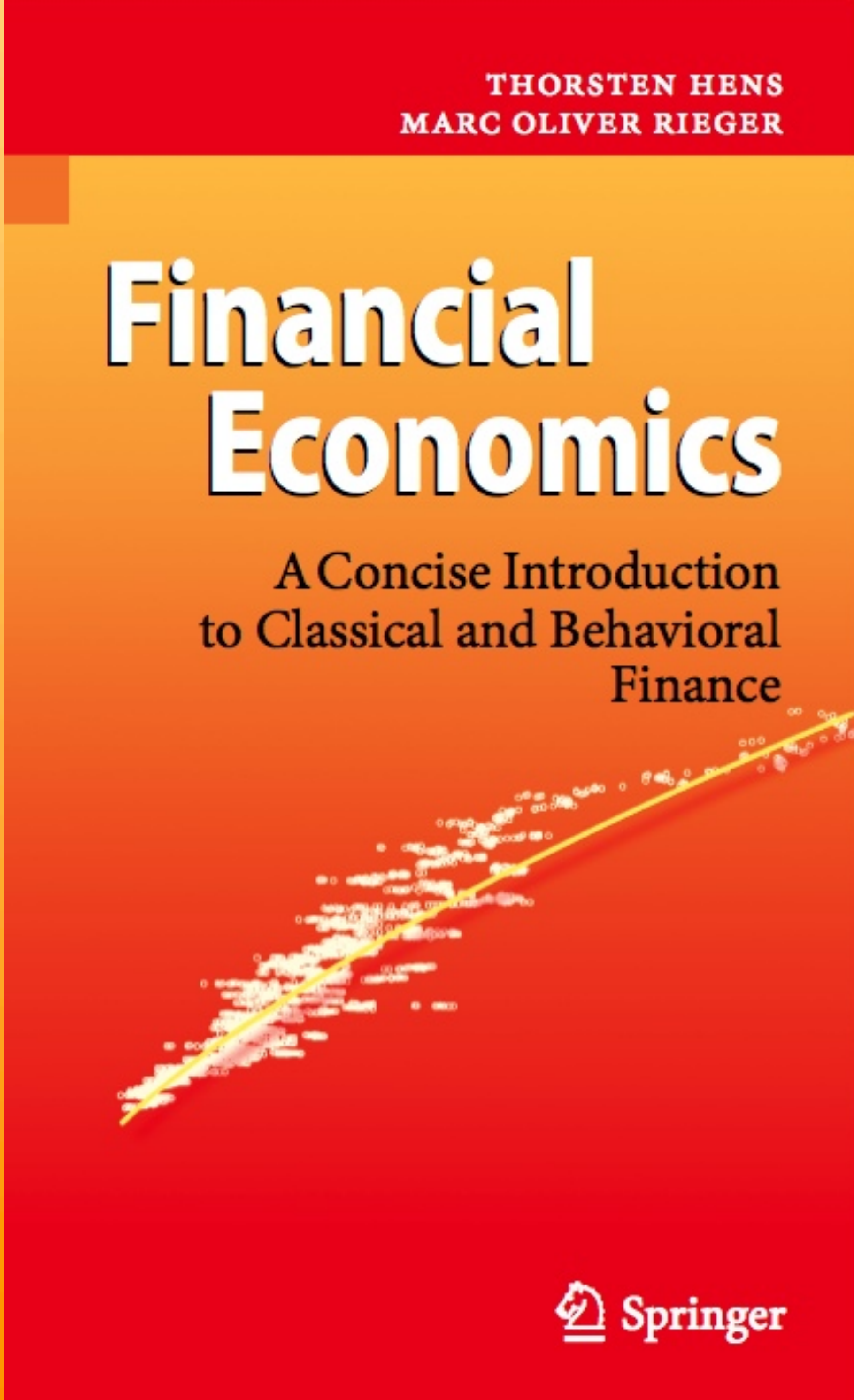

HENS · RIEGER

Financial Economics

Financial economics is a fascinating topic where ideas from economics, mathematics and, most recently, psychology are combined to understand financial markets. This book gives a concise introduction into this field and includes for the first time recent results from behavioral finance that help to understand many puzzles in traditional finance. The book is tailor made for master and PhD students and includes tests and exercises that enable the students to keep track of their progress. Parts of the book can also be used on a bachelor level. Researchers will find it particularly useful as a source for recent results in behavioral finance and decision theory.

