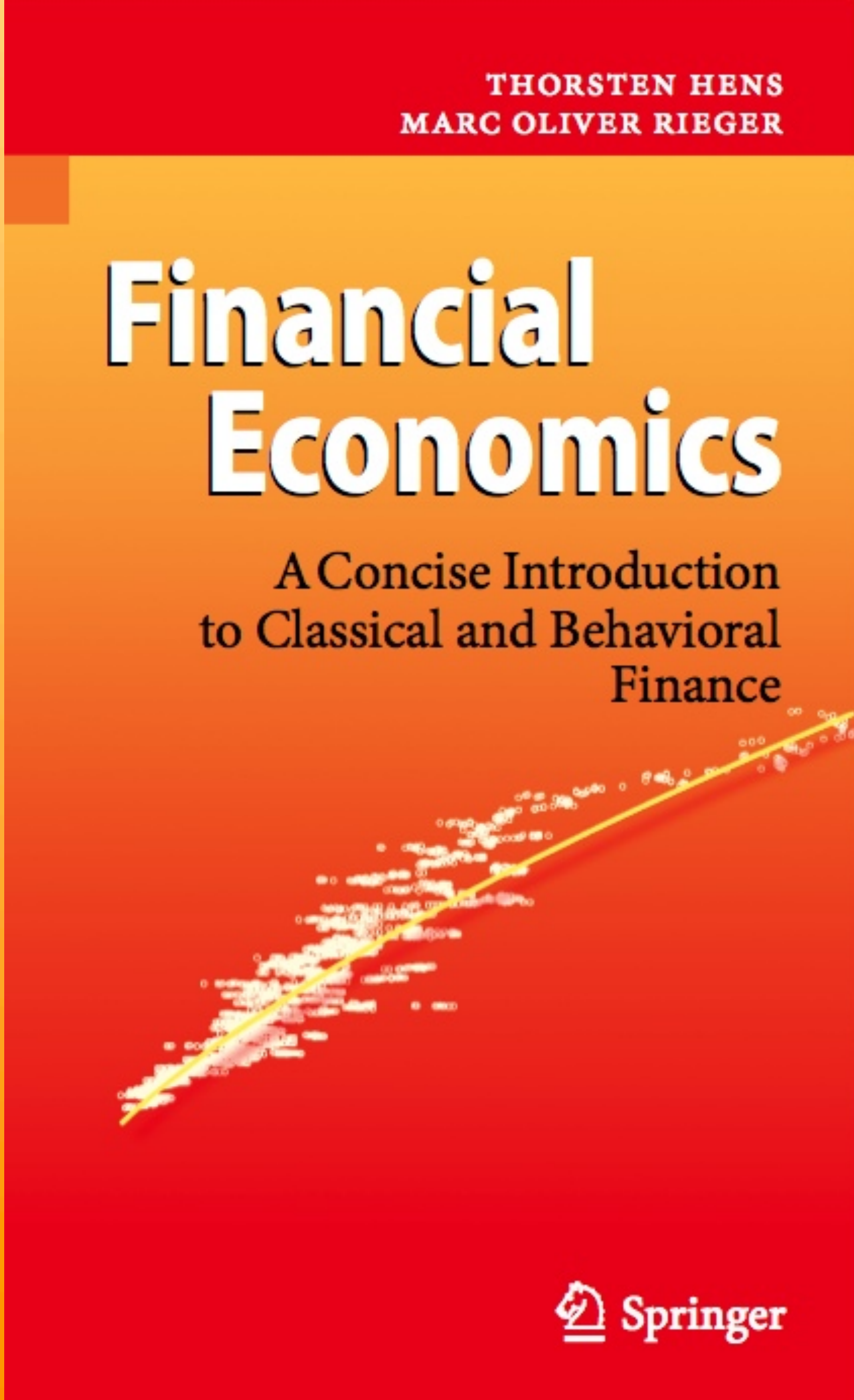

HENS · RIEGER

Financial Economics

Financial economics is a fascinating topic where ideas from economics, mathematics and, most recently, psychology are combined to understand financial markets. This book gives a concise introduction into this field and includes for the first time recent results from behavioral finance that help to understand many puzzles in traditional finance. The book is tailor made for master and PhD students and includes tests and exercises that enable the students to keep track of their progress. Parts of the book can also be used on a bachelor level. Researchers will find it particularly useful as a source for recent results in behavioral finance and decision theory.