The text book to this class is available at www.springer.com

On the book's homepage at www.financial-economics.de there is further material available to this lecture, e.g. corrections and updates.

Financial Economics

A Concise Introduction to Classical and Behavioral Finance
Chapter 2

Thorsten Hens and Marc Oliver Rieger

Swiss Banking Institute, University of Zurich / BWL, University of Trier

July 31, 2010

Decision Theory

"As soon as questions of will or decision or reason or choice of action arise, human science is at a loss." Noam Chomsky

Decisions

- Two central questions of Decision Theory:
  - *Prescriptive (rational)* approach: How rational decisions should be made
  - *Descriptive (behavioral)* approach: Model the actual decisions made by individuals.

  More in the book on page 15.

- In this book choices between alternatives involving risk and uncertainty.
  - *Risk* means here that a decision leads to consequences that are not precisely predictable, but follow a known probability distribution.
  - *Uncertainty* or *ambiguity* means that this probability distribution is at least partially unknown to the decision maker.

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Preference Relations between Lotteries

- A lottery is a given set of states together with their respective outcomes and probabilities.
- A preference relation is a set of rules that states how we make pairwise decisions between lotteries.
- Example (see page 16 in the book):

State preference approach:

state	probability	stock	bond
Boom	$prob_1$	a_1^s	a_1^b
Recession	$prob_2$	a_2^s	a_2^b

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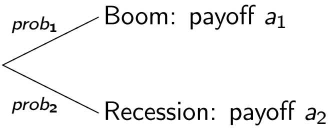

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Preference Relations between Lotteries

- Lottery approach: add the probabilities of all states where our asset has the same payoff:

$$p_c = \sum_{\{i=1,\dots,S \,|\, a_i=c\}} prob_i.$$

- A *preference* compares lotteries.
- $A \succ B$: we prefer lottery A over B.
- $A \sim B$: we are indifferent between A and B.

Preference Relation

Definition

A preference relation $\succeq$ on $\mathcal{P}$ satisfies the following conditions:

(i) *It is complete, i.e., for all lotteries A, $B \in \mathcal{P}$, either $A \succeq B$ or $B \succeq A$ or both.*

(ii) *It is transitive, i.e., for all lotteries A, B, $C \in \mathcal{P}$ with $A \succeq B$ and $B \succeq C$ we have $A \succeq C$.*

More in the book on page 17.

Woody Allen:

> *Money is better than poverty, if only for financial reasons.*

State Dominance

Definition

If, for all states $s = 1, \ldots S$*, we have* $a_s^A \geq a_s^B$ *and there is at least one state* $s \in \{1, \ldots, S\}$ *with* $a_s^A > a_s^B$*, then we say that A state dominates B. We sometimes write* $A \succeq_{SD} B$*.*
We say that a preference relation $\succeq$ *respects (or is compatible with) state dominance if* $A \succeq_{SD} B$ *implies* $A \succeq B$*. If* $\succeq$ *does not respect state dominance, we say that it violates state dominance.*

More in the book on page 18.

Utility Functional

Definition

Let U be a map that assigns a real number to every lottery. We say that U is a utility functional for the preference relation $\succeq$ if for every pair of lotteries A and B, we have $U(A) \geq U(B)$ if and only if $A \succeq B$. In the case of state independent preference relations, we can understand U as a map that assigns a real number to every probability measure on the set of possible outcomes, i.e., $U : \mathcal{P} \to \mathbb{R}$.

More in the book on page 19.