The text book to this class is available at www.springer.com

On the book's homepage at www.financial-economics.de there is further material available to this lecture, e.g. corrections and updates.

Financial Economics

A Concise Introduction to Classical and Behavioral Finance
Chapter 8

Thorsten Hens and Marc Oliver Rieger

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August 3, 2010

Time-Continuous Model

Πάντα χωρεῖ καὶ οὐδὲν μένει.
(All is flux and nothing stays still.)
Heraklite, *as quoted by* Platon.

Time-Continuous Model

- Trading on a stock market
- Transactions in a high frequency
- Model applications in a time-continuous setting
- We will derive the famous Black-Scholes model for asset pricing by Fischer Black and Myron Scholes [Black and Scholes, 1973] and by Robert C. Merton [Merton, 1973].
- Nobel prize in 1997
- Classical reference for further studies: book by Duffie [Duffie, 1996], a lighter source is [Korn and Korn, 2001].

Assumptions

- First assumption:
  Price of the asset changes when new information reaches the market. The information is assumed to be random, its influence on the return of the asset is assumed to be normally distributed:

$$\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}B(t), \tag{1}$$

  where μ and σ are mean and standard deviation and $B(t)$ is a *Brownian motion*.
- Second fundamental assumption: no-arbitrage principle.

A Rough Path to the Black-Scholes Formula

- We want to price an option on the asset S.
- $V(S, t)$: value of this option at time t.
- Goal: derive a partial differential equation for the function V and boundary conditions imposed by our knowledge of the price at maturity.
- Incremental change of V (using the Itō formula, see Lemma 8.6 in the book on page 303):

$$\begin{aligned} \mathrm{d}V(S,t) &= \left(\mu S(t)\frac{\partial V(S,t)}{\partial S} + \frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S,t)}{\partial S^2}\right)\mathrm{d}t \\ &\quad + \sigma S(t)\frac{\partial V(S,t)}{\partial S}\,\mathrm{d}B(t). \end{aligned} \tag{2}$$

A Rough Path to the Black-Scholes Formula

- Define a trading strategy that replicate exactly the option. At every time t hold one option and $-\partial V(S,t)/\partial S$ stock
- Incremental profit:

$$\mathrm{d}R(t) = \mathrm{d}V - \frac{\partial V(S,t)}{\partial S}\,\mathrm{d}S.$$

- Insert (2) and (1) into this formula and get:

$$\mathrm{d}R(t) = \left(\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S,t)}{\partial S(t)^2}\right)\mathrm{d}t.$$

- Notice that the incremental returns of the delta hedge portfolio are not stochastic, they are risk-free, and should not be different from the returns of the risk-free asset.

A Rough Path to the Black-Scholes Formula

- Therefore,

$$r\left(V(S,t) - S(t)\frac{\partial V(S,t)}{\partial S(t)}\right)\,\mathrm{d}t = \left(\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S,t)}{\partial S(t)^2}\right)\,\mathrm{d}t.$$

- What are our boundary conditions?
- Let us price a simple call option with exercise price K.
- Boundary value problem:

$$\begin{cases} \frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S,t)}{\partial S(t)^2} + rS(t)\frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0, \\ V(0,t) = 0 \quad \text{for all } t, \\ V(S,t)/S \to 1 \quad \text{as } S \to \infty, \text{ for all } t, \\ V(S,T) = \max(S-K,0) \quad \text{for all } S. \end{cases}$$

- This problem can in fact be solved. More details can be found in the book on page 300.