Origins of Expected Utility Theory

- The concept of probabilities was developed in the 17th century by Pierre de Fermat, Blaise Pascal and Christiaan Huygens, among others.
- Expected value of a lottery A having outcomes x_i with probabilities p_i:

$$\mathbb{E}(A) = \sum_i x_i p_i.$$

St. Petersburg Paradox

Daniel Bernoulli:
After paying a fixed entrance fee, a fair coin is tossed repeatedly until a "tails" first appears. This ends the game. If the number of times the coin is tossed until this point is k, you win 2^{k-1} ducats.

$$\begin{aligned} p_k &= P(\text{"head" on 1st toss}) \cdot P(\text{"head" on 2nd toss}) \cdots \\ &\quad \cdots P(\text{"tail" on } k\text{-th toss}) \\ &= \left(\tfrac{1}{2}\right)^k . \\ \mathbb{E}(A) &= \sum_{k=1}^{\infty} x_k p_k = \sum_{k=1}^{\infty} 2^{k-1} \left(\frac{1}{2}\right)^k = \sum_{k=1}^{\infty} \frac{1}{2} = +\infty . \end{aligned}$$

Most people would be willing to pay not more than a couple of ducats.

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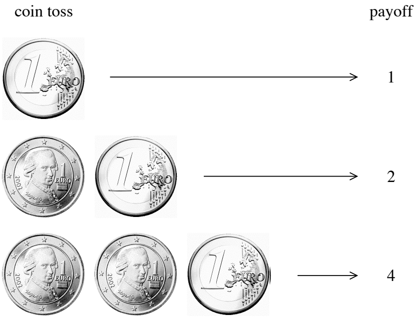
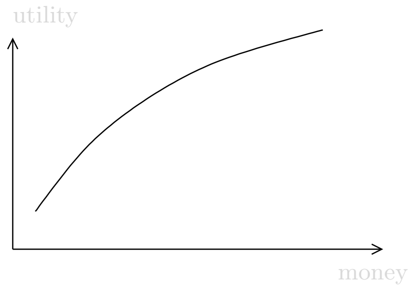
St. Petersburg Lottery



Daniel Bernoulli noticed, it is not at all clear why twice the money should always be twice as good.

St. Petersburg Paradox

- In Bernoulli's own words:
  "There is no doubt that a gain of one thousand ducats is more significant to the pauper than to a rich man though both gain the same amount."
- Utility function:



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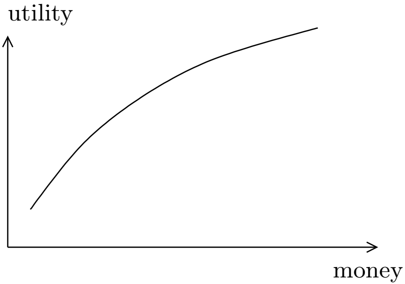


St. Petersburg Paradox

- We want to maximize the expected value of the utility, in other words, our utility functional becomes

$$U(p) = \mathbb{E}(u) = \sum_i u(x_i)p_i,$$

- This resolves the St. Petersburg Paradox.
- Assume $u(x) := \ln(x)$, then

$$EUT(\text{Lottery}) = \sum_k u(x_k)p_k = \sum_k \ln(2^{k-1}) \left(\frac{1}{2}\right)^k$$
$$= (\ln 2) \sum_k \frac{k-1}{2^k} < +\infty.$$

This is caused by the "diminishing marginal utility of money", see in the book on page 24.

Definitions

Definition (Concavity)

We call a function $u: \mathbb{R} \to \mathbb{R}$ *concave on the interval* (a, b) *(which might be* $\mathbb{R}$*) if for all* $x_1, x_2 \in (a, b)$ *and* $\lambda \in (0, 1)$ *the following inequality holds:*

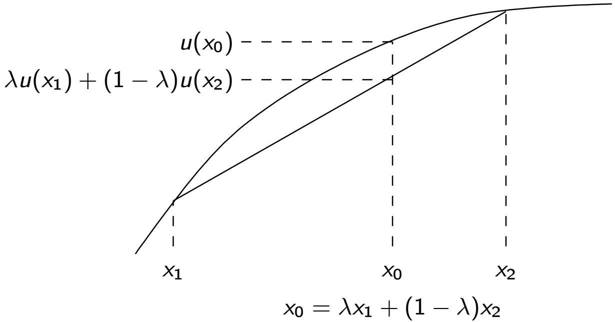
$$\lambda u(x_1) + (1 - \lambda)u(x_2) \leq u\left(\lambda x_1 + (1 - \lambda)x_2\right).$$

We call u *strictly concave if the above inequality is always strict (for* $x_1 \neq x_2$*).*

Definition (Risk-averse behavior)

We call a person risk-averse if he prefers the expected value of every lottery over the lottery itself.