Final result

$$V(S,t) = S\phi(d_1) - K\mathrm{e}^{-r(T-t)}\phi(d_2),$$

where ϕ is the normal cumulative distribution function, i.e.,

$$\phi(x) := \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) \mathrm{d}y,$$

and the auxiliary variables d_1 and d_2 are given by

$$d_1 = \frac{\ln(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Brownian Motion

We consider a state space Ω with a probability measure p.

Definition (Process)

A process X is a measurable function $X : \Omega \times [0, \infty) \to \mathbb{R}$. We call $X(t) := X(\cdot, t)$ the value of X at time t.

Historically, this idea goes back to the year 1900 and Louis Bachelier's seminal and unfortunately long forgotten work [Bachelier, 1900].

Brownian Motion

Definition (Brownian motion)

A standard Brownian motion is a process B defined by the properties:

(a) $B(0) = 0$ *a.s.*

(b) *For any times* t_0, t_1 *with* $t_0 < t_1$, *the difference* $B(t_1) - B(t_0)$ *is normally distributed with mean zero and variance* $t_1 - t_0$.

(c) *For any times* $0 \leq t_0 < t_1 < t_2 < \cdots < t_n < \infty$, *the random variables* $B(t_0), B(t_1) - B(t_0), \ldots, B(t_n) - B(t_{n-1})$ *are independently distributed.*

(d) *For each* $w \in \Omega$, *the sample path* $t \mapsto B(w,t)$ *is continuous.*

More about the idea of Brownian motion can be found in the book on page 302.

Definitions

Definition (Filtration)

A filtration of a measurable space Ω with σ-algebra $\mathcal{F}$ is a family of σ-algebras $\{\mathcal{F}(t)\}_{t\in(0,\infty)}$ such that

(i) $\mathcal{F}(t) \subset \mathcal{F}$ *for all* t,

(ii) $\mathcal{F}(t_1) \subset \mathcal{F}(t_2)$ *for all* $t_1 \leq t_2$.

Definition (adapted process)

A process X is called adapted to the filtration $\mathcal{F}(t)$ of Ω if $X(t)\colon \Omega \to \mathbb{R}$ is a $\mathcal{F}(t)$-measurable function for each $t \in [0,\infty)$.

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Trading Strategy

- A *trading strategy* is described by a process that prescribes in every state w and at any time t the assets a person should hold.
- With only one asset, the trading strategy is $\theta\colon \Omega \times [0,\infty) \to \mathbb{R}$.
- For general θ we need to assume

$$\int_0^T \theta(t)^2 \, \mathrm{d}t < \infty$$

- Define the total gain:

$$\int_0^T \theta(t) \, \mathrm{d}B(t).$$

- More details in the book on page 302.

Itō processes

Definition (Itō process)

Let B be a Brownian motion, $x \in \mathbb{R}$, $\sigma \in L^2$, i.e., σ is an adapted process with $\int_0^T \sigma(t)^2 \, \mathrm{d}t < \infty$ a.s. for all t, and $\mu \in L^1$, i.e., μ is an adapted process with $\int_0^T |\mu(t)| \, \mathrm{d}t < \infty$ a.s. for all t. Then the Itō process S is defined for $t \in [0, \infty)$ as

$$S(t) = S_0 + \int_0^t \mu(s) \, \mathrm{d}s + \int_0^t \sigma(s) \, \mathrm{d}B(s).$$

Informally and shorter, we denote $dS(t) = \mu(t) \, \mathrm{d}t + \sigma(t) \, \mathrm{d}B(t)$, $S(0) = S_0$.

Itō processes

- One can prove that

$$\frac{\mathrm{d}}{\mathrm{d}r} E_t(S(r))|_{r=t} = \mu_t, \quad \frac{\mathrm{d}}{\mathrm{d}r} \mathrm{var}_t(S(r))|_{r=t} = \sigma_t^2.$$

- μ is called the *drift process* and σ the *diffusion process* of S.

Lemma (Itō formula)

Let S be an Itō process and $f: \mathbb{R}^2 \to \mathbb{R}$ twice continuously differentiable, then the process $Y(t) := f(S(t), t)$ is an Itō process satisfying

$$\mathrm{d}Y(t) = \left(\frac{\partial f(S(t),t)}{\partial S}\mu(t) + \frac{\partial f(S(t),t)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(S(t),t)}{\partial S^2}\sigma(t)^2\right)\mathrm{d}t + \frac{\partial f(S(t),t)}{\partial S}\sigma(t)\,\mathrm{d}B(t).$$

The complete proof can be found in [Duffie, 1996], an intuition is described in the book on page 304.

Itō processes

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The complete proof can be found in [Duffie, 1996], an intuition is described in the book on page 304.

Derivation of the Black-Scholes Formula for Call Options

- Describe an underlying asset S by a *geometric Brownian motion with drift*, i.e.

$$\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}B(t),$$

  σ is called the *volatility*.

- Consider a second asset, a bond, with fixed interest rate with

$$\beta(t) = \beta_0 \mathrm{e}^{rt}.$$

- Itō process:

$$\mathrm{d}\beta(t) = r\beta(t)\,\mathrm{d}t.$$

Derivation of the Black-Scholes Formula for Call Options

- We call a *trading strategy* with a portfolio that contains $a(t)$ shares of stocks and $b(t)$ shares of bonds at time t *self-financing* if for all t:

$$a(t)S(t)+b(t)\beta(t) = a(0)S_0+b(0)\beta_0+\int_0^t a(s)\,\mathrm{d}S(s)+\int_0^t b(s)\,\mathrm{d}\beta(s).$$

- We want to find a self-financing strategy (a, b) that replicates the payoff structure at maturity, i.e.

$$a(T)S(T)+b(T)\beta(T) = \max(S(T)-K, 0).$$

- Given the existence of such a strategy, the price of the option at time t has to be $a(t)S(t)+b(t)\beta(t)$.

Derivation of the Black-Scholes Formula for Call Options

- Assume first that the price of the option at time t equals some function $V(S(t), t)$ and that V is twice differentiable.
- Apply the Itō formula:

$$\mathrm{d}V(S,t) = \left(\mu S(t)\frac{\partial V(S,t)}{\partial S} + \frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V(S,t)}{\partial S^2}\right)\mathrm{d}t + \sigma S(t)\frac{\partial V(S,t)}{\partial S}\,\mathrm{d}B(t). \quad (3)$$

- We have seen this formula before, but this time a rigorous derivation led us here.
- Assume the existence of this self-financing trading strategy (a, b) with

$$\mathrm{d}V(S,t) = a(t)dS(t) + b(t)d\beta(t). \quad (4)$$

Derivation of the Black-Scholes Formula for Call Options

- Inserting the expressions for $S(t)$ and $\beta(t)$, we obtain

$$\mathrm{d}V(S,t) = (a(t)\mu S(t) + b(t)\beta(t)r)\,\mathrm{d}t + a(t)\sigma S(t)\,\mathrm{d}B(t). \quad (5)$$

- By matching the coefficients in the two expressions we get

$$\sigma S(t)\frac{\partial V(S,t)}{\partial S} = a(t)\sigma S(t).$$

- From this we obtain

$$a(t) = \frac{\partial V(S,t)}{\partial S}.$$

Derivation of the Black-Scholes Formula for Call Options

- Insert into (4) to get

$$\mathrm{d}V(S,t) = \frac{\partial V(S,t)}{\partial S} dS(t) + b(t)\,\mathrm{d}\beta(t).$$

- Solve for $b(t)$:

$$b(t) = \frac{1}{\beta(t)}\left(V(S,t) - \frac{\partial V(S,t)}{\partial S} S(t)\right).$$

- Match the coefficients of $\mathrm{d}t$ in (5) and (3):

$$-rV(S,t) + \frac{\partial V(S,t)}{\partial t} + rS(t)\frac{\partial V(S,t)}{\partial S} - \frac{1}{2}\sigma^2\frac{\partial^2 V(S,t)}{\partial S^2} = 0.$$