A Strictly Concave Function



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We call u *strictly convex if the above inequality is always strict (for* $x_1 \neq x_2$*).*

Definition (Risk-seeking behavior)

We call a person risk-seeking if he prefers every lottery over its expected value.

Propositions

Proposition

If u is strictly concave, then a person described by the Expected Utility Theory with the utility function u is risk-averse. If u is strictly convex, then a person described by the Expected Utility Theory with the utility function u is risk-seeking.

U is fixed only up to monotone transformations and u only up to positive affine transformations:

Proposition

Let $\lambda > 0$ and $c \in \mathbb{R}$. If u is a utility function that corresponds to the preference relation $\succeq$, i.e., $A \succeq B$ implies $U(A) \geq U(B)$, then $v(x) := \lambda u(x) + c$ is also a utility function corresponding to $\succeq$.

More in the book on page 27.

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More in the book on page 27.

Axiomatic Definition

We can derive EUT from a set of much simpler assumptions on an individual's decisions.

Axiom (Completeness)

For every pair of possible alternatives, A, B, either $A \prec B$, $A \sim B$ or $A \succ B$ holds.

Axiom (Transitivity)

For every A,B,C with $A \preceq B$ and $B \preceq C$, we have $A \preceq C$.

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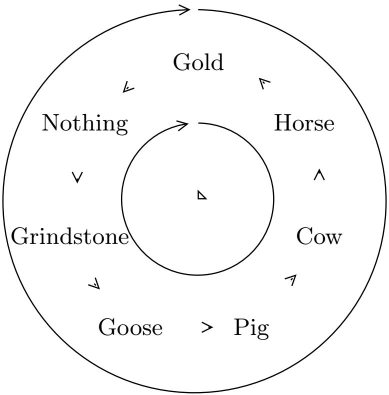
The Cycle of the "Lucky Hans", Violating Transitivity



Axiomatic Definition

Definition

Let A and B be lotteries and $\lambda \in [0,1]$, then $\lambda A + (1-\lambda)B$ denotes a new combined lottery where with probability λ the lottery A is played, and with probability $1-\lambda$ the lottery B is played.

An example can be found in the book on page 31.

Axiom (Independence)

Let A and B be two lotteries with $A \succ B$, and let $\lambda \in (0,1]$ then for any lottery C, it must hold

$$\lambda A + (1-\lambda)C \succ \lambda B + (1-\lambda)C.$$

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Axiomatic Definition

Axiom (Continuity)

Let A, B, C be lotteries with $A \succeq B \succeq C$ then there exists a probability p such that $B \sim pA + (1-p)C$.

Theorem (Expected Utility Theory)

A preference relation that satisfies the Completeness Axiom 1, the Transitivity Axiom 2, the Independence Axiom 3 and the Continuity Axiom 4, can be represented by an EUT functional. EUT always satisfies these axioms.

The proof can be found in the book on page 33.

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Which Utility Functions are "Suitable"?

- Risk aversion measure:

$$r(x) := -\frac{u''(x)}{u'(x)},$$

  first introduced by J.W. Pratt.

- The larger r, the more a person is risk-averse.

Which Utility Functions are "Suitable"?

Proposition

Let p be the outcome distribution of a lottery with $\mathbb{E}(p) = 0$, in other words, p is a fair bet. Let w be the wealth level of the person, then, neglecting higher order terms in $r(w)$ and p,

$$EUT(w + p) = u\left(w - \frac{1}{2}\,\mathrm{var}(p) r(w)\right),$$

where $\mathrm{var}(p)$ denotes the variance of p. We could say that the "risk premium", i.e., the amount the person is willing to pay for an insurance against a fair bet, is proportional to $r(w)$.

The proof can be found in the book on page 37.