Theorem (Black-Scholes formula)

The value of a European call option with strike K, maturity T and underlying asset S (described by a geometric Brownian motion with drift μ and volatility σ) is given by

$$V(S,t) = S\phi(d_1) - K\mathrm{e}^{-r(T-t)}\phi(d_2),$$

where ϕ is the normal cumulative distribution function, i.e.,

$$\phi(x) := \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) \mathrm{d}y,$$

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$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Put-Call Parity

- To derive this parity, we consider the following two portfolios:
  - One put option and one share.
  - One call option and K bonds that pay each 1 at maturity.
- Both pay K if $S \leq K$ and S if $S \geq K$, therefore both portfolios have the same value – also at times $t < T$.
- $P(t)$: value of the put option at time t
- $C(t)$: value of the call option
- Then,

$$C(t) + KR(t) = P(t) + S(t).$$

- Thus, the value of a put option is

$$P(t) = C(t) - S(t) + K\mathrm{e}^{-r(T-t)}.$$

Exotic Options

- Options we have priced so far: "plain vanilla options".
- Only one underlying and their value at maturity only depends on the value of their underlying at maturity.
- Exotic options:
  - Barrier option
  - Asian option
  - Fixed-strike average
  - Variance swap
  - Rainbow option

  More details in the book on page 308.
- How to price exotic options:
  - Numerical approximation methods
  - Monte Carlo method
- We can in fact derive the Black-Scholes model also as a limit of the finite time steps models of Chapter 5. See Section 8.5 for connections to the Multi-Period Model.

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Mutual Fund Theorem

- How should an investor choose his trading strategy?
- We will show that in continuous-time trading the so-called *Mutual Fund Theorem* holds, which is in a certain sense a generalization of the Two-Fund Theorem for a large class of rational investors.
- Merton [Merton, 1972]
- Assume that there are $N \geq 1$ underlying assets driven by a $D \geq 1$ dimensional geometric Brownian motion $B(t)$. Let $S(t) \in \mathbb{R}^N$ be the price vector of the assets and $\sigma \in \mathbb{R}^{D \times N}$ the volatility matrix.
- We need to distinguish two utility functions:
  - $u_1 : \mathbb{R} \times [0, \infty) \to [-\infty, +\infty)$: describes the utility derived from consumption during the investment time
  - $u_2 : [0, \infty) \to [-\infty, +\infty)$: describes the utility derived from final wealth.

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Mutual Fund Theorem

- $c(t)$: consumption over time
- $\theta(t)$: trading strategy
- Maximize

$$U := \mathbb{E}\left(\int_0^T u_1(t, c(t))\,\mathrm{d}t + u_2(X(T))\right)$$

Assumption

We assume that u_1 and u_2 satisfy the following conditions:

(i) $I_1 := (u_1')^{-1}$ *and* $I_1 := (u_2')^{-1}$ *have polynomial growth.*

(ii) $u_1(I_1)$ *and* $u_2(I_2)$ *have polynomial growth.*

(iii) I_1 *is Hölder continuous.*

(iv) *Either* $\partial I_1(t, y)/\partial y$ *is strictly negative for a.e.* y *or* $\partial I_2'(y)/\partial y$ *is strictly negative a.e. (or both).*

Mutual Fund Theorem

Theorem (Mutual Fund Theorem)

Assume that the volatility of the underlying assets is given by σ and the dividend process is given by δ. Assume that the investor can invest risk-free for a return of r and borrow money for a return of b. Assume that σ, δ, r and b are smooth functions of the time t. Then any agent with preferences satisfying assumption 1 should hold a mutual fund containing the assets in the proportion

$$(\sigma'(t))^{-1}\theta(t) = (\sigma(t)\sigma'(t))^{-1}(b(t) + \delta(t) - r(t)\vec{1})$$

plus a risk-free asset in order to maximize the expected utility U.