CARA (Constant Absolute Risk Aversion)

- One standard assumption: risk aversion measure r is constant for all wealth levels.
- Example:

$$u(x) := -\mathrm{e}^{-Ax}.$$

- We can verify this by computing

$$r(x) = -\frac{u''(x)}{u'(x)} = \frac{A^2\mathrm{e}^{-Ax}}{A\mathrm{e}^{-Ax}} = A.$$

- Realistic values: $A \approx 0.0001$.

CRRA (Constant Relative Risk Aversion)

- Another standard approach: $r(x)$ should be proportional to x.
- Relative risk aversion:

$$rr(x) := xr(x) = -x\frac{u''(x)}{u'(x)}$$

is constant for all x.

- Examples:

$$u(x) := \frac{x^R}{R}, \quad \text{where } R < 1,\ R \neq 0,$$
$$u(x) := \ln x.$$

- Typical values for R are between -1 and -3.

HARA (Hyperbolic Absolute Risk Aversion)

- Generalization of the classes of utility functions.
- All functions where the *reciprocal of absolute risk aversion*, $T := 1/r(x)$, is an affine function of x.

Proposition

A function $u\colon \mathbb{R} \to \mathbb{R}$ *is HARA if and only if it is an affine transformation of one of these functions:*

$$v_1(x) := \ln(x+a), \quad v_2(x) := -a\mathrm{e}^{-x/a}, \quad v_3(x) := \frac{(a+bx)^{(b-1)/b}}{b-1},$$

where a *and* b *are arbitrary constants (*$b \notin \{0,1\}$ *for* v_3*). If we define* $b := 1$ *for* v_1 *and* $b := 0$ *for* v_2*, we have in all three cases* $T = a + bx$*.*

Classes of Utility Functions

Class of utilities	Definition	ARA $r(x)$	RRA $rr(x)$	Special properties
Logarithmic	$\ln(x+c)$, $c \geq 0$	decr.	const.	"Bernoulli utility"
Power	$\frac{1}{\alpha}x^{\alpha}$, $\alpha \neq 0$	decr.	const.	risk-averse if $\alpha < 1$, bounded if $\alpha < 0$
Quadratic	$x - \alpha x^2$, $\alpha > 0$	incr.	incr.	bounded, monotone only up to $x = (2\alpha)^{-1}$
Exponential	$-e^{-\alpha x}$, $\alpha > 0$	const.	incr.	bounded

"Super St. Petersburg Paradox", see in the book on page 41.

Theorem (St. Petersburg Lottery)

Let p be the outcome distribution of a lottery. Let $u : \mathbb{R} \to \mathbb{R}$ be a utility function.

(i) *If u is bounded, then $EUT(p) := \int u(x)\,\mathrm{d}p < \infty$.*

(ii) *Assume that $\mathbb{E}(p) < \infty$. If u is asymptotically concave, i.e., there is a $C > 0$ such that u is concave on the interval $[C, +\infty)$, then $EUT(p) < \infty$.*

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Measuring the Utility Function

- Midpoint certainty equivalent method.
- A subject is asked to state a monetary equivalent to a lottery with two outcomes.
- Each with probability 1/2.
- Such a monetary equivalent is called a *Certainty Equivalent (CE)*.
- Set $u(x_0) := 0$ and $u(x_1) := 1$, then $u(CE) = 0.5$.
- Set $x_{0.5} := CE$ and iterate this method by comparing a lottery with the outcomes x_0 and $x_{0.5}$ and probabilities 1/2 each etc.
- An example can be found in the book on page 44.

Definition and Fundamental Properties

- Introduced in 1952 by Markowitz
- Key idea: measure the risk of an asset by only one parameter, the variance σ.

Definition (Mean-Variance approach)

A mean-variance utility function u is a utility function $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ *which corresponds to a utility functional* $U : \mathcal{P} \to \mathbb{R}$ *that only depends on the mean and the variance of a probability measure p, i.e.,* $U(p) = u(\mathbb{E}(p), \mathrm{var}(p))$.

Definition

A mean-variance utility function $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ *is called monotone if* $u(\mu, \sigma) \geq u(\nu, \sigma)$ *for all* μ, ν, σ *with* $\mu > \nu$. *It is called strictly monotone if even* $u(\mu, \sigma) > u(\nu, \sigma)$.