Mutual Fund Theorem

- What about its validity in reality?
- First, observations about investment decisions show a strong heterogeneity.
- Possible explanation: expectations of investors are heterogeneous.
- But there are situations where expectations should be homogeneous, e.g. the case of a client advisor at a bank.
- Discrepancy between real life on financial markets and the theory helps us to improve at least one of them – either the model or the real life. . .

Mutual Fund Theorem

- What could be possible explanations for this puzzle?
- Model overlooks market friction, e.g., transaction costs.
- Investment decisions and markets are far away from being rational.
- Preferences that are not covered by the Mutual Fund Theorem.
- To sum up: the puzzle gives rise to improving the theory.
- More about market equilibria in continuous time can be found in the book on page 318.

Assumptions

1. Trading in the assets is continuous in time.
2. The price of the underlying asset follows a geometric Brownian motion with drift.
3. The market is arbitrage free.
4. There are no short-sell constraints.
5. Assets are arbitrarily divisible.
6. There are no frictions, like transaction costs or taxes.
7. There is a fixed risk-free rate for which money can be invested or borrowed.
8. There is no dividend payment.

Volatility Smile and other Unfriendly Effects

- At-the-money options tend to have lower implied volatility than options with a strike far away from the current price of the underlying. This curved shape reminded some researchers of a smile, thus the name *volatility smile*.
- There is also a time-dependence of the implied volatility: different maturities lead to different implied volatilities, an effect which is called *term structure of volatility*.

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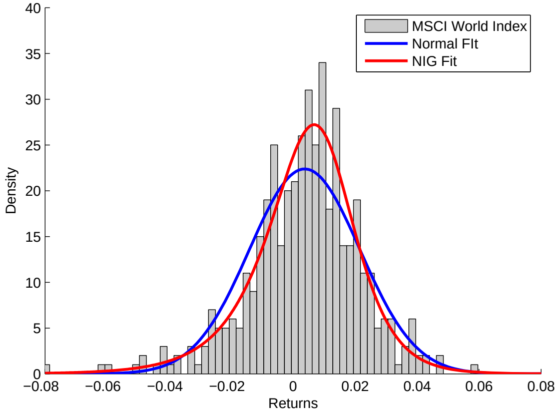
Not Normal – Alternatives to Normally Distributed Returns

Standard assets like stocks and bonds do not usually have normally distributed returns.



Not Normal – Alternatives to Normally Distributed Returns

- Example for a class of very versatile distributions: normal inverse Gaussian distributions.
- They are specified by four parameters that roughly correspond to the first four moments.
- Other approach: Lévy skew alpha-stable distribution, see page 325 in the book for more detail.

Jumping Up and Down – Lévy Processes

Definition (Lévy process)

A Lévy process is a process X defined by the properties:

(a) $X(0) = 0$ *a.s.*

(b) *For any times t_0, t_1 with $t_0 < t_1$, the difference $X(t_1) - X(t_0)$ follows a fixed distribution.*

(c) *For any times $0 \leq t_0 < t_1 < t_2 < \cdots < t_n < \infty$, the random variables $X(t_0), X(t_1) - X(t_0), \ldots, X(t_n) - X(t_{n-1})$ are independently distributed.*

More details in the book on page 327.

One of many technically simpler subclasses of Lévy processes are *stable Lévy processes with exponential decay*, encompasses in particular Brownian motion, NIG processes, hyperbolic processes etc.

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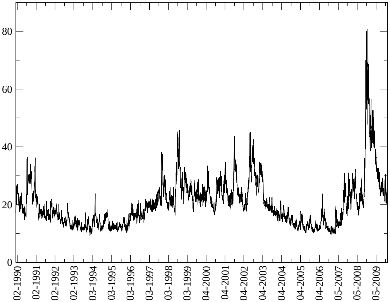
Drifting Away – Heston and GARCH Models

- Brownian motion assume constant volatility.
- But there are periods with high and periods with low volatility (*volatility drift*).
- Volatility index (VIX) from 1990 to 2009:



Drifting Away – Heston and GARCH Models

- We need to describe the volatility itself by a random process.
- Stochastic differential equations:

$$dS(t) = \mu S(t)\,\mathrm{d}t + \sqrt{\sigma(t)}S(t)\,\mathrm{d}B_1(t),$$
$$d\sigma(t) = \alpha(\sigma(t))\,\mathrm{d}t + \beta(\sigma(t))\,\mathrm{d}B_2(t),$$

  where α and β are given and B_1, B_2 are both Brownian motions that may correlate with each other with correlation $\rho \in [-1, +1]$.
- Fluctuation of the volatility tends to be larger when the volatility is large.
- Standard models with $\alpha(\sigma(t)) = \theta(\omega - \sigma(t))$ and $\beta(\sigma(t)) = \xi\sigma(t)^\gamma$.
- θ describes how strongly the process tends to return to its mean, ξ is the (constant part of the) volatility of σ. Finally, γ is an exponent that describes how strong the volatility of σ increases when σ increases.

Drifting Away – Heston and GARCH Models

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- Standard models with $\alpha(\sigma(t)) = \theta(\omega - \sigma(t))$ and $\beta(\sigma(t)) = \xi\sigma(t)^\gamma$.
- θ describes how strongly the process tends to return to its mean, ξ is the (constant part of the) volatility of σ. Finally, γ is an exponent that describes how strong the volatility of σ increases when σ increases.

Drifting Away – Heston and GARCH Models

- We need to describe the volatility itself by a random process.
- Stochastic differential equations:

$$dS(t) = \mu S(t)\,\mathrm{d}t + \sqrt{\sigma(t)}S(t)\,\mathrm{d}B_1(t),$$
$$d\sigma(t) = \alpha(\sigma(t))\,\mathrm{d}t + \beta(\sigma(t))\,\mathrm{d}B_2(t),$$

  where α and β are given and B_1, B_2 are both Brownian motions that may correlate with each other with correlation $\rho \in [-1, +1]$.
- Fluctuation of the volatility tends to be larger when the volatility is large.
- Standard models with $\alpha(\sigma(t)) = \theta(\omega - \sigma(t))$ and $\beta(\sigma(t)) = \xi\sigma(t)^\gamma$.
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Drifting Away – Heston and GARCH Models

- Three frequently used models
  - The *Heston model* assumes $\gamma = 1/2$. The variance process is in this case called a *CIR process*, named after its inventors John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross [Cox et al., 1985].
  - The *Generalized Autoregressive Conditional Heteroskedasticity model* (short: *GARCH*) assumes $\gamma = 1$.
  - The 3/2 *model* assumes $\gamma = 3/2$.
- One more interesting feature about the volatility: volatility tends to be lower in bull markets, and higher in bear markets.
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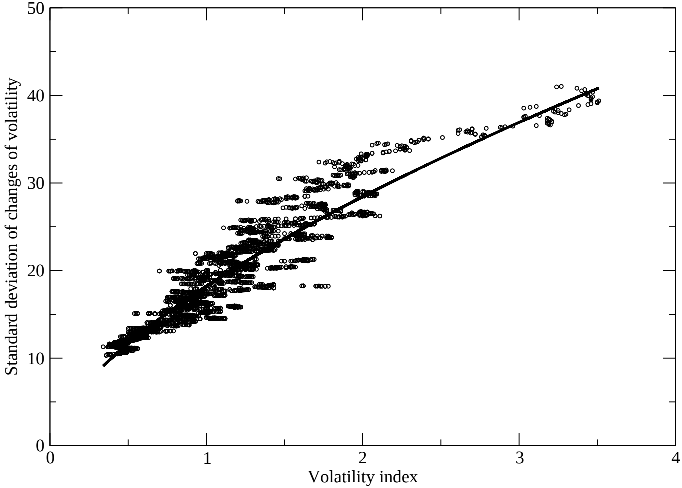
Drifting Away – Heston and GARCH Models



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