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Remark

Let u be a mean-variance function. Then the preference induced by u is risk-averse if and only if $u(\mu,\sigma) < u(\mu,0)$ *for all* μ,σ. *The preference is risk-seeking if and only if* $u(\mu,\sigma) > u(\mu,0)$.

Example:

$$u_1(\mu,\sigma) := \mu - \sigma^2.$$

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Success and Limitation

- Main advantage of the mean-variance approach: simplicity.
- Allows us to use (μ, σ)-diagrams.
- Nevertheless certain problems and limitations of the Mean-Variance Theory.
- Example: the following two assets have identical mean and variance:

$A :=$

payoff	0€	1010€
probability	99.5%	0.05%

$B :=$

payoff	-1000€	10€
probability	0.05%	99.5%

Success and Limitation

There are strong theoretical limitations:

Theorem (Mean-Variance Paradox)

For every continuous mean-variance utility function $u(\mu, \sigma)$ which corresponds to a risk-averse preference, there exist two assets A and B where A state dominates B, but B is preferred over A.

Proof and Corollary can be found in the book on page 50 and 51.

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Prospect Theory

- How do people *really* decide?
- As if they were maximizing their expected utility?
- Or as if they were following the mean-variance approach?

Example: Allais Paradox

- Consider four lotteries A, B, C and D.
- In each lottery a random number is drawn from the set $\{1, 2, \ldots, 100\}$ where each number occurs with the same probability of 1%.
- The lotteries assign outcomes to every of these 100 possible numbers (states).
- The test persons are asked to decide between the two lotteries A and B and then between C and D. Most people prefer B over A and C over D.
- This behavior is not rational and violates the Independence Axiom.

The four lotteries of Allais' Paradox

Lottery A	State	1–33	34–99	100
	Outcome	2500	2400	0
Lottery B	State	1–100		
	Outcome	2400		
Lottery C	State	1–33	34–100	
	Outcome	2500	0	
Lottery D	State	1–33	34–99	100
	Outcome	2400	0	2400

Observed Facts

- People tend to buy insurances (risk-averse behavior) and take part in lotteries (risk-seeking behavior) at the same time.
- People are usually risk-averse even for small-stake gambles and large initial wealth. This would predict a degree on risk aversion for high-stake gambles that is far away from standard behavior.
- Does this mean, that the "homo economicus" is dead and that all models of humans as rational deciders are obsolete?
- Is "science at a loss" when it comes to people's decisions?
- The "homo economicus" is still a central concept, and there are modifications that describe human decisions.

Observed Facts

- Framing effect
- in gains: people behave risk-averse
- losses: people tend to behave risk-seeking.
- People tend to systematically *overweight* small probabilities.
- Risk attitudes depending on probability and frame:

	Losses	Gains
Medium probabilities	risk-seeking	risk-averse
Low probabilities	risk-averse	risk-seeking

- We explain Allais' Paradox with this idea.

Original Prospect Theory

- Kahneman and Tversky
- Instead of the final wealth we consider the gain and loss (framing effect) instead of the real probabilities we consider weighted probabilities
- Subjective utility:

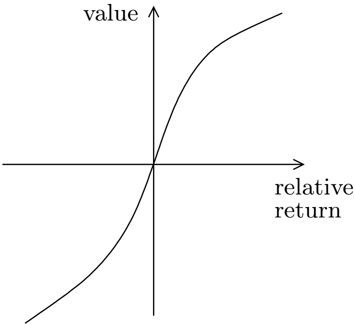
$$PT(A) := \sum_{i=1}^{n} v(x_i)w(p_i),$$

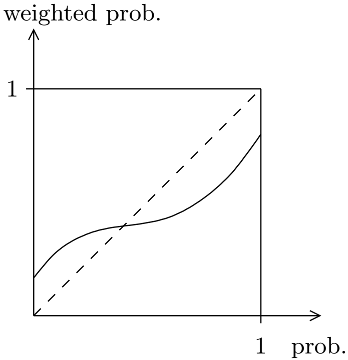
where $v : \mathbb{R} \to \mathbb{R}$ is the *value function*, defined on losses and gains. $w : [0,1] \to [0,1]$ is the *probability weighting function*.

Original Prospect Theory

- v is continuous and monotone increasing.
- The function v is strictly concave for values larger than zero, i.e., in gains, but strictly convex for values less than zero, i.e., in losses.
- At zero, the function v is "steeper" in losses than in gains, i.e., its slope at $-x$ is bigger than its slope at x.
- The function w is continuous and monotone increasing.
- $w(p) > p$ for small values of $p > 0$ (probability overweighting) and $w(p) < p$ for large values of $p < 1$ (probability underweighting), $w(0) = 0$, $w(1) = 1$ (no weighting for sure outcomes).

Original Prospect Theory





Original Prospect Theory

- If we have many events, all of them will probably be overweighted and the sum of the weighted probabilities will be large.
- Alternative formulation of Prospect Theory that fixes the problem:

$$PT(A) = \frac{\sum_{i=1}^{n} v(x_i)w(p_i)}{\sum_{i=1}^{n} w(p_i)}.$$

- More about this and the four-fold pattern in the book on page 58.

Original Prospect Theory

Definition (Stochastic dominance)

A lottery A is stochastically dominant over a lottery B if, for every payoff x, the probability to obtain more than x is larger or equal for A than for B and there is at least some payoff x such that this probability is strictly larger.

An example can be found in the book on page 59.

- PT violates stochastic dominance.
- Another limitation: PT can only be applied for finitely many outcomes.
- In finance, however, we are interested ininfinitely many outcomes.

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Cumulative Prospect Theory

Key idea: Replace the probabilities by *differences of cumulative probabilities*.

Definition (Cumulative Prospect Theory)

For a lottery A with n outcomes $x_1, \ldots, x_n$ *and the probabilities* $p_1, \ldots, p_n$ *where* $x_1 < x_2 < \cdots < x_n$ *and* $\sum_{i=1}^{n} p_i = 1$ *we define*

$$CPT(A) := \sum_{i=1}^{n} \left(w(F_i) - w(F_{i-1})\right) v(x_i), \tag{1}$$

where $F_0 := 0$ *and* $F_i := \sum_{j=1}^{i} p_j$ *for* $i = 1, \ldots, n$.

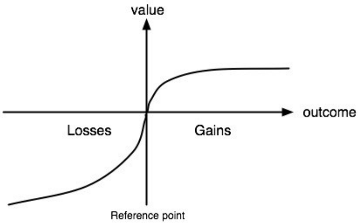
How this formula is connected to Prospect Theory is written in the book on page 60.

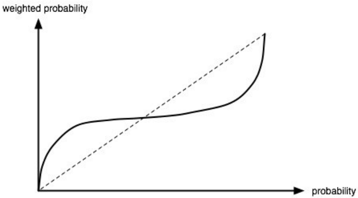
Cumulative Prospect Theory

- On average, events are neither over- nor underweighted in CPT, see page 61 in the book.
- Prototypical example of a value function v:

$$v(x) := \begin{cases} x^{\alpha} & , x \geq 0, \\ -\lambda(-x)^{\beta} & , x < 0, \end{cases} \tag{2}$$

- Value and weighting function:





Cumulative Prospect Theory

- Probability weighting function w:

$$w(p) := \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}},$$

- Experimental values:

Study	Estimate for α,β	Estimate for γ, δ
Tversky and Kahneman		
gains:	0.88	0.61
losses:	0.88	0.69
Camerer and Ho	0.37	0.56
Tversky and Fox	0.88	0.69
Wu and Gonzalez		
gains:	0.52	20.71
Abdellaoui		
gains:	0.89	0.60
losses:	0.92	0.70
Bleichrodt and Pinto	0.77	0.67/0.55
Kilka and Weber	0.76-1.00	0.30-0.51

Cumulative Prospect Theory

- Extend CPT to arbitrary lotteries.
- we can describe lotteries by probability measures, see Appendix A.4 for details.

Definition

Let p be an arbitrary probability measure, then the generalized form of CPT reads as

$$CPT(p) := \int_{-\infty}^{+\infty} v(x) \left(\frac{\mathrm{d}}{\mathrm{d}t} w(F(t))|_{t=x} \right) \mathrm{d}x, \quad (3)$$

where

$$F(t) := \int_{-\infty}^{t} \mathrm{d}p.$$

Cumulative Prospect Theory

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Cumulative Prospect Theory

Proposition

CPT does not violate stochastic dominance, i.e., if A is stochastic dominant over B then $CPT(A) > CPT(B)$.

The proof can be found in the book on page 64.

- Since the value function has the same convex-concave shape in CPT as in PT, the four-fold pattern of risk-attitudes can be explained in exactly the same manner.
- Choice of value and weighting function, see book on page 67.

Cumulative Prospect Theory

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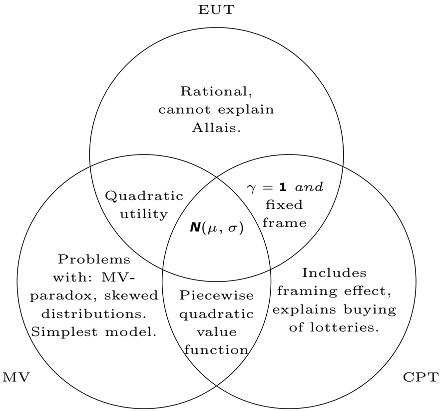
EUT, Mean-Variance Theory and CPT

- EUT is the "rational benchmark". We will use it as a reference of rational behavior and as a prescriptive theory when we want to find an objectively optimal decision.
- Mean-Variance Theory is the "pragmatic solution". The theory is widely used in finance.
- CPT model "real life behavior". We will use it to describe behavior patterns of investors.

Differences and Agreements of EUT, PT and Mean-Variance



Ambiguity and Uncertainty

- Often probabilities are known. We call this *ambiguity* or *uncertainty*.
- Example:
  There is an urn with 300 balls. 100 of them are red, 200 are blue or green. You can pick red or blue and then take one ball (blindly, of course). If it is of the color you picked, you win 100€, else you don't win anything. Which color do you choose?
- Most people choose red.
- Example:
  Same situation, you pick again a color (either red or blue) and then take a ball. This time, if the ball is *not* of the color you picked, you win 100€, else you don't win anything. Which color do you choose?
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Ambiguity and Uncertainty

- People go both times for the "sure" option, the option where they know their probabilities to win.
- Uncertainty-aversity, see book on page 81.
- People are often not very knowledgeable about the chances and risks of financial investments.
- This explains why many people invest into very few stocks (that they are familiar with) or even only into the stock of their own company (even if their company is not performing well).

Time Discounting

- Original utility:

$$u(t) = u(0)\mathrm{e}^{-\delta t},$$

- Classical time discounting leads to a time-consistent preference. More details in the book on page 82.
- Hyperbolic discounting:

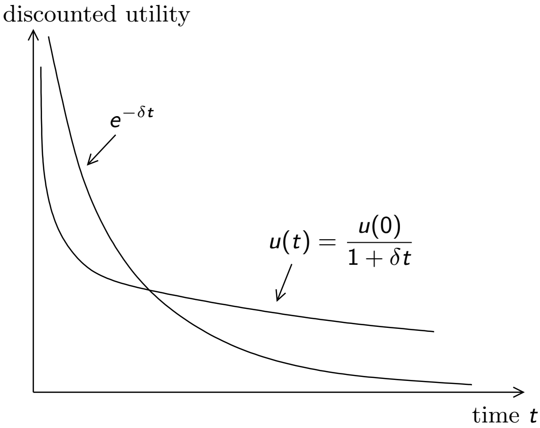
$$u(t) = \frac{u(0)}{1 + \delta t}$$

- Quasi-hyperbolic discounting:

$$u(t) = \begin{cases} u(0) & , \text{ for } t = 0, \\ \frac{1}{1+\beta} u(0) e^{-\delta t} & , \text{ for } t > 0, \end{cases} \qquad \text{where } \beta > 0.$$

- An example can be found in the book on page 84.

Rational versus Hyperbolic Time Discounting



